THE ROTATION OF A SYMMETRICAL RIGID BODY UNDER THE ACTION OF SUPERPOSITION OF A FOLLOWING MOMENT AND A CONSTANT MOMENT

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The rotation of a symmetrical rigid body around its center of mass is considered. A following moment, having a constant module and directed along the axis of symmetry of the body, and a constant moment act on the body. Method of investigation of the problem is based on using of a special representation of the tensor of turn. The method allows to reduce the problem to the solution of the 4th order differential equation with respect to the moment of momentum vector squared. It is proved, that at the large values of time the motion of the body tends to the rotation around its axis of symmetry and direction of the axis of symmetry of the body tends to the direction of the constant moment.

1. The turn-tensor. The left and right the angular velocity vectors. The Darboux problem.

This section contains known facts which will be used to solve considered problem. Information about the turn-tensor in more detail can be found in [3], [4].

Definition. A properly orthogonal tensor which, is a solution of the equations

$$\underline{\underline{P}} \cdot \underline{\underline{P}}^T = \underline{\underline{P}}^T \cdot \underline{\underline{P}} = \underline{\underline{E}}\,, \qquad det\,\underline{\underline{P}} = +1 \tag{1.1}$$

is called a turn-tensor.

Definition. he tensors $\underline{\underline{S}}(t)$ and $\underline{\underline{S}}_r(t)$, calculated by the formulae

$$\underline{\underline{S}}(t) = \dot{\underline{\underline{P}}}(t) \cdot \underline{\underline{P}}^T(t)\,, \qquad \underline{\underline{S}}_r(t) = \underline{\underline{P}}^T(t) \cdot \dot{\underline{\underline{P}}}(t) \tag{1.2}$$

are called respectively the left spin tensor and the right spin tensor.

Definition. The accompanying vector of the left spin tensor $\underline{\omega}(t)$, defining by the formula

$$\underline{\underline{S}}(t) = \underline{\omega}(t) \times \underline{\underline{E}} \tag{1.3}$$

is called the left angular velocity vector.

Definition. The accompanying vector of the right spin tensor $\underline{\Omega}(t)$, defining by the formula

$$\underline{\underline{S}}_r(t) = \underline{\Omega}(t) \times \underline{\underline{E}} \tag{1.4}$$

is called the right angular velocity vector.

The left Darboux problem. Let the left angular velocity vector $\underline{\omega}(t)$ be known and the turn-tensor should be found. This problem is formulated as follows

$$\dot{\underline{\underline{P}}}(t) = \underline{\omega}(t) \times \underline{\underline{P}}(t)\,, \qquad \underline{\underline{P}}(t_0) = \underline{\underline{P}}_0 \tag{1.5}$$

The right Darboux problem. Let the right angular velocity vector $\underline{\Omega}(t)$ be known and the turn-tensor should be found. This problem is formulated as follows

$$\dot{\underline{\underline{P}}}(t) = \underline{\underline{P}}(t) \times \underline{\Omega}(t)\,, \qquad \underline{\underline{P}}(0) = \underline{\underline{P}}_0 \tag{1.6}$$

Theorem: Every turn-tensor can be represented as the composition of any number of the turn-tensors:

$$\underline{\underline{P}}(t) = \underline{\underline{P}}_n(t) \cdot \underline{\underline{P}}_{n-1}(t) \cdot \ \dots \ \cdot \underline{\underline{P}}_3(t) \cdot \underline{\underline{P}}_2(t) \cdot \underline{\underline{P}}_1(t) \qquad (1.7)$$

Theorem: If the turn-tensor $\underline{\underline{P}}(t)$ is represented as the composition of two turn-tensors $\underline{\underline{P}}_1(t)$ and $\underline{\underline{P}}_2(t)$, then the angular velocity vector $\underline{\omega}(t)$, corresponding to the turn-tensor $\underline{\underline{P}}(t)$, is expressed in terms of the angular velocity vectors $\underline{\omega}_1(t)$ and $\underline{\omega}_2(t)$, corresponding to the turn-tensors $\underline{\underline{P}}_1(t)$ and $\underline{\underline{P}}_2(t)$, as follows:

$$\underline{\underline{P}}(t) = \underline{\underline{P}}_2(t) \cdot \underline{\underline{P}}_1(t) \qquad \Longrightarrow \qquad \underline{\omega}(t) = \underline{\omega}_2(t) + \underline{\underline{P}}_2(t) \cdot \underline{\omega}_1(t) \qquad (1.8)$$

Euler theorem: Every turn-tensor $\underline{\underline{P}}(t) \neq \underline{\underline{E}}$ can be represented by uniquely in the form

$$\underline{\underline{P}}(\theta \underline{m}) = (1 - \cos\theta(t))\, \underline{m}(t)\underline{m}(t) + \cos\theta(t)\, \underline{\underline{E}} + \sin\theta(t)\, \underline{m}(t) \times \underline{\underline{E}}\,, \quad |\underline{m}(t)| = 1 \qquad (1.9)$$

Using Euler representation of the turn-tensor, it is easy to derive expressions for the angular velocities

$$\begin{aligned} \underline{\omega}(t) &= \dot{\theta}(t)\, \underline{m}(t) + \sin\theta(t)\, \dot{\underline{m}}(t) + (1 - \cos\theta(t))\, \underline{m}(t) \times \dot{\underline{m}}(t) \\ \underline{\Omega}(t) &= \dot{\theta}(t)\, \underline{m}(t) + \sin\theta(t)\, \dot{\underline{m}}(t) - (1 - \cos\theta(t))\, \underline{m}(t) \times \dot{\underline{m}}(t) \end{aligned} \qquad (1.10)$$

2. Representation of the turn-tensor of an symmetrical rigid body by using of the kinetic moment vector.

Theorem: Let the inertia tensor of a rigid body in the reference position be

$$\underline{\underline{\theta}} = \lambda\, \underline{k}\underline{k} + \mu\, (\underline{\underline{E}} - \underline{k}\underline{k}) \qquad (2.1)$$

Let the actual position of the rigid body be determined by the turn-tensor $\underline{\underline{P}}(t)$ and the rigid body have the angular velocity $\underline{\omega}(t)$, and the kinetic moment vector (the moment of momentum vector) of the rigid body, calculated with respect to some point of the body, be $\underline{L}(t)$:

$$\underline{L}(t) = \underline{\underline{P}}(t) \cdot \underline{\underline{\theta}} \cdot \underline{\underline{P}}^T(t) \cdot \underline{\omega}(t) \qquad (2.2)$$

In this case the turn-tensor of the rigid body $\underline{\underline{P}}(t)$ can be represented as the composition of two turn-tensors

$$\underline{\underline{P}}(t) = \underline{\underline{P}}_L(t) \cdot \underline{\underline{P}}_*(t) \qquad (2.3)$$

The turn-tensor $\underline{\underline{P}}_L(t)$ is determined by the kinetic moment vector of the rigid body $\underline{L}(t)$ as solution of the left Darboux problem

$$\dot{\underline{\underline{P}}}_L(t) = \underline{\omega}_L(t) \times \underline{\underline{P}}_L(t)\,, \qquad \underline{\omega}_L(t) = \mu^{-1} \underline{L}(t)\,, \qquad \underline{\underline{P}}_L(0) = \underline{\underline{P}}(0) \qquad (2.4)$$

The turn-tensor $\underline{\underline{P}}_*(t)$ has the form

$$\underline{\underline{P}}_*(t) = (1-\cos\beta(t))\,\underline{k}\underline{k} + \cos\beta(t)\,\underline{\underline{E}} + \sin\beta(t)\,\underline{k}\times\underline{\underline{E}} \qquad (2.5)$$

where angle $\beta(t)$ is expressed in terms of the right angular velocity corresponding to turn-tensor $\underline{\underline{P}}_L(t)$

$$\beta(t) = \int (\mu-\lambda)\,\lambda^{-1}\,\underline{k}\cdot\underline{\Omega}_L(t)\,dt\,, \quad \underline{\Omega}_L(t) = \underline{\underline{P}}_L^T(t)\cdot\underline{\omega}_L(t)\,, \quad \beta(0)=0 \qquad (2.6)$$

Proof of the theorem can be found in [3].

3. The rotation of a symmetrical rigid body under the action of a half-tangential moment.

Let us consider a symmetrical rigid body. The center of mass of the body is fixed. A half-tangential moment $\underline{M}(t) = M^*[\underline{n}(t)+\underline{e}]$ acts on the body. Here $\underline{n}(t)$ and $\underline{e}$ are the unit vectors, defining directions of axis of symmetry of the body in the actual position and in the initial position (see Figure 1). Equations of the motion of the body have the form

$$\dot{\underline{L}}(t) = M^*\,[\underline{n}(t)+\underline{e}]\,,$$

$$\underline{L}(t) = \underline{\underline{P}}(t)\cdot[\lambda\underline{e}\underline{e}+\mu(\underline{\underline{E}}-\underline{e}\underline{e})]\cdot\underline{\underline{P}}^T(t)\cdot\underline{\omega}(t),$$

$$\underline{n}(t) = \underline{\underline{P}}(t)\cdot\underline{e}, \qquad \dot{\underline{\underline{P}}}(t) = \underline{\omega}(t)\times\underline{\underline{P}}(t),$$

$$\underline{\underline{P}}(0) = \underline{\underline{E}}, \qquad \underline{\omega}(0) = \underline{\omega}_0. \qquad (3.1)$$

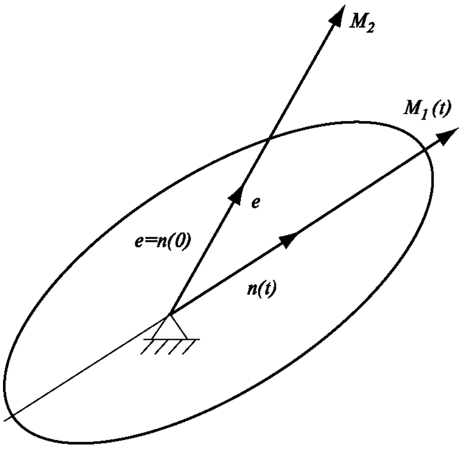

Fig. 1. A half-tangential moment.

Using the theorem of representation of the turn-tensor (2.11) – (2.18), the problem (2.19) can be rewritten in the form

$$\dot{\underline{\omega}}_L(t) = M\,[\underline{n}(t)+\underline{e}]\,, \qquad \underline{\omega}_L(0) = [(\lambda/\mu-1)\underline{e}\underline{e}+\underline{\underline{E}}]\cdot\underline{\omega}_0\,, \qquad M = M^*/\mu$$

$$\underline{n}(t) = \underline{\underline{P}}_L(t)\cdot\underline{e}\,, \qquad \dot{\underline{\underline{P}}}_L(t) = \underline{\omega}_L(t)\times\underline{\underline{P}}_L(t)\,, \qquad \underline{\underline{P}}_L(0) = \underline{\underline{E}}$$

$$\underline{\underline{P}}(t) = \underline{\underline{P}}_L(t)\cdot\underline{\underline{P}}(\beta\underline{e})\,, \qquad \beta(t) = (\mu/\lambda-1)\int\underline{e}\cdot\underline{\Omega}_L(t)\,dt\,, \quad \beta(0)=0$$

$$(3.2)$$

It is easy to show, that then problem (2.20) can be reduced to the following form

$$\ddot{\underline{\omega}}_L(t) - \underline{\omega}_L(t)\times\dot{\underline{\omega}}_L(t) - M\underline{e}\times\underline{\omega}_L(t) = 0$$

$$\dot{\underline{\omega}}_L(0) = 2M\underline{e}\,, \qquad \underline{\omega}_L(0) = \underline{\omega}_{L0}\,, \qquad \underline{\omega}_{L0} = [(\lambda/\mu-1)\underline{e}\underline{e}+\underline{\underline{E}}]\cdot\underline{\omega}_0 \qquad (3.3)$$

If the angular velocity vector $\underline{\omega}_L(t)$ is known, then the vector $\underline{n}(t)$ can be calculated by the formula

$$\underline{n}(t) = M^{-1}\dot{\underline{\omega}}_L(t) - \underline{e} \tag{3.4}$$

The following scalar relations can be obtained from eq. (2.21)

$$\begin{aligned} &\underline{e}\cdot\underline{\omega}_L = \tfrac{1}{4M}(\omega_L^2)^{\cdot}\,, \quad \dot{\underline{\omega}}_L\cdot\dot{\underline{\omega}}_L = \tfrac{1}{2}(\omega_L^2)^{\cdot\cdot}\,, \quad \underline{e}\cdot(\underline{\omega}_L\times\dot{\underline{\omega}}_L) = \tfrac{1}{4M}(\omega_L^2)^{\cdot\cdot\cdot} \\ &\underline{e}\cdot\underline{\Omega}_L = \tfrac{1}{4M}(\Omega_L^2)^{\cdot}\,, \quad \dot{\underline{\Omega}}_L\cdot\dot{\underline{\Omega}}_L = \tfrac{1}{2}(\Omega_L^2)^{\cdot\cdot}\,, \quad \underline{e}\cdot(\underline{\Omega}_L\times\dot{\underline{\Omega}}_L) = -\tfrac{1}{4M}(\Omega_L^2)^{\cdot\cdot\cdot} \end{aligned} \tag{3.5}$$

The relations (2.23) allow to represent the vector $\underline{\omega}_L(t)$ as a function of its squared

$$\begin{gathered} \underline{\omega}_L(t) = z(t)\,\underline{e} + r(t)\,\underline{e}_r[\varphi(t)] \\ z(t) = \frac{(\omega_L^2)^{\cdot}}{4M}\,, \quad r(t) = \sqrt{\omega_L^2 - z^2}\,, \quad \varphi(t) = \frac{1}{4M}\int\frac{(\omega_L^2)^{\cdot\cdot\cdot}}{\omega_L^2 - z^2}\,dt\,, \quad \varphi(0) = 0 \\ \underline{e}_r[\varphi(t)] = \cos\varphi(t)\,\underline{i} + \sin\varphi(t)\,\underline{j}\,, \qquad \underline{i} = \frac{\underline{\omega}_{L0} - (\underline{e}\cdot\underline{\omega}_{L0})\underline{e}}{\sqrt{\omega_{L0}^2 + (\underline{e}\cdot\underline{\omega}_{L0})^2}}\,, \quad \underline{j} = \underline{e}\times\underline{i} \end{gathered} \tag{3.6}$$

Using the eqs. (2.21) and the relations (2.23), it is easy to obtain the Cauchy problem for determination of the function $\omega_L^2(t)$

$$\begin{gathered} (\omega_L^2)^{IV} + \omega_L^2\,(\omega_L^2)^{\cdot\cdot} - \tfrac{1}{4}\left[(\omega_L^2)^{\cdot}\right]^2 - 4M^2\omega_L^2 = 0 \\ (\omega_L^2)^{\cdot\cdot\cdot}\big|_{t=0} = 0\,, \quad (\omega_L^2)^{\cdot\cdot}\big|_{t=0} = 8M^2\,, \quad (\omega_L^2)^{\cdot}\big|_{t=0} = 4M\,\underline{e}\cdot\underline{\omega}_{L0}\,, \quad \omega_L^2\big|_{t=0} = \omega_{L0}^2 \end{gathered} \tag{3.7}$$

The differential equation (2.23) has a first integral

$$\begin{gathered} \left[(\omega_L^2)^{\cdot\cdot\cdot}\right]^2 = \Big((\omega_L^2)^{\cdot\cdot} - 8M^2\Big)\Big(\frac{1}{2}\left[(\omega_L^2)^{\cdot}\right]^2 - \omega_L^2\,(\omega_L^2)^{\cdot\cdot}\Big) \\ (\omega_L^2)^{\cdot\cdot}\big|_{t=0} = 8M^2\,, \quad (\omega_L^2)^{\cdot}\big|_{t=0} = 4M\,\underline{e}\cdot\underline{\omega}_{L0}\,, \quad \omega_L^2\big|_{t=0} = \omega_{L0}^2 \end{gathered} \tag{3.8}$$

Let us introduce new variables

$$y = \frac{1}{8M^2}\,(\omega_L^2)^{\cdot\cdot}\,, \qquad \tau = \frac{1}{8M^{3/2}}\,(\omega_L^2)^{\cdot} \tag{3.9}$$

Using eq. (2.23) it is easy to show, that the function $\tau(t)$ increases. Therefore, the problem (3.8) can be rewritten in new variables (3.9). As a result, the Cauchy problem in function $y(\tau)$ is obtained

$$\begin{gathered} (1-y)y^3y''_\tau + \tfrac{1}{2}y^2y'^2_\tau = 2\tau^2(1-y)^2, \\ y'_\tau\big|_{\tau=\tau_0} = 0, \quad y\big|_{\tau=\tau_0} = 1, \quad \tau_0 = \tfrac{1}{2\sqrt{M}}\,\underline{e}\cdot\underline{\omega}_{L0}. \end{gathered} \tag{3.10}$$

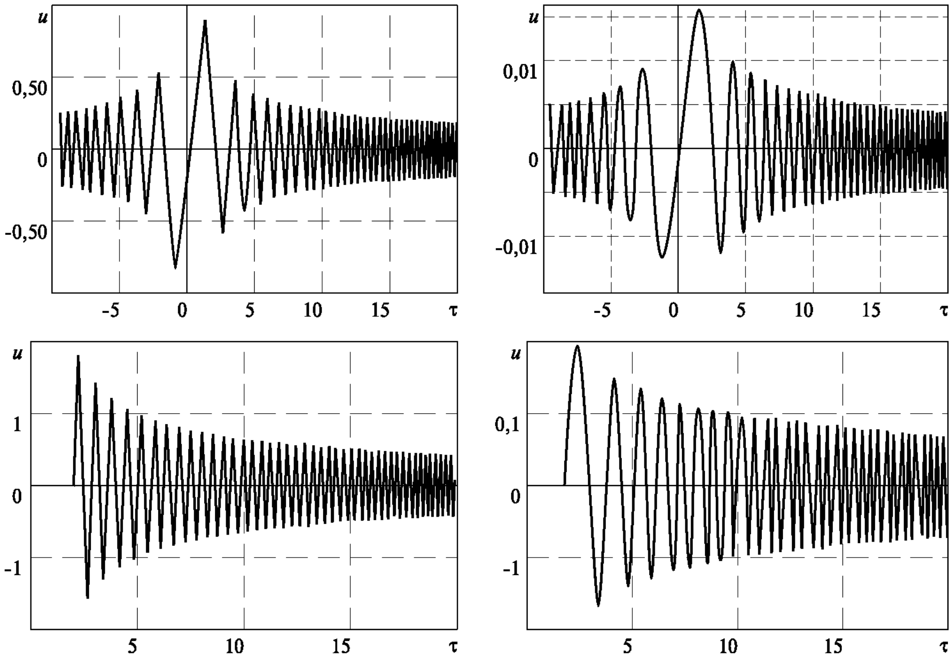

Fig. 2. A half-tangential moment. Function $u(\tau)$.

If the problem (3.10) is solved, then function $\omega_L^2(t)$ can be calculated by the formulae

$$t = \frac{1}{8M^2} \int\limits_{\tau_0}^{\tau} \frac{d\tau}{y(\tau)}\,, \qquad \omega_L^2(t) = \omega_{L0}^2 + \int\limits_0^t \tau(t)\,dt \tag{3.11}$$

The relations (2.23), (3.9) allow to show, that $0 \leqslant y \leqslant 1$. Hence new variables $y = \cos^2\theta$, $u = \mathrm{tg}\,\theta$ can be used. Formulation of the problem in variable $u(\tau)$ is

$$u''_\tau = -\tau^2 u\,(1+u^2)^4 + \frac{4u}{1+u^2}\,u'^2_\tau\,, \quad u(\tau_0) = 0\,, \quad u'_\tau(\tau_0) = \sqrt{\frac{\omega_{L0}^2 - (\underset{\sim}{e}\cdot\underset{\sim}{\omega}_{L0})^2}{4M}} \tag{3.12}$$

Let us introduce notification γ for the angle between the actual and the initial positions of the axis of symmetry of the body. The following relations show the physical sense of variables y , θ and u

$$y = \cos^2\frac{\gamma}{2} \quad\Longrightarrow\quad \theta = \pm\Big(\frac{\gamma}{2} + \pi k\Big)\,, \quad u = \pm\mathrm{tg}\,\frac{\gamma}{2} \qquad (\cos\gamma = \underset{\sim}{e}\cdot\underset{\sim}{n}) \tag{3.13}$$

An exact analytical solution of the eq. (3.2) does not known. However, analytical investigation of this equation allows to determine the behaviour of its solution. It is easy to show, that the function $u(\tau)$ is an oscillating function. If $\tau_0 > 0$ (initial value of the angle α_0 between the external moment and the kinetic moment is less than $\pi/2$), then period and amplitude of oscillations of $u(\tau)$ decrease and $u(\tau) \xrightarrow{\tau\to+\infty} 0$. Let

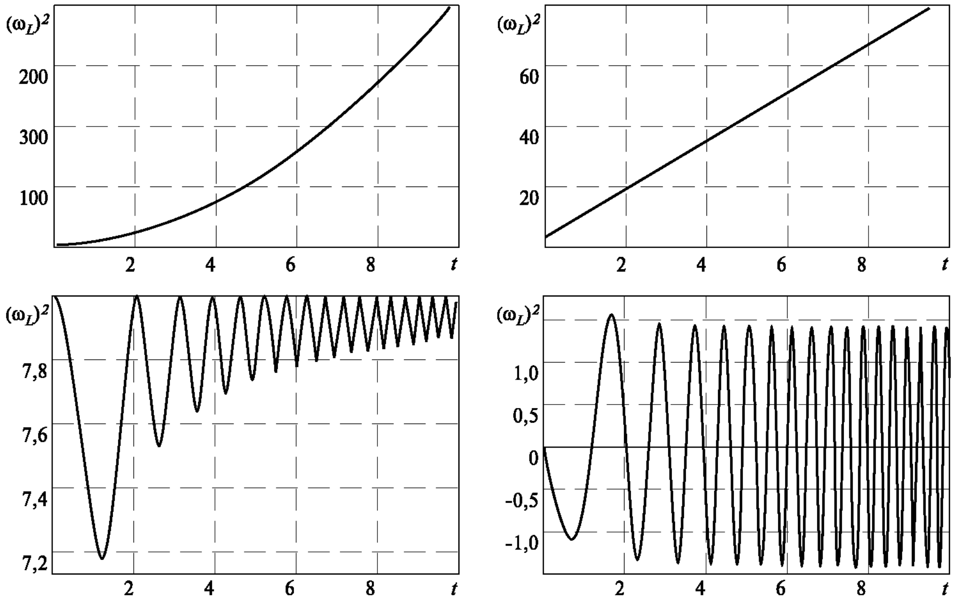

Fig. 3. A half-tangential moment. Functions ω_L^2, $(\omega_L^2)^{\cdot}$, $(\omega_L^2)^{\cdot\cdot}$, $(\omega_L^2)^{\cdot\cdot\cdot}$.

us note amplitude of function $u(\tau)$ as $|u(\tau_n)|$. The following estimations take place

$$|u(\tau_n)| \leqslant |u(\tau_{n-1})|\,, \qquad |u(\tau_n)| \leqslant \frac{u'_\tau(\tau_0)}{\tau_0}\,, \qquad |u(\tau_n)| \geqslant \frac{u'_\tau(\tau_0)}{\tau_n} \tag{3.14}$$

Using the estimations (3.4), it is easy to show, that the angle $\gamma(t)$ between the actual and the initial directions of the axis of symmetry of the body will not be more than $2\alpha_0$. If $\tau_0 = 0$ ($\alpha_0 = \pi/2$), then properties of the function $u(\tau)$ do not differ from them in the case $\tau_0 > 0$, but condition of boundedness above has no sense because it takes the form $|u(\tau)| < +\infty$. If $\tau_0 < 0$ (initial value of the angle α_0 between the external moment and the kinetic moment is more than $\pi/2$), then in the interval τ $[\tau_0, 0]$ period and amplitude of oscillations of $u(\tau)$ increase and in the interval τ $[0, +\infty]$ period and amplitude of oscillations of $u(\tau)$ decrease and $u(\tau) \xrightarrow{\tau\to+\infty} 0$. The following estimations take place

$$\begin{array}{llll}
\tau_0 \leqslant \tau \leqslant 0: & |u(\tau_n)| \geqslant |u(\tau_{n-1})|\,, & |u(\tau_n)| \geqslant \frac{u'(\tau_0)}{|\tau_0|}\,, & |u(\tau_n)| \leqslant \frac{u'(\tau_0)}{|\tau_n|} \\
0 \leqslant \tau \leqslant |\tau_0|: & |u(\tau_n)| \leqslant |u(\tau_{n-1})|\,, & |u(\tau_n)| \geqslant \frac{u'(\tau_0)}{|\tau_0|}\,, & |u(\tau_n)| \leqslant \frac{u'(\tau_0)}{|\tau_n|} \\
|\tau_0| \leqslant \tau < +\infty: & |u(\tau_n)| \leqslant |u(\tau_{n-1})|\,, & |u(\tau_n)| \leqslant \frac{u'(\tau_0)}{|\tau_0|}\,, & |u(\tau_n)| \geqslant \frac{u'(\tau_0)}{|\tau_n|}
\end{array} \tag{3.15}$$

The behaviour of the function $u(\tau)$ is illustrated by Figure 2. The graphics of Figure 2 correspond the following parameters: 1. $\tau_0 = -10$, $u'(\tau_0) = 2.5$; 2. $\tau_0 = -10$, $u'(\tau_0) = 0.05$; 3. $\tau_0 = 2$, $u'(\tau_0) = 4.0$; 4. $\tau_0 = 2$, $u'(\tau_0) = 0.5$.

The bounds of variation of the function ω_L^2 and its derivations are given by the formulae

$$
\begin{gathered}
8M^2\cos^2\alpha_0 \leqslant (\omega_L^2)^{\cdot\cdot} \leqslant 8M^2 \\
8M^2\cos^2\alpha_0\, t + 4M\,\underline{e}\cdot\underline{\omega}_{L0} \leqslant (\omega_L^2)^{\cdot} \leqslant 8M^2 t + 4M\,\underline{e}\cdot\underline{\omega}_{L0} \qquad (3.16) \\
4M^2\cos^2\alpha_0\, t^2 + 4M\,\underline{e}\cdot\underline{\omega}_{L0}\, t + \omega_{L0}^2 \leqslant \omega_L^2 \leqslant 4M^2 t^2 + 4M\,\underline{e}\cdot\underline{\omega}_{L0}\, t + \omega_{L0}^2
\end{gathered}
$$

Hence, the following asymptotic estimations take place at the large values of t

$$
\omega_L^2 \sim t^2\,, \qquad (\omega_L^2)^{\cdot} \sim t\,, \qquad (\omega_L^2)^{\cdot\cdot} \sim 1 \tag{3.17}
$$

Asymptotic solution of the differential equation (3.6) at the large values of t has the form

$$
\begin{aligned}
&\omega_L^2 = (2Mt + D)^2 + O(t^{-3})\,, \qquad (\omega_L^2)^{\cdot} = 4M(2Mt + D) + O(t^{-2}) \\
&(\omega_L^2)^{\cdot\cdot} = 8M^2 + t^{-1}\big[A\cos(Mt^2) + B\sin(Mt^2)\big] + O(t^{-2}) \\
&(\omega_L^2)^{\cdot\cdot\cdot} = 2M\left[-A\sin(Mt^2) + B\cos(Mt^2)\right] + O(t^{-1}) \\
&(\omega_L^2)^{IV} = -4M^2 t\left[A\cos(Mt^2) + B\sin(Mt^2)\right] + O(1)
\end{aligned} \tag{3.18}
$$

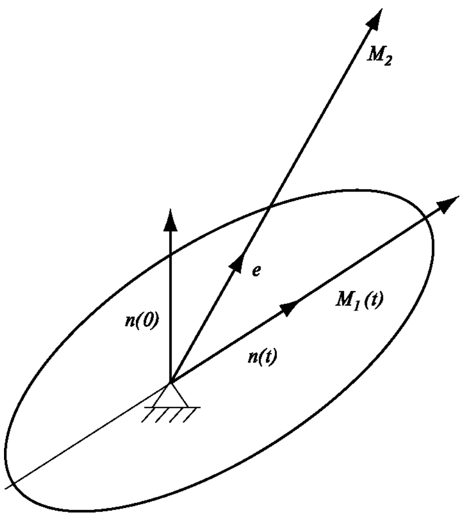

Fig. 4. A superposition of the following and a constant moments.

where A, B, D are constants. The behavior of the function ω_L^2 and its derivations is illustrated by Figure 3. Substituting expressions (3.8) into (3.5), we obtain

$$
\underline{\omega}_L(t) \xrightarrow{t\to+\infty} 2Mt\,\underline{e} \tag{3.19}
$$

Using relation $\underline{e}\cdot\underline{\Omega}_L = \frac{1}{4M}(\Omega_L^2)^{\cdot}$ (see eq. (2.23)) and taking into account relation $\Omega_L^2 = \omega_L^2$, we obtain

$$
\underline{\Omega}_L(t) \xrightarrow{t\to+\infty} 2Mt\,\underline{e} \tag{3.20}
$$

The formulae (3.9), (3.20) prove the fact, that at the large values of t the motion of the rigid body tends to the rotation around its axis of symmetry and the axis of symmetry of the body tends to its initial position: $\underline{n}(t) \xrightarrow{\tau\to+\infty} \underline{e}$.

4. The rotation of a symmetrical rigid body under the action of a superposition of the following moment and a constant moment.

Let us consider a symmetrical rigid body. The center of mass of the body is fixed. Two moments act on the body (see Figure 4). The first one is the following moment $\underline{M}_1(t) = M_1^*\underline{n}(t)$ directed along the axis of symmetry of the body ($M_1^* = \text{const}$). The second one is a constant moment $\underline{M}_2 = M_2^* \underline{e}$. Equations of the motion of the body have the form

$$\dot{\underline{L}}(t) = M_1^* \, \underline{n}(t) + M_2^* \, \underline{e} \,, \qquad \underline{L}(t) = \underline{\underline{P}}(t) \cdot [\lambda \underline{e}\underline{e} + \mu(\underline{\underline{E}} - \underline{e}\underline{e})] \cdot \underline{\underline{P}}^T(t) \cdot \underline{\omega}(t)$$
$$\underline{n}(t) = \underline{\underline{P}}(t) \cdot \underline{e} \,, \qquad \dot{\underline{\underline{P}}}(t) = \underline{\omega}(t) \times \underline{\underline{P}}(t) \,, \qquad \underline{\underline{P}}(0) = \underline{\underline{P}}_0 \,, \qquad \underline{\omega}(0) = \underline{\omega}_0 \tag{4.1}$$

Using the theorem of representation of the turn-tensor (2.11) $-$ (2.18), we obtain

$$\dot{\underline{\omega}}_L(t) = M_1 \, \underline{n}(t) + M_2 \, \underline{e} \,, \qquad \underline{n}(t) = \underline{\underline{P}}_L(t) \cdot \underline{e} \,, \qquad M_1 = M_1^*/\mu \,, \quad M_2 = M_2^*/\mu$$
$$\dot{\underline{\underline{P}}}_L(t) = \underline{\omega}_L(t) \times \underline{\underline{P}}_L(t) \,, \quad \underline{\underline{P}}_L(0) = \underline{\underline{P}}_0 \,, \quad \underline{\omega}_L(0) = \underline{\underline{P}}_0 \cdot [(\lambda/\mu - 1)\underline{e}\underline{e} + \underline{\underline{E}}] \cdot \underline{\underline{P}}_0^T \cdot \underline{\omega}_0$$
$$\underline{\underline{P}}(t) = \underline{\underline{P}}_L(t) \cdot \underline{\underline{P}}(\beta \underline{e}) \,, \qquad \beta(t) = (\mu/\lambda - 1) \int \underline{e} \cdot \underline{\Omega}_L(t) \, dt \,, \qquad \beta(0) = 0 \tag{4.2}$$

It is easy to show, that the problem (4.2) can be reduced to the following form

$$\ddot{\underline{\omega}}_L(t) - \underline{\omega}_L(t) \times \dot{\underline{\omega}}_L(t) - M_2 \underline{e} \times \underline{\omega}_L(t) = 0$$
$$\dot{\underline{\omega}}_L(0) = M_1 \, \underline{\underline{P}}_0 \cdot \underline{e} + M_2 \, \underline{e} \,, \;\; \underline{\omega}_L(0) = \underline{\omega}_{L0} \,, \;\; \underline{\omega}_{L0} = \underline{\underline{P}}_0 \cdot [(\lambda/\mu - 1)\underline{e}\underline{e} + \underline{\underline{E}}] \cdot \underline{\underline{P}}_0^T \cdot \underline{\omega}_0 \tag{4.3}$$

If the angular velocity vector $\underline{\omega}_L(t)$ is known, then the vector $\underline{n}(t)$ can be calculated by the formula

$$\underline{n}(t) = M_1^{-1} \big[\dot{\underline{\omega}}_L(t) - M_2 \, \underline{e} \big] \tag{4.4}$$

The following scalar relations can be obtained from the eq. (4.3).

$$\underline{e} \cdot \underline{\omega}_L = \tfrac{1}{4M_2} (\omega_L^2)^{\cdot} + \tfrac{M_2^2 - M_1^2}{2M_2} \, t + \tfrac{C}{2M_2} \,, \quad \underline{e} \cdot \underline{\Omega}_L = \tfrac{1}{4M_1} (\Omega_L^2)^{\cdot} + \tfrac{M_1^2 - M_2^2}{2M_1} \, t - \tfrac{C}{2M_1}$$
$$\dot{\underline{\omega}}_L \cdot \dot{\underline{\omega}}_L = \tfrac{1}{2} (\omega_L^2)^{\cdot\cdot} \,, \quad \dot{\underline{\Omega}}_L \cdot \dot{\underline{\Omega}}_L = \tfrac{1}{2} (\Omega_L^2)^{\cdot\cdot}$$
$$\underline{e} \cdot (\underline{\omega}_L \times \dot{\underline{\omega}}_L) = \tfrac{1}{4M_2} (\omega_L^2)^{\cdot\cdot\cdot} \,, \quad \underline{e} \cdot (\underline{\Omega}_L \times \dot{\underline{\Omega}}_L) = \tfrac{1}{4M_1} (\Omega_L^2)^{\cdot\cdot\cdot} \tag{4.5}$$

where $C = \underline{\omega}_{L0} \cdot (M_2 \, \underline{e} - M_1 \, \underline{\underline{P}}_0 \cdot \underline{e})$. Eqs. (4.5) allow to represent the vector $\underline{\omega}_L(t)$ as a function of its squared

$$\underline{\omega}_L(t) = z(t) \, \underline{e} + r(t) \, \underline{e}_r [\varphi(t)]$$
$$z(t) = \frac{1}{4M_2} (\omega_L^2)^{\cdot} + \frac{M_2^2 - M_1^2}{2M_2} \, t + \frac{C}{2M_2} \,, \qquad r(t) = \sqrt{\omega_L^2 - z^2}$$
$$\underline{e}_r [\varphi(t)] = \cos\varphi(t) \, \underline{i} + \sin\varphi(t) \, \underline{j} \,, \quad \varphi(t) = \frac{1}{4M_2} \int \frac{(\omega_L^2)^{\cdot\cdot\cdot}}{\omega_L^2 - z^2} \, dt \,, \quad \varphi(0) = 0 \tag{4.6}$$

where vectors $\underset{\sim}{i}$ ш $\underset{\sim}{j}$ are calculated by the formulae (3.5). The Cauchy problem in function $\omega_L^2(t)$, which also can be obtained from (4.3), has the form

$$(\omega_L^2)^{IV} + \omega_L^2\,(\omega_L^2)^{\cdot\cdot} - \tfrac{1}{4}\left[(\omega_L^2)^{\cdot}\right]^2 - 2\,(M_1^2 + M_2^2)\,\omega_L^2 + \left[(M_1^2 - M_2^2)\,t + C\right]^2 = 0$$

$$(\omega_L^2)^{\cdot\cdot\cdot}\big|_{t=0} = 0\,, \qquad (\omega_L^2)^{\cdot\cdot}\big|_{t=0} = 2\,(M_1\,\underline{\underline{P}}_0 \cdot \underset{\sim}{e} + M_2\,\underset{\sim}{e})^2$$

$$(\omega_L^2)^{\cdot}\big|_{t=0} = 2\,\underset{\sim}{\omega}_{L0} \cdot (M_1\,\underline{\underline{P}}_0 \cdot \underset{\sim}{e} + M_2\,\underset{\sim}{e})\,, \qquad \omega_L^2\big|_{t=0} = \omega_{L0}^2 \qquad (4.7)$$

Analysis of the eqs. (4.2), (4.5) allows to find the bounds of variation of the function $\omega_L^2(t)$ and its derivations

$$2(M_1 - M_2)^2 \leqslant (\omega_L^2)^{\cdot\cdot} \leqslant 2(M_1 + M_2)^2$$

$$2(M_1 - M_2)^2 t + (\omega_L^2)^{\cdot}\big|_{t=0} \leqslant (\omega_L^2)^{\cdot} \leqslant 2(M_1 + M_2)^2 t + (\omega_L^2)^{\cdot}\big|_{t=0} \qquad (4.8)$$

$$(M_1 - M_2)^2 t^2 + (\omega_L^2)^{\cdot}\big|_{t=0} t + \omega_{L0}^2 \leqslant \omega_L^2 \leqslant 2(M_1 + M_2)^2 t^2 + (\omega_L^2)^{\cdot}\big|_{t=0} t + \omega_{L0}^2$$

The relations (4.8) prove the fact, that at the large values of t asymptotic estimations (3.7) for the functions ω_L^2 , $(\omega_L^2)^{\cdot}$, $(\omega_L^2)^{\cdot\cdot}$ take place. Asymptotic solution of the differential equation (4.7) at the large values of t has the following form

$$\omega_L^2 = (M_1 + M_2)^2 t^2 + O(t)\,, \qquad (\omega_L^2)^{\cdot} = 2(M_1 + M_2)^2 t + O(1)$$

$$\underset{\sim}{e} \cdot \underset{\sim}{\omega}_L = (M_1 + M_2)t + O(1) \qquad (4.9)$$

Using the asymptotic solution (4.9), it is easy to show, that

$$\underset{\sim}{n}(t) \xrightarrow{t \to +\infty} \underset{\sim}{e}\,, \qquad \underline{\underline{P}}(t) \xrightarrow{t \to +\infty} \underline{\underline{P}}(\psi \underset{\sim}{e})\,, \qquad \dot{\psi} = (M_1 + M_2)\lambda^{-1}\mu\, t \qquad (4.10)$$

Formulae (4.10) proves, that at the large values of t the motion of the rigid body tends to the rotation around its axis of symmetry and the axis of symmetry of the body tends to the direction of the constant moment.

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