Rotation of rigid body under the action of motor moment and friction moment

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Abstract

The rotation of a symmetrical rigid body around its center of mass is considered. A motor moment and a friction moment act on the body. The friction moment vector is supposed to be proportional the kinetic moment of the body. Two types of the motor moment are considered. The first one is the constant moment. The second one is the following moment directed along the axis of symmetry of the body. Exact analytical solutions of the problems are constructed in the form of the exponential series. Method of solution of the problems is based on using of the representation of turn-tensor by the kinetic moment vector and the representation of turn-tensor by the left and the right angular velocity vectors.

1 Turn-tensor and angular velocity vectors.

This section contains known facts which will be used to solve considered problems. Information about the turn-tensor in more detail can be found in [1], [2].

Definition. A properly orthogonal tensor which is a solution of the equations

$$\underline{\underline{P}} \cdot \underline{\underline{P}}^T = \underline{\underline{P}}^T \cdot \underline{\underline{P}} = \underline{\underline{E}}\,, \qquad\qquad det\,\underline{\underline{P}} = +1 \tag{1}$$

is called a turn-tensor.

Definition. The tensors $\underline{\underline{S}}(t)$ and $\underline{\underline{S}}_r(t)$, calculated by the formulae

$$\underline{\underline{S}}(t) = \dot{\underline{\underline{P}}}(t) \cdot \underline{\underline{P}}^T(t)\,, \qquad\qquad \underline{\underline{S}}_r(t) = \underline{\underline{P}}^T(t) \cdot \dot{\underline{\underline{P}}}(t)\,, \tag{2}$$

are called respectively the left spin tensor and the right spin tensor.

Definition. The accompanying vector of the left spin tensor $\underline{\omega}(t)$, defining by the formula

$$\underline{\underline{S}}(t) = \underline{\omega}(t) \times \underline{\underline{E}}\,, \tag{3}$$

is called the left angular velocity vector.

Definition. The accompanying vector of the right spin tensor $\underline{\Omega}(t)$, defining by the formula

$$\underline{\underline{S}}_r(t) = \underline{\Omega}(t) \times \underline{\underline{E}}\,, \tag{4}$$

is called the right angular velocity vector.

The left Darboux problem. Let the left angular velocity vector $\underline{\omega}(t)$ be known and the turn-tensor should be found. This problem is formulated as follows

$$\dot{\underline{\underline{P}}}(t) = \underline{\omega}(t) \times \underline{\underline{P}}(t)\,, \qquad \underline{\underline{P}}(t_0) = \underline{\underline{P}}_0\,. \tag{5}$$

The right Darboux problem. Let the right angular velocity vector $\underline{\Omega}(t)$ be known and the turn-tensor should be found. This problem is formulated as follows

$$\dot{\underline{\underline{P}}}(t) = \underline{\underline{P}}(t) \times \underline{\Omega}(t)\,, \qquad \underline{\underline{P}}(0) = \underline{\underline{P}}_0\,. \tag{6}$$

Theorem: Every turn-tensor can be represented as the composition of any number of the turn-tensors:

$$\underline{\underline{P}}(t) = \underline{\underline{P}}_n(t) \cdot \underline{\underline{P}}_{n-1}(t) \cdot \ \ldots\ \cdot \underline{\underline{P}}_3(t) \cdot \underline{\underline{P}}_2(t) \cdot \underline{\underline{P}}_1(t)\,. \tag{7}$$

Theorem: If the turn-tensor $\underline{\underline{P}}(t)$ is represented as the composition of two turn-tensors $\underline{\underline{P}}_1(t)$ and $\underline{\underline{P}}_2(t)$, then the angular velocity vector $\underline{\omega}(t)$, corresponding to the turn-tensor $\underline{\underline{P}}(t)$, is expressed in terms of the angular velocity vectors $\underline{\omega}_1(t)$ and $\underline{\omega}_2(t)$, corresponding to the turn-tensors $\underline{\underline{P}}_1(t)$ and $\underline{\underline{P}}_2(t)$, as follows:

$$\underline{\underline{P}}(t) = \underline{\underline{P}}_2(t) \cdot \underline{\underline{P}}_1(t) \qquad \Longrightarrow \qquad \underline{\omega}(t) = \underline{\omega}_2(t) + \underline{\underline{P}}_2(t) \cdot \underline{\omega}_1(t)\,. \tag{8}$$

Euler theorem: Every turn-tensor $\underline{\underline{P}}(t) \neq \underline{\underline{E}}$ can be represented by uniquely in the form

$$\underline{\underline{P}}(\theta \underline{m}) = (1 - \cos\theta(t))\, \underline{m}(t)\underline{m}(t) + \cos\theta(t)\, \underline{\underline{E}} + \sin\theta(t)\, \underline{m}(t) \times \underline{\underline{E}}\,, \quad |\underline{m}(t)| = 1\,. \tag{9}$$

Using Euler representation of the turn-tensor, it is easy to derive expressions for the angular velocities

$$\begin{aligned} \underline{\omega}(t) &= \dot{\theta}(t)\, \underline{m}(t) + \sin\theta(t)\, \dot{\underline{m}}(t) + (1-\cos\theta(t))\, \underline{m}(t)\times\dot{\underline{m}}(t)\,, \\ \underline{\Omega}(t) &= \dot{\theta}(t)\, \underline{m}(t) + \sin\theta(t)\, \dot{\underline{m}}(t) - (1-\cos\theta(t))\, \underline{m}(t)\times\dot{\underline{m}}(t)\,. \end{aligned} \tag{10}$$

2 Representation of turn-tensor of symmetrical rigid body by the kinetic moment vector.

Theorem: Let the inertia tensor of a rigid body in the reference position be

$$\underline{\underline{\theta}} = \lambda\, \underline{e}\underline{e} + \mu\, (\underline{\underline{E}} - \underline{e}\underline{e})\,. \tag{11}$$

Let the actual position of the rigid body be determined by the turn-tensor $\underline{\underline{P}}(t)$ and the rigid body have the angular velocity $\underline{\omega}(t)$, and the kinetic moment vector (the moment of momentum vector) of the rigid body, calculated with respect to some point of the body, be $\underline{L}(t)$:

$$\underline{L}(t) = \underline{\underline{P}}(t)\cdot\underline{\underline{\theta}}\cdot\underline{\underline{P}}^T(t)\cdot\underline{\omega}(t)\,. \tag{12}$$

In this case the turn-tensor of the rigid body $\underline{\underline{P}}(t)$ can be represented as the composition of two turn-tensors

$$\underline{\underline{P}}(t) = \underline{\underline{P}}_L(t)\cdot\underline{\underline{P}}_*(t)\,. \tag{13}$$

The turn-tensor $\underline{\underline{P}}_L(t)$ is determined by the kinetic moment vector of the rigid body $\underline{L}(t)$ as solution of the left Darboux problem

$$\dot{\underline{\underline{P}}}_L(t) = \underline{\omega}_L(t)\times\underline{\underline{P}}_L(t)\,, \qquad \underline{\omega}_L(t) = \mu^{-1}\underline{L}(t)\,, \qquad \underline{\underline{P}}_L(0) = \underline{\underline{P}}(0)\,. \tag{14}$$

The turn-tensor $\underline{\underline{P}}_*(t)$ has the form

$$\underline{\underline{P}}_*(t) = (1-\cos\beta(t))\, \underline{e}\underline{e} + \cos\beta(t)\, \underline{\underline{E}} + \sin\beta(t)\, \underline{e}\times\underline{\underline{E}}\,, \tag{15}$$

where angle $\beta(t)$ is expressed in terms of the right angular velocity corresponding to turn-tensor $\underline{\underline{P}}_L(t)$

$$\beta(t) = \int (\mu-\lambda)\,\lambda^{-1}\, \underline{e}\cdot\underline{\Omega}_L(t)\, dt\,, \quad \underline{\Omega}_L(t) = \underline{\underline{P}}_L^T(t)\cdot\underline{\omega}_L(t)\,, \quad \beta(0) = 0\,. \tag{16}$$

Proof of the theorem can be found in [3].

3 Representation of turn-tensor by left and right angular velocity vectors. Alternative formulations of Darboux problem.

Theorem: Let both angular velocity vectors be known. Then the turn-tensor can be found without additional integration. It is calculated by formula

$$\begin{aligned} &\underline{\underline{P}} = A\, \underline{\omega}\,\underline{\Omega} + B\,\big(\underline{\omega}\,\underline{\Omega}\big)^{\cdot} + C\, \dot{\underline{\omega}}\,\dot{\underline{\Omega}} + D\,\big(\underline{\omega}\times\dot{\underline{\omega}}\big)\big(\underline{\Omega}\times\dot{\underline{\Omega}}\big)\,, \\ &A = \dot{\underline{\Omega}}\cdot\dot{\underline{\Omega}}\, D = \dot{\underline{\omega}}\cdot\dot{\underline{\omega}}\, D\,, \qquad B = -\tfrac{1}{2}\big(\Omega^2\big)^{\cdot}\, D = -\tfrac{1}{2}\big(\omega^2\big)^{\cdot}\, D \\ &C = \Omega^2\, D = \omega^2\, D\,, \qquad D = 1/\big(\underline{\Omega}\times\dot{\underline{\Omega}}\big)^2 = 1/\big(\underline{\omega}\times\dot{\underline{\omega}}\big)^2\,. \end{aligned} \tag{17}$$

Proof of this theorem can be found in [4]. Theorem (17) allows to formulate Darboux problem as follows.

Left Darboux problem. The left angular velocity vector $\underline{\omega}(t)$ is known. The right angular velocity vector $\underline{\Omega}(t)$ should be found (see [4]).

Formulation I. The right angular velocity vector is determined as solution of Cauchy problem

$$\begin{aligned} &\ddot{\underline{\Omega}} + a(\underline{\omega})\,\dot{\underline{\Omega}} + b(\underline{\omega})\,\underline{\Omega} = c(\underline{\omega})\,\underline{\Omega}\times\dot{\underline{\Omega}}\,, \qquad \underline{\Omega}(t_0) = \underline{\underline{P}}_0^T\cdot\underline{\omega}_0\,, \quad \dot{\underline{\Omega}}(t_0) = \underline{\underline{P}}_0^T\cdot\dot{\underline{\omega}}_0\,, \\ &a(\underline{\omega}) = -(\ln|\underline{\omega}\times\dot{\underline{\omega}}|)^{\cdot}, \quad b(\underline{\omega}) = \frac{(\underline{\omega}\times\dot{\underline{\omega}})\cdot(\dot{\underline{\omega}}\times\ddot{\underline{\omega}})}{(\underline{\omega}\times\dot{\underline{\omega}})^2}\,, \quad c(\underline{\omega}) = \frac{\ddot{\underline{\omega}}\cdot(\underline{\omega}\times\dot{\underline{\omega}})}{(\underline{\omega}\times\dot{\underline{\omega}})^2} - 1\,. \end{aligned} \tag{18}$$

Formulation II. The right angular velocity vector is determined as solution of Cauchy problem

$$\begin{aligned} &\dddot{\underline{\Omega}} + A(\underline{\omega})\,\ddot{\underline{\Omega}} + B(\underline{\omega})\,\dot{\underline{\Omega}} + C(\underline{\omega})\,\underline{\Omega} = 0\,, \qquad \underline{\Omega}(t_0) = \underline{\underline{P}}_0^T\cdot\underline{\omega}_0\,, \qquad \dot{\underline{\Omega}}(t_0) = \underline{\underline{P}}_0^T\cdot\dot{\underline{\omega}}_0\,, \\ &\ddot{\underline{\Omega}}(t_0) = \underline{\underline{P}}_0^T\cdot\left[c(\underline{\omega}_0)\,\underline{\omega}_0\times\dot{\underline{\omega}}_0 - a(\underline{\omega}_0)\,\dot{\underline{\omega}}_0 - b(\underline{\omega}_0)\,\underline{\omega}_0\right]\,, \\ &A(\underline{\omega}) = 2a(\underline{\omega}) - [\ln c(\underline{\omega})]^{\cdot}, \;\; B(\underline{\omega}) = \dot{a}(\underline{\omega}) + a(\underline{\omega})\,(a(\underline{\omega}) - [\ln c(\underline{\omega})]^{\cdot}) + b(\underline{\omega}) + c^2(\underline{\omega})\,\omega^2\,, \\ &C(\underline{\omega}) = \dot{b}(\underline{\omega}) + b(\underline{\omega})\,(a(\underline{\omega}) - [\ln c(\underline{\omega})]^{\cdot}) - \tfrac{1}{2}c^2(\underline{\omega})\,(\omega^2)^{\cdot}. \end{aligned} \tag{19}$$

Right Darboux problem. The right angular velocity vector $\underline{\Omega}(t)$ is known. The left angular velocity vector $\underline{\omega}(t)$ should be found (see [4]).

Formulation I. The left angular velocity vector is determined as solution of Cauchy problem

$$
\begin{gathered}
\ddot{\underline{\omega}} + a(\underline{\Omega})\,\dot{\underline{\omega}} + b(\underline{\Omega})\,\underline{\omega} = c(\underline{\Omega})\,\underline{\omega}\times\dot{\underline{\omega}}\,, \qquad \underline{\omega}(0) = \underline{\underline{P}}_0\cdot\underline{\Omega}_0\,, \qquad \dot{\underline{\omega}}(0) = \underline{\underline{P}}_0\cdot\dot{\underline{\Omega}}_0\,, \\
a(\underline{\Omega}) = -(\ln|\underline{\Omega}\times\dot{\underline{\Omega}}|)^{\cdot}, \quad b(\underline{\Omega}) = \frac{(\underline{\Omega}\times\dot{\underline{\Omega}})\cdot(\dot{\underline{\Omega}}\times\ddot{\underline{\Omega}})}{(\underline{\Omega}\times\dot{\underline{\Omega}})^2}\,, \quad c(\underline{\Omega}) = \frac{\ddot{\underline{\Omega}}\cdot(\underline{\Omega}\times\dot{\underline{\Omega}})}{(\underline{\Omega}\times\dot{\underline{\Omega}})^2} + 1\,.
\end{gathered} \tag{20}
$$

Formulation II. The left angular velocity vector is determined as solution of Cauchy problem

$$
\begin{gathered}
\dddot{\underline{\omega}} + A(\underline{\Omega})\,\ddot{\underline{\omega}} + B(\underline{\Omega})\,\dot{\underline{\omega}} + C(\underline{\Omega})\,\underline{\omega} = 0\,, \qquad \underline{\omega}(t_0) = \underline{\underline{P}}_0\cdot\underline{\Omega}_0\,, \qquad \dot{\underline{\omega}}(t_0) = \underline{\underline{P}}_0\cdot\dot{\underline{\Omega}}_0\,, \\
\ddot{\underline{\omega}}(t_0) = \underline{\underline{P}}_0\cdot\left[c(\underline{\Omega}_0)\,\underline{\Omega}_0\times\dot{\underline{\Omega}}_0 - a(\underline{\Omega}_0)\,\dot{\underline{\Omega}}_0 - b(\underline{\Omega}_0)\,\underline{\Omega}_0\right]\,, \\
A(\underline{\Omega}) = 2a(\underline{\Omega}) - [\ln c(\underline{\Omega})]^{\cdot},\; B(\underline{\Omega}) = \dot{a}(\underline{\Omega}) + a(\underline{\Omega})\,(a(\underline{\Omega}) - [\ln c(\underline{\Omega})]^{\cdot}) + b(\underline{\Omega}) + c^2(\underline{\Omega})\,\Omega^2, \\
C(\underline{\Omega}) = \dot{b}(\underline{\Omega}) + b(\underline{\Omega})\,(a(\underline{\Omega}) - [\ln c(\underline{\Omega})]^{\cdot}) - \tfrac{1}{2}c^2(\underline{\Omega})\,(\Omega^2)^{\cdot}.
\end{gathered} \tag{21}
$$

4 Rotation of rigid body under the action of constant motor moment.

Let us consider a symmetrical rigid body. The center of mass of the body is fixed. A motor moment $\underline{M}_*$ and a friction moment $\underline{M}_{fr}(t) = -k\,\underline{L}(t)$ act on the body. Equations of the motion of the body have the form

$$
\begin{gathered}
\dot{\underline{L}}(t) = -k\,\underline{L}(t) + \underline{M}_*\,, \qquad \underline{L}(t) = \underline{\underline{P}}(t)\cdot[\lambda\underline{e}\underline{e} + \mu(\underline{\underline{E}} - \underline{e}\underline{e})]\cdot\underline{\underline{P}}^T(t)\cdot\underline{\omega}(t)\,, \\
\underline{M}_* = \text{const}\,, \qquad \dot{\underline{\underline{P}}}(t) = \underline{\omega}(t)\times\underline{\underline{P}}(t)\,, \qquad \underline{\underline{P}}(0) = \underline{\underline{E}}\,, \qquad \underline{\omega}(0) = \underline{\omega}_0\,.
\end{gathered} \tag{22}
$$

Using the theorem of representation of turn-tensor (11) – (16), the problem (22) can be rewritten in the form

$$
\begin{gathered}
\dot{\underline{\omega}}_L(t) = -k\,\underline{\omega}_L(t) + \underline{M}\,, \qquad \underline{\omega}_L(0) = \underline{\omega}_{L0}\,, \quad \underline{\omega}_{L0} = [(\lambda/\mu - 1)\underline{e}\underline{e} + \underline{\underline{E}}]\cdot\underline{\omega}_0\,, \\
\underline{\underline{P}}(t) = \underline{\underline{P}}_L(t)\cdot\underline{\underline{P}}(\beta\underline{e})\,, \qquad \dot{\underline{\underline{P}}}_L(t) = \underline{\omega}_L(t)\times\underline{\underline{P}}_L(t)\,, \qquad \underline{\underline{P}}_L(0) = \underline{\underline{E}}\,, \\
\underline{M} = \underline{M}_*/\mu\,, \qquad \beta(t) = (\mu/\lambda - 1)\int \underline{e}\cdot\underline{\Omega}_L(t)\,dt\,, \qquad \beta(0) = 0\,.
\end{gathered} \tag{23}
$$

It is easy to see, that problem (23) can be reduced to solution of the left Darboux problem:

$$
\begin{gathered}
\dot{\underline{\underline{P}}}_L(t) = \underline{\omega}_L(t)\times\underline{\underline{P}}_L(t)\,, \qquad \underline{\underline{P}}_L(0) = \underline{\underline{E}}\,, \\
\underline{\omega}_L(t) = \underline{a}\,e^{-k\,t} + \underline{b}\,, \qquad \underline{a} = \underline{\omega}_{L0} - \underline{M}/k\,, \qquad \underline{b} = \underline{M}/k\,.
\end{gathered} \tag{24}
$$

Using the theorem of representation of turn-tensor (17) and formulation of the left Darboux problem (18), problem (24) can be formulated as Cauchy problem in term of the right angular velocity vector $\underline{\Omega}_L(t)$:

$$
\ddot{\underline{\Omega}}_L(t) + k\,\dot{\underline{\Omega}}_L(t) = -\underline{\Omega}_L(t)\times\dot{\underline{\Omega}}_L(t)\,, \qquad \underline{\Omega}_L(0) = \underline{a} + \underline{b}\,, \qquad \dot{\underline{\Omega}}_L(0) = -k\,\underline{a}\,. \tag{25}
$$

Using formulation of the left Darboux problem (19), problem (25) can be rewritten as follows:

$$
\begin{gathered}
\dddot{\underline{\Omega}}_L(t) + 2k\,\ddot{\underline{\Omega}}_L(t) + (k^2 + b^2 + 2\,\underline{a}\cdot\underline{b}\,e^{-k\,t} + a^2e^{-2k\,t})\,\dot{\underline{\Omega}}_L(t) + \\
+ k\,(\underline{a}\cdot\underline{b}\,e^{-k\,t} + a^2e^{-2k\,t})\,\underline{\Omega}_L(t) = 0\,, \\
\underline{\Omega}_L(0) = \underline{a} + \underline{b}\,, \qquad \dot{\underline{\Omega}}_L(0) = -k\,\underline{a}\,, \qquad \ddot{\underline{\Omega}}_L(0) = k^2\,\underline{a} - k\,\underline{a}\times\underline{b}\,.
\end{gathered} \tag{26}
$$

Let us look for solution of differential equation (26) in the form of exponential series. It is easy to show that solution, represented by the exponential series, takes the form

$$\begin{gathered}
\underline{\Omega}_L(t) = \underline{A}_0 \sum_{n=0}^{\infty} C_n^{(0)}\, e^{-nkt} + (\underline{B}_1 \cos bt + \underline{B}_2 \sin bt) \sum_{n=1}^{\infty} \operatorname{Re} C_n\, e^{-nkt} + \\
+(-\underline{B}_1 \sin bt + \underline{B}_2 \cos bt) \sum_{n=1}^{\infty} \operatorname{Im} C_n\, e^{-nkt}\,, \\
C_0^{(0)} = C_1 = 1, \quad C_1^{(0)} = \frac{\underline{a}\cdot\underline{b}}{b^2}, \quad C_n^{(0)} = -\frac{(2n-3)\,\underline{a}\cdot\underline{b}\,C_{n-1}^{(0)} + (n-3)\,a^2\,C_{n-2}^{(0)}}{n\,[(n-1)^2k^2+b^2]}\,, \\
C_2 = -\frac{\underline{a}\cdot\underline{b}}{k(2k-ib)}, \quad C_n = -\frac{[(2n-3)k-2ib]\,\underline{a}\cdot\underline{b}\,C_{n-1} + [(n-3)k-ib]\,a^2\,C_{n-2}}{(n-1)k\,[(n-1)k-2ib](nk-ib)}\,.
\end{gathered} \tag{27}$$

Here constant vectors $\underline{A}_0$, $\underline{B}_1$, $\underline{B}_2$ are determined by the initial conditions (26). Proof of the fact that the system of exponents is complete on the class of functions which are solution of differential equation (26) and proof of uniform convergence of series (27) can be found in [5], where analogous differential equation is solved.

Analysis of solution of the problem. An asymptotic analysis of the solution at the large values of time shows that at all initial conditions the following properties of the body motion take place:

$$\underline{\omega}_L(t) \xrightarrow{t\to+\infty} \frac{1}{k}\underline{M}\,, \qquad \underline{\Omega}_L(t) \xrightarrow{t\to+\infty} \underline{A}_0\,, \qquad \dot{\varphi}(t) \xrightarrow{t\to+\infty} \left(\frac{\mu}{\lambda}-1\right)\underline{e}\cdot\underline{A}_0\,. \tag{28}$$

Hence, at the large values of time the body motion tends to the regular precession, which axis is directed along the motor moment. In the case $\underline{\omega}_{L0} = \omega_{L0}^0 \underline{m}$, $\underline{m} = \underline{M}/M$, solution of considered problem takes the form

$$\begin{gathered}
\underline{\underline{P}}(t) = \underline{\underline{P}}_{L0}(\psi \underline{m}) \cdot \underline{\underline{P}}(\beta_0 \underline{e})\,, \qquad \dot{\psi}(t) = \omega_0(t)\,, \qquad \psi(0) = 0\,, \\
\dot{\beta}_0(t) = \left(\frac{\mu}{\lambda}-1\right)\omega_0(t)\,\underline{m}\cdot\underline{e}\,, \qquad \beta(0) = 0\,, \qquad \omega_0(t) = \frac{1}{k}M + \left(\omega_{L0}^0 - \frac{1}{k}M\right)e^{-k\,t}\,.
\end{gathered} \tag{29}$$

Let us investigate stability of motion (29). Let us suppose that $\underline{\omega}_{L0} = \omega_{L0}^0 \underline{m} + \tilde{\underline{\omega}}_{L0}$, where $|\tilde{\underline{\omega}}_{L0}| \sim \varepsilon$. Then

$$\underline{\omega}_L(t) = \omega_0(t)\,\underline{m} + \tilde{\underline{\omega}}_{L0} e^{-k\,t}\,, \tag{30}$$

where $\omega_0(t)$ is determined by formula (29). Expression (30) proves that $|\underline{\omega}_L(t) - \omega_0(t)\,\underline{m}| \sim \varepsilon$. Let us represent turn-tensor $\underline{\underline{P}}_L$ in the form

$$\underline{\underline{P}}_L(t) = \underline{\underline{P}}_{L0}(\psi(t)\underline{m}) \cdot \tilde{\underline{\underline{P}}}(t), \qquad \tilde{\underline{\underline{P}}}(0) = \underline{\underline{P}}_{L0}^T(\psi(0)\underline{m}) \cdot \underline{\underline{P}}_L(0) = \underline{\underline{E}}\,. \tag{31}$$

To prove stability of motion of axis of symmetry of the body it is necessary to prove that $\tilde{\underline{\underline{P}}}$ is the tensor of small turn. According to formulae (1) – (4), (8)

$$\underline{\omega}_L = \omega_0 \underline{m} + \underline{\underline{P}}_{L0} \cdot \tilde{\underline{\omega}}\,, \qquad \underline{\Omega}_L = \tilde{\underline{\underline{P}}}^T \cdot \omega_0 \underline{m} + \tilde{\underline{\Omega}}\,. \tag{32}$$

It is easy to show that

$$\underline{\Omega}_L - \underline{\underline{P}}_{L0}^T \cdot \underline{\omega}_L = \underline{\underline{Q}}^T \cdot \omega_0 \underline{m} - \underline{\underline{Q}} \cdot \tilde{\underline{\Omega}}\,, \qquad \underline{\underline{Q}} = \tilde{\underline{\underline{P}}} - \underline{\underline{E}}\,. \tag{33}$$

If $\tilde{\underline{\underline{P}}}$ is the tensor of small turn, then $\tilde{\underline{\omega}} \sim \varepsilon$, $\tilde{\underline{\Omega}} \sim \varepsilon$, $\underline{\underline{Q}} \sim \varepsilon$. As follows from (33), $|\underline{\Omega}_L - \underline{\underline{P}}_{L0}^T \cdot \underline{\omega}_L| \sim \underline{\underline{Q}}$. Hence, condition $\underline{\underline{Q}} \sim \varepsilon$ and condition $|\underline{\Omega}_L - \underline{\underline{P}}_{L0}^T \cdot \underline{\omega}_L| \sim \varepsilon$ are equivalent. Let us consider the difference

$$\omega_0(\underline{\Omega}_L - \underline{\underline{P}}_{L0}^T \cdot \underline{\omega}_L)^{\cdot} - \dot{\omega}_0(\underline{\Omega}_L - \underline{\underline{P}}_{L0}^T \cdot \underline{\omega}_L) = \omega_0^2 \underline{m} \times \tilde{\underline{\omega}} + \omega_0^2 \underline{\underline{Q}}^T \cdot (\underline{m} \times \tilde{\underline{\omega}}) + \underline{\underline{Q}} \cdot (\dot{\omega}_0 \tilde{\underline{\Omega}} - \omega_0 \dot{\tilde{\underline{\Omega}}})\,. \tag{34}$$

According to (30), (32)

$$\tilde{\underline{\omega}} = \underline{\underline{P}}_{L0}^T \cdot \underline{\omega}_L - \omega_0 \underline{m} = \underline{\underline{P}}_{L0}^T \cdot \tilde{\underline{\omega}}_{L0}\, e^{-k\,t}\,. \tag{35}$$

Neglecting the terms of ε^2 order in equation (34) and using expression (35), we obtain a linear equation in term of $(\underline{\Omega}_L - \underline{\underline{P}}_{L0}^T \cdot \underline{\omega}_L)$:

$$\omega_0(\underline{\Omega}_L - \underline{\underline{P}}_{L0}^T \cdot \underline{\omega}_L)^{\cdot} - \dot{\omega}_0(\underline{\Omega}_L - \underline{\underline{P}}_{L0}^T \cdot \underline{\omega}_L) = \omega_0^2 \underline{\underline{P}}_{L0}^T \cdot (\underline{m} \times \tilde{\underline{\omega}}_{L0})\, e^{-k\,t}\,. \tag{36}$$

As it was shown above, if solution of equation (36) is ε order, then motion of axis of symmetry of the body is stable. Solution of equation (36) has the form

$$\underline{\Omega}_L - \underline{\underline{P}}_{L0}^T \cdot \underline{\omega}_L = \omega_0 \int_0^t \underline{\underline{P}}_{L0}^T \cdot (\underline{m} \times \tilde{\underline{\omega}}_{L0})\, e^{-k\,t}\, dt\,. \tag{37}$$

As $|\tilde{\underline{\omega}}_{L0}| \sim \varepsilon$ and $|\underline{m}| = 1$, according to (37), $|\underline{\Omega}_L - \underline{\underline{P}}_{L0}^T \cdot \underline{\omega}_L| \sim \varepsilon$. Hence, motion of axis of symmetry of the body is stable.

5 Rotation of rigid body under the action of following motor moment.

A symmetrical rigid body is considered. The center of mass of the body is fixed. A motor moment $\underline{M}_*(t) = M_* \underline{n}(t)$, directed along the axis of symmetry of the body, and a friction moment $\underline{M}_{fr}(t) = -k\,\underline{L}(t)$ act on the body. Equations of the motion of the body have the form

$$\dot{\underline{L}}(t) = -k\,\underline{L}(t) + M_* \underline{n}(t)\,, \qquad \underline{L}(t) = \underline{\underline{P}}(t)\cdot[\lambda \underline{e}\underline{e} + \mu(\underline{\underline{E}} - \underline{e}\underline{e})]\cdot \underline{\underline{P}}^T(t)\cdot\underline{\omega}(t)\,,$$
$$M_* = \text{const}\,, \quad \underline{n}(t) = \underline{\underline{P}}(t)\cdot\underline{e}\,, \quad \dot{\underline{\underline{P}}}(t) = \underline{\omega}(t)\times\underline{\underline{P}}(t)\,, \quad \underline{\underline{P}}(0) = \underline{\underline{E}}\,, \quad \underline{\omega}(0) = \underline{\omega}_0\,. \tag{38}$$

Using the theorem of representation of turn-tensor (11) – (16), the problem (38) can be rewritten in the form

$$\dot{\underline{\omega}}_L(t) = -k\,\underline{\omega}_L(t) + M\,\underline{n}(t)\,, \qquad \underline{\omega}_L(0) = \underline{\omega}_{L0}\,, \quad \underline{\omega}_{L0} = [(\lambda/\mu - 1)\underline{e}\underline{e} + \underline{\underline{E}}]\cdot\underline{\omega}_0\,,$$
$$\underline{\underline{P}}(t) = \underline{\underline{P}}_L(t)\cdot\underline{\underline{P}}(\beta\underline{e})\,, \qquad \dot{\underline{\underline{P}}}_L(t) = \underline{\omega}_L(t)\times\underline{\underline{P}}_L(t)\,, \qquad \underline{\underline{P}}_L(0) = \underline{\underline{E}}\,,$$
$$M = M_*/\mu\,, \quad \underline{n}(t) = \underline{\underline{P}}_L(t)\cdot\underline{e}\,, \quad \beta(t) = (\mu/\lambda - 1)\int \underline{e}\cdot\underline{\Omega}_L(t)\,dt\,, \quad \beta(0) = 0\,. \tag{39}$$

It is easy to see, that the first equation from (39) can be reduced to the form

$$\dot{\underline{\Omega}}_L(t) = -k\,\underline{\Omega}_L(t) + M\,\underline{e}\,. \tag{40}$$

Hence, problem (39) can be reduced to the solution of the right Darboux problem:

$$\dot{\underline{\underline{P}}}_L(t) = \underline{\underline{P}}_L(t)\times\underline{\Omega}_L(t)\,, \qquad \underline{\underline{P}}_L(0) = \underline{\underline{E}}\,,$$
$$\underline{\Omega}_L(t) = \underline{a}\,e^{-k\,t} + \underline{b}\,, \qquad \underline{a} = \underline{\omega}_{L0} - M\underline{e}/k\,, \qquad \underline{b} = M\underline{e}/k\,. \tag{41}$$

Using the theorem of representation of turn-tensor (17) and formulation of the right Darboux problem (20), the problem (41) can be formulated as Cauchy problem in term of the left angular velocity vector $\underline{\omega}_L(t)$:

$$\ddot{\underline{\omega}}_L(t) + k\,\dot{\underline{\omega}}_L(t) = \underline{\omega}_L(t)\times\dot{\underline{\omega}}_L(t)\,, \qquad \underline{\omega}_L(0) = \underline{a} + \underline{b}\,, \qquad \dot{\underline{\omega}}_L(0) = -k\,\underline{a}\,. \tag{42}$$

Using formulation of the right Darboux problem (21), the problem (42) can be rewritten as follows:

$$\dddot{\underline{\omega}}_L(t) + 2k\,\ddot{\underline{\omega}}_L(t) + (k^2 + b^2 + 2\,\underline{a}\cdot\underline{b}\,e^{-k\,t} + a^2 e^{-2k\,t})\,\dot{\underline{\omega}}_L(t) +$$
$$+k\,(\underline{a}\cdot\underline{b}\,e^{-k\,t} + a^2 e^{-2k\,t})\,\underline{\Omega}_L(t) = 0\,,$$
$$\underline{\omega}_L(0) = \underline{a} + \underline{b}\,, \qquad \dot{\underline{\omega}}_L(0) = -k\,\underline{a}\,, \qquad \ddot{\underline{\omega}}_L(0) = k^2\,\underline{a} + k\,\underline{a}\times\underline{b}\,. \tag{43}$$

It easy to see, that the differential equation (43) in term of $\underline{\omega}_L$ is analogous to the differential equation (26) in term of $\underline{\Omega}_L$, which was obtained in the case of constant motor moment. Hence, solution of equation (43) can be represent by the exponential series (27).

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