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Derivation of theory of thermoviscoelasticity by means of two-component medium

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Abstract We consider the mechanical model of a two-component medium whose first component is the classical continuum and the other component is the continuum having only the rotational degrees of freedom. We show that the proposed model can be used for description of thermal and dissipative phenomena. In special cases, the mathematical description of the proposed model is proved to reduce to well-known equations such as the heat conduction equation, the self-diffusion equation and the equations of the coupled problem of thermoelasticity. In the context of the proposed theory, we consider the original model of internal damping and give interpretation of the volume viscosity.

1 Introduction

At present, thermodynamics covers a widespread frame including gas dynamics, thermoelasticity, thermoviscoelasticity, thermoelectric and thermomagnetic phenomena, phase changes and chemical reactions. At the same time, it constitutes a set of science areas which are not connected to each other and differ by both interpretation of the fundamental concepts and applied mathematical methods. Dealing with the mathematical methods, we should refer to the thermodynamic potential theory underlying the chemical and electrochemical thermodynamics, continuum mechanics within the framework of which the models of thermoelastic and thermoviscoelastic media have been developed, the methods of crystal lattice dynamics underlying the description of transport phenomena in solids, and also the classical and quantum statistics. In view of the aforesaid, it is important to develop a unified theory for description of all thermodynamical phenomena which are studied now in different science areas by using the different methods. We are firmly convinced that it can be made on the basis of the fundamental laws of mechanics by using the continuum mechanics method.

The proposed theory is based on the continual mechanical model whose mathematical description in special cases is proved to reduce to the well-known equations of thermodynamics and thermoelasticity. It is the presence of additional rotational degrees of freedom and, accordingly, additional inertia and elastic characteristics which can be interpreted as the thermodynamical constants that distinguish the proposed model among other continual models. In fact, this model is a two-component medium. The first component is the classical continuum with translational and rotational degrees of freedom and the other component is the continuum having only the rotational degrees of freedom and the moment interactions. The idea of mathematical description of various physical phenomena in microcosm by using the continual models based on rotational degrees of freedom and the moment interactions was repeatedly asserted by Zhilin [1], [2], [3], [4]. The model proposed in the present paper is a realization of this idea as applied to the description of thermal and dissipative phenomena.

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The proposed model does not pretend to be an explanation of the physical nature of thermal phenomena. Interpretation of the temperature, entropy and other thermodynamic quantities given in accordance with the proposed model is no more than the mechanical analogy. However, use of these mechanical counterparts allows one to obtain the well-known equations describing the thermal and diffusion processes within the framework of the proposed model. The mathematical description of the proposed mechanical model includes as special cases not only the classical formulation of coupled problem of thermoelasticity but also the formulation of the coupled problem of thermoelasticity with the hyperbolic type heat conduction equation. In the context of the proposed mechanical model, an original interpretation of the volume (acoustic) viscosity is offered. Just due to our interpretation of the volume viscosity, we treat the proposed theory as a theory of thermoviscoelasticity.

2 Fundamental laws of mechanics

2.1 Body-point and its dynamic structures

Constructing a model of continuum, we will use a body-point (a one-spin particle) as the base material object. The body-point, unlike a point mass, undergoes to not only translational but also rotational (spinor) motions; it is the material object occupying zero volume in space. Position of a body-point is considered to be determined if the position vector $\mathbf{R}(t)$ and the rotation tensor $\mathbf{P}(t)$ are assigned. For the first time, a body-point (a one-spin particle) has been introduced into consideration by Zhilin, see e. g. [1], [2], [3].

Definition 1 The tensor of rotation is a properly orthogonal tensor which represents the solution of the equations

$$\mathbf{P}\cdot\mathbf{P}^T = \mathbf{P}^T\cdot\mathbf{P} = \mathbf{E}, \qquad \det\mathbf{P} = 1, \tag{1}$$

where $\mathbf{E}$ is the unit tensor. The translational and angular velocities of a body-point are calculated by the formulas

$$\mathbf{v}(t) = \frac{d\mathbf{R}(t)}{dt}, \qquad \boldsymbol{\omega}(t) = -\frac{1}{2}\left(\frac{d\mathbf{P}(t)}{dt}\cdot\mathbf{P}^T(t)\right)_\times, \tag{2}$$

where $(\ )_\times$ denotes the vector invariant of a tensor.

Definition 2 The kinetic energy of a body-point is a quadratic form of its translational and angular velocities, see [1], [2], [3]:

$$K = \frac{1}{2}m\,\mathbf{v}\cdot\mathbf{v} + \mathbf{v}\cdot m\mathbf{B}\cdot\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}\cdot m\mathbf{J}\cdot\boldsymbol{\omega}. \tag{3}$$

Here, the second-rank tensors $m\mathbf{B}$, $m\mathbf{J}$ are the inertia tensors of a body-point, and m is the mass of a body-point, respectively. The inertia tensors are frame-indifferent characteristics of a body-point, therefore, they should depend on the rotation tensor $\mathbf{P}(t)$ as

$$m\mathbf{B}(t) = \mathbf{P}(t)\cdot m\mathbf{B}_0\cdot\mathbf{P}^T(t), \qquad m\mathbf{J}(t) = \mathbf{P}(t)\cdot m\mathbf{J}_0\cdot\mathbf{P}^T(t), \tag{4}$$

where $m\mathbf{B}_0$, $m\mathbf{J}_0$ are the inertia tensors at the reference position, i. e. for those values t_0 at which $\mathbf{P}(t_0) = \mathbf{E}$.

Definition 3 The momentum of a body-point is the linear form of its translational and angular velocities, see [1], [2], [3]:

$$\mathbf{K}_1 = \frac{\partial K}{\partial\mathbf{v}} = m\,\mathbf{v} + m\mathbf{B}\cdot\boldsymbol{\omega}. \tag{5}$$

Definition 4 The proper angular momentum (dynamic spin) of a body-point is the linear form of its translational and angular velocities, see [1], [2], [3]:

$$\mathbf{K}_2 = \frac{\partial K}{\partial\boldsymbol{\omega}} = \mathbf{v}\cdot m\mathbf{B} + m\mathbf{J}\cdot\boldsymbol{\omega}. \tag{6}$$

Definition 5 The angular momentum of a body-point calculated with respect to a fixed reference point Q is defined by the following formula, see [1], [2], [3]:

$$\mathbf{K}_2^Q = (\mathbf{R} - \mathbf{R}_Q) \times \frac{\partial K}{\partial \mathbf{v}} + \frac{\partial K}{\partial \boldsymbol{\omega}}. \tag{7}$$

The first term on the right-hand side of Eq. (7) is the moment of momentum and the second one is the dynamic spin.

The kinetic energy, the momentum and the angular momentum are the additive functions of mass. This implies that if the body A consists of body-points A_i then

$$K(A) = \sum_i K(A_i), \qquad \mathbf{K}_1(A) = \sum_i \mathbf{K}_1(A_i), \qquad \mathbf{K}_2^Q(A) = \sum_i \mathbf{K}_2^Q(A_i). \tag{8}$$

In the case of distributed mass the sums in the right-hand sides of Eqs. (8) should be replaced by the corresponding integrals.

2.2 Interactions in a system of body-points

Below we enunciate the concept of interactions of bodies of the general form, as suggested in works by Zhilin, see e. g. [1], [2], [3]. Let us consider a collection of body-points, which we call a body A. All remaining body-points are called the environment of body A and denoted by a symbol A^e. To model the action of the environment A^e on the body A we should assign a pair of vectors: a force vector and a moment vector. The force and moment vectors are additive on both the bodies constituting the body A and the bodies constituting its environment A^e. We denote the force acting on body A from body B by the vector $\mathbf{F}(A, B)$. Thus, the force $\mathbf{F}(A, B)$ is the reaction of body B to the change of a spatial position of body A.

Definition 6 The moment $\mathbf{M}^Q(A, B)$ acting on body A from body B, which is calculated with respect to a reference point Q, can be expressed as follows:

$$\mathbf{M}^Q(A, B) = (\mathbf{R}_P - \mathbf{R}_Q) \times \mathbf{F}(A, B) + \mathbf{L}^P(A, B), \tag{9}$$

where the vector $\mathbf{R}_Q$ defines the position of a reference point Q, and the vector $\mathbf{R}_P$ defines the position of a datum point P. The reference point Q can be chosen arbitrary but it should be fixed (motionless). The first term on the right-hand side of Eq. (9) is called the moment of force. The vector $\mathbf{L}^P(A, B)$ is called the proper moment. It depends on the choice of a datum point P but not on the choice of a reference point Q. The proper moment $\mathbf{L}^P(A, B)$ is reaction of body B to the rotation of body A about the datum point P.

By definition, the full moment $\mathbf{M}^Q(A, B)$ does not depend on the choice of a datum point. In other words, the datum point being changed, the properly moment vector varies in such a way that the full moment vector $\mathbf{M}^Q(A, B)$ remains unchanged. Consequently,

$$\mathbf{L}^S(A, B) = (\mathbf{R}_P - \mathbf{R}_S) \times \mathbf{F}(A, B) + \mathbf{L}^P(A, B). \tag{10}$$

Equation (10) provides us with the relation between the proper moments calculated with respect to different datum points.

2.3 The balance equations in Euler's mechanics

In Newton's mechanics, the equations of momentum balance, angular momentum balance and energy balance for a system of point masses follow from the second Newton's law. In Euler's mechanics that considers the motion of the particles possessing rotational degrees of freedom and an internal structure, all the balance equations are independent laws. The successive statement of Euler's mechanics can be found in works by Zhilin [1], [2], [3]. Here, we briefly formulate three fundamental laws of Euler's mechanics.

The equation of momentum balance: The rate in the momentum change of body A is equal to the force acting on the body A from its environment plus the rate of the momentum supply in body A, namely:

$$\frac{d\mathbf{K}_1(A)}{dt} = \mathbf{F}(A, A^e) + \mathbf{k}_1(A). \tag{11}$$

The equation of angular momentum balance: The rate in the angular momentum change of body A, calculated with respect to a reference point Q, is equal to the moment acting on body A from its environment, calculated with respect to the same reference point Q, plus the rate of the angular momentum supply in a body A, namely:

$$\frac{d\mathbf{K}_2^Q(A)}{dt} = \mathbf{M}^Q(A, A^e) + \mathbf{k}_2^Q(A). \tag{12}$$

The equation of energy balance: The rate in the total energy change of body A is equal to the external force and moment power $N(A, A^e)$ plus the rate of supply of the energy of non-mechanical nature $\varepsilon(A)$, see [1]:

$$\frac{dE(A)}{dt} = N(A, A^e) + \varepsilon(A). \tag{13}$$

The total energy of a body $E(A)$ is a sum of the kinetic energy and the internal energy: $E(A) = K(A) + U(A)$. The power of external actions on body A consisting of body-points A_i is the bilinear form of velocities and actions, see [1]:

$$N(A, A^e) = \sum_i \Big[\mathbf{F}(A_i, A^e) \cdot \mathbf{v}_i + \mathbf{L}(A_i, A^e) \cdot \boldsymbol{\omega}_i\Big]. \tag{14}$$

2.4 Model of a nonclassical particle

If a body-point is not identified with the infinitesimal rigid body then the inertia tensor $m\mathbf{B}$ can be considered as being arbitrary. For the first time that sort of body-points has been introduced into consideration by Zhilin, see [1], [2], [4]. We will use term "nonclassical" for the particles consisting of body-points whose inertia tensor $m\mathbf{B}$ has both symmetric and antisymmetric parts. Below we construct a model of continuum whose elementary volume is represented by a nonclassical particle. Now we consider a body-point possessing the spherical inertia tensors, whose kinetic energy is

$$K = m_* \left(\frac{1}{2}\mathbf{v}_* \cdot \mathbf{v}_* + B\, \mathbf{v}_* \cdot \boldsymbol{\omega}_* + \frac{1}{2} J\, \boldsymbol{\omega}_* \cdot \boldsymbol{\omega}_*\right). \tag{15}$$

The momentum and the properly angular momentum of this body-point are

$$\mathbf{K}_1 = m_* \,(\mathbf{v}_* + B\, \boldsymbol{\omega}_*), \qquad \mathbf{K}_2 = m_* \,(B\, \mathbf{v}_* + J\, \boldsymbol{\omega}_*). \tag{16}$$

Let us note that the problem of free motion of the particle described by Eqs. (15), (16) has been solved in [1], [2], [4]. Free motion of this particle has been proved to be uniform motion along the helical curve whose axis is directed along the momentum vector.

3 Continuum of one-rotor gyrostats

3.1 Quasi-rigid body and its approximate model

Let us consider a particle (see Fig. 1 a) which is a quasi-rigid body made of body-points with Eqs. (15), (16). This particle is a rigid body in the sense that the distances between any two points of this particle are kept unchanged under arbitrary motions of this particle. However, unlike the standard rigid body, each point of a quasi-rigid body is a body-point which can rotate independently of rotations of other body-points. In other words, the quasi-rigid body is a multi-rotor gyrostat whose rotors are body-points that are governed by Eqs. (15) and (16) under arbitrary rotational motion. The carrier body of a gyrostat is considered to be inertialless and rotors are supposed to be distributed continuously. The motion of a carrier body is defined by the position

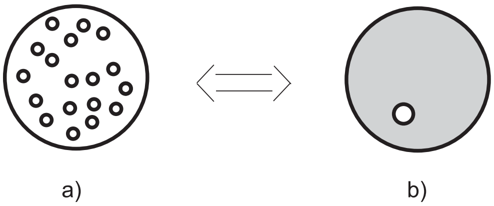
a) b)

Fig. 1 Quasi-rigid body and its approximate model

vector of the mass center $\mathbf{R}(t)$ of a quasi-rigid body and the rotation tensor $\tilde{\mathbf{P}}(t)$. The mass center velocity and the angular velocity of the carrier body are given by the formulas

$$\mathbf{v}(t)=\frac{d\mathbf{R}(t)}{dt},\qquad \tilde{\boldsymbol{\omega}}(t)=-\frac{1}{2}\left(\frac{d\tilde{\mathbf{P}}(t)}{dt}\cdot\tilde{\mathbf{P}}^T(t)\right)_\times, \tag{17}$$

where $(\ )_\times$ denotes the vector invariant of a tensor.

Let us consider a rotor whose position with respect to the mass center in the reference configurations is defined by a position vector $\mathbf{r}_*$. According to the fundamental theorem of the rigid body kinematics, the vector $\mathbf{R}_*$ defining the rotor position in the actual configuration is calculated by the formula

$$\mathbf{R}_*(\mathbf{r}_*,t)=\mathbf{R}(t)+\tilde{\mathbf{P}}(t)\cdot\mathbf{r}_*. \tag{18}$$

The rotation of this rotor is defined by the rotation tensor $\mathbf{P}_*(\mathbf{r}_*,t)$; the translational and angular velocities are calculated by the formulas

$$\mathbf{v}_*(\mathbf{r}_*,t)=\mathbf{v}(t)+\tilde{\boldsymbol{\omega}}(t)\times\tilde{\mathbf{P}}(t)\cdot\mathbf{r}_*,\qquad \boldsymbol{\omega}_*(\mathbf{r}_*,t)=-\frac{1}{2}\left(\frac{d\mathbf{P}_*(\mathbf{r}_*,t)}{dt}\cdot\mathbf{P}_*^T(\mathbf{r}_*,t)\right)_\times. \tag{19}$$

According to the axioms of additivity (8), the kinetic energy, the momentum and the angular momentum of a quasi-rigid body A are calculated as follows:

$$\begin{aligned}K(A)=\int_{(m)}\Big[\frac{1}{2}\mathbf{v}_*\cdot\mathbf{v}_*+B\,\mathbf{v}_*\cdot\boldsymbol{\omega}_*+\frac{1}{2}J\,\boldsymbol{\omega}_*\cdot\boldsymbol{\omega}_*\Big]dm=\frac{1}{2}m\left[\mathbf{v}\cdot\mathbf{v}+\tilde{\boldsymbol{\omega}}\cdot\mathbf{I}_*\cdot\tilde{\boldsymbol{\omega}}\right]+\\ +B\left[\mathbf{v}\cdot\int_{(m)}\boldsymbol{\omega}_*dm+\tilde{\boldsymbol{\omega}}\cdot\big(\tilde{\mathbf{P}}\cdot\int_{(m)}\mathbf{r}_*\big)\times\boldsymbol{\omega}_*dm\right]+\frac{1}{2}J\int_{(m)}\boldsymbol{\omega}_*\cdot\boldsymbol{\omega}_*dm,\end{aligned} \tag{20}$$

$$\mathbf{K}_1(A)=\int_{(m)}\left(\mathbf{v}_*+B\boldsymbol{\omega}_*\right)dm=m\,\mathbf{v}+B\int_{(m)}\boldsymbol{\omega}_*dm, \tag{21}$$

$$\begin{aligned}\mathbf{K}_2^Q(A)=\int_{(m)}\Big[\mathbf{R}_*\times\left(\mathbf{v}_*+B\,\boldsymbol{\omega}_*\right)+B\,\mathbf{v}_*+J\,\boldsymbol{\omega}_*\Big]dm=\mathbf{R}\times m\,\mathbf{v}+m\mathbf{I}_*\cdot\tilde{\boldsymbol{\omega}}+\\ +B\Big[\mathbf{R}\times\int_{(m)}\boldsymbol{\omega}_*dm+\big(\tilde{\mathbf{P}}\cdot\int_{(m)}\mathbf{r}_*\big)\times\boldsymbol{\omega}_*dm+m\,\mathbf{v}\Big]+J\int_{(m)}\boldsymbol{\omega}_*dm,\end{aligned} \tag{22}$$

where the symbol $\mathbf{I}_*$ is used to denote the geometrical inertia tensor of the quasi-rigid body:

$$m\mathbf{I}_*=\tilde{\mathbf{P}}\cdot m\mathbf{I}_*^{(0)}\cdot\tilde{\mathbf{P}}^T,\qquad m\mathbf{I}_*^{(0)}=\int_{(m)}\Big(r_*^2\mathbf{E}-\mathbf{r}_*\mathbf{r}_*\Big)dm. \tag{23}$$

Let us represent the angular velocity vector $\boldsymbol{\omega}_*(\mathbf{r}_*,t)$ of the rotors in the form

$$\boldsymbol{\omega}_*(\mathbf{r}_*,t)=\boldsymbol{\omega}(t)+\hat{\boldsymbol{\omega}}(\mathbf{r}_*,t),\qquad \boldsymbol{\omega}(t)=\frac{1}{m}\int_{(m)}\boldsymbol{\omega}_*(\mathbf{r}_*,t)dm. \tag{24}$$

Here, $\boldsymbol{\omega}(t)$ is the average angular velocity of the rotors of a quasi-rigid body; $\hat{\boldsymbol{\omega}}(\mathbf{r}_*, t)$ is the deviation of the angular velocity of the given rotor from the average angular velocity. Now we assume that

$$|\hat{\boldsymbol{\omega}}(\mathbf{r}_*, t)| \ll |\boldsymbol{\omega}(t)| \qquad \Rightarrow \qquad \boldsymbol{\omega}_*(\mathbf{r}_*, t) \approx \boldsymbol{\omega}(t). \tag{25}$$

In view of notation (24) and assumption (25) expressions for the kinetic energy, the momentum and the angular momentum (20)–(22) take the following simplified form:

$$K(A) = m\left(\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}\,\tilde{\boldsymbol{\omega}}\cdot\mathbf{I}_*\cdot\tilde{\boldsymbol{\omega}} + B\,\mathbf{v}\cdot\boldsymbol{\omega} + \frac{1}{2}J\,\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right), \tag{26}$$

$$\mathbf{K}_1(A) = m\big(\mathbf{v} + B\boldsymbol{\omega}\big), \tag{27}$$

$$\mathbf{K}_2^Q(A) = m\Big[\mathbf{R}\times(\mathbf{v} + B\,\boldsymbol{\omega}) + \mathbf{I}_*\cdot\tilde{\boldsymbol{\omega}} + B\,\mathbf{v} + J\,\boldsymbol{\omega}\Big]. \tag{28}$$

An examination of Eqs. (26)–(28) reveals that the approximate expressions for the kinetic energy, the momentum and the angular momentum of a quasi-rigid body, obtained under assumption (25), are coincident with those of a one-rotor gyrostat (see Fig. 1 b) whose rotor is located in the mass center of the quasi-rigid body. A carrier body of the gyrostat is the classical rigid body which inertial properties is characterized by inertia tensor $m\mathbf{I}_*$, and a rotor is a non-classical particle which is similar to the body-points constituting the quasi-rigid body (see Fig. 1 a).

3.2 Dynamical structure of continuum

The material medium (see Fig. 2) consisting of one-rotor gyrostats (26)–(28) is considered. To derive the dynamic equations of the continuum, we apply the spatial description. Let vector $\mathbf{r}$ determine the position of some point of space. We introduce following notations: $\rho(\mathbf{r}, t)$ is the mass density of the material medium at a given point of space; $\mathbf{v}(\mathbf{r}, t)$ is the velocity field; $\mathbf{u}(\mathbf{r}, t)$ is the displacement field; $\tilde{\mathbf{P}}(\mathbf{r}, t)$, $\tilde{\boldsymbol{\omega}}(\mathbf{r}, t)$ are the fields of the rotation tensors and the angular velocity vectors of the carrier bodies; $\mathbf{P}(\mathbf{r}, t)$ and $\boldsymbol{\omega}(\mathbf{r}, t)$ are fields of the rotation tensors and the angular velocity vectors of the rotors. In the spatial description, the formulas relating the velocity vector to the displacement vector and also the angular velocity vectors to the rotation tensors are written down by means of the material derivative, see [2], [3]:

$$\mathbf{v} = \frac{\delta\mathbf{u}}{\delta t}, \qquad \tilde{\boldsymbol{\omega}} = -\frac{1}{2}\left(\frac{\delta\tilde{\mathbf{P}}}{\delta t}\cdot\tilde{\mathbf{P}}^T\right)_\times, \qquad \boldsymbol{\omega} = -\frac{1}{2}\left(\frac{\delta\mathbf{P}}{\delta t}\cdot\mathbf{P}^T\right)_\times. \tag{29}$$

Here, the material derivative $\dfrac{\delta}{\delta t}$ is defined as follows, see [2], [3]:

$$\frac{\delta}{\delta t}\mathbf{u}(\mathbf{r}, t) = \lim_{\triangle t\to 0}\frac{\mathbf{u}(\mathbf{r} + \mathbf{v}\triangle t,\, t + \triangle t) - \mathbf{u}(\mathbf{r}, t)}{\triangle t} \equiv \frac{d}{dt}\mathbf{u}(\mathbf{r}, t) + \mathbf{v}(\mathbf{r}, t)\cdot\nabla\mathbf{u}(\mathbf{r}, t). \tag{30}$$

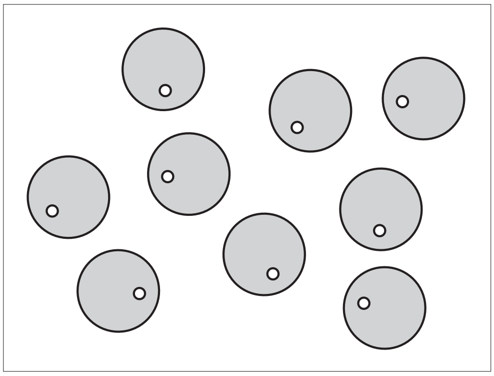

Fig. 2 Elementary volume of continuum consisting of one-rotor gyrostats

In accordance with Eqs. (26)–(28) the densities of the kinetic energy, the momentum and the angular momentum of continuum under consideration take a form

$$\rho K = \rho\left(\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}\,\tilde{\boldsymbol{\omega}}\cdot\mathbf{I}_*\cdot\tilde{\boldsymbol{\omega}} + B\,\mathbf{v}\cdot\boldsymbol{\omega} + \frac{1}{2}J\,\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right), \tag{31}$$

$$\rho\mathbf{K}_1 = \rho\big(\mathbf{v} + B\,\boldsymbol{\omega}\big), \qquad \rho\mathbf{K}_2 = \rho\Big[\mathbf{r}\times(\mathbf{v} + B\,\boldsymbol{\omega}) + \mathbf{I}_*\cdot\tilde{\boldsymbol{\omega}} + B\,\mathbf{v} + J\,\boldsymbol{\omega}\Big], \tag{32}$$

where the angular momentum density is calculated with respect to the origin of the reference frame. The particles of continuum under consideration possess the internal degrees of freedom. Therefore, in order to describe the motion of this continuum, it is not sufficient to formulate the balance equations of the momentum and the angular momentum for the control volume of the continuum. It is necessary to add these equations to the balance equation of the angular momentum for the rotors in control volume of the continuum. Therefore below we need the densities of the momentum and the angular momentum of the carrier bodies

$$\rho\mathbf{K}_1^{(cb)} = \rho(1-\epsilon)\,\mathbf{v}, \qquad \rho\mathbf{K}_2^{(cb)} = \rho\Big[\mathbf{r}\times(1-\epsilon)\,\mathbf{v} + \mathbf{I}_*\cdot\tilde{\boldsymbol{\omega}}\Big], \tag{33}$$

and the momentum and the angular momentum of the rotors

$$\rho\mathbf{K}_1^{(rot)} = \rho(\epsilon\,\mathbf{v} + B\,\boldsymbol{\omega}), \qquad \rho\mathbf{K}_2^{(rot)} = \rho\Big[\mathbf{r}\times(\epsilon\,\mathbf{v} + B\,\boldsymbol{\omega}) + B\,\mathbf{v} + J\,\boldsymbol{\omega}\Big]. \tag{34}$$

The dimensionless parameter ϵ in Eqs. (33), (34) characterizes the distribution of mass in the gyrostat: if m is the mass of the gyrostat then $(1-\epsilon)m$ is the mass of its carrier body and ϵm is the mass of its rotor. Below we will see that the value of the parameter ϵ is not important. It is evident that Eqs. (33), (34) do not follow from Eqs. (32) and they should be considered as additional assumptions. However, it is easy to see that

$$\rho\mathbf{K}_1 = \rho\mathbf{K}_1^{(cb)} + \rho\mathbf{K}_1^{(rot)}, \qquad \rho\mathbf{K}_2 = \rho\mathbf{K}_2^{(cb)} + \rho\mathbf{K}_2^{(rot)}. \tag{35}$$

3.3 Law of mass conservation and equations of dynamics

Let V and S denote some fixed region in the reference frame (control volume) and its surface, respectively. Let us formulate the law of mass conservation for the control volume:

$$\frac{d}{dt}\int_{(V)}\rho(\mathbf{r},t)\,dV = -\int_{(S)}\mathbf{n}\cdot\mathbf{v}(\mathbf{r},t)\rho(\mathbf{r},t)\,dS. \tag{36}$$

Here, $\mathbf{n}$ denotes the unit vector of a normal to the surface S. Using standard line of reasoning we derive from the Eq. (36) the law of mass conservation in the local form

$$\frac{\delta\rho}{\delta t} + \rho\,\nabla\cdot\mathbf{v} = 0. \tag{37}$$

Now, we formulate the equation of balance of momentum (11) for the carrier bodies in the control volume V:

$$\frac{d}{dt}\int_{(V)}\rho\mathbf{K}_1^{(cb)}dV = \int_{(V)}\rho(\mathbf{f}+\mathbf{F})dV + \int_{(S)}\boldsymbol{\tau}_n dS - \int_{(S)}(\mathbf{n}\cdot\mathbf{v})\rho\mathbf{K}_1^{(cb)}dS. \tag{38}$$

Here $\mathbf{f}$ is the mass density of external forces; $\mathbf{F}$ is the mass density of forces modelling the effect of the rotors; $\boldsymbol{\tau}_n$ is the force vector modelling the influence of surrounding medium on the carrier bodies of gyrostats being on surface S of control volume V. Next, we formulate the equation of balance of momentum (11) for the rotors in the control volume V:

$$\frac{d}{dt}\int_{(V)}\rho\mathbf{K}_1^{(rot)}dV = -\int_{(V)}\rho\mathbf{F}dV - \int_{(S)}(\mathbf{n}\cdot\mathbf{v})\rho\mathbf{K}_1^{(rot)}dS. \tag{39}$$

The rotors are assumed to interact only by moment rather than forces. External force actions on the rotors are also supposed to be absent.

By standard reasoning we introduce the concept of stress tensor concerned with the force vector $\boldsymbol{\tau}_n$ by relation $\boldsymbol{\tau}_n = \mathbf{n} \cdot \boldsymbol{\tau}$ and from Eqs. (38), (39) we derive the local form of the momentum balance equations for the carrier bodies and the rotors:

$$\nabla \cdot \boldsymbol{\tau} + \rho(\mathbf{f} + \mathbf{F}) = \rho(1 - \epsilon)\,\frac{\delta \mathbf{v}}{\delta t}, \qquad -\rho\mathbf{F} = \rho\frac{\delta}{\delta t}\big(\epsilon\,\mathbf{v} + B\boldsymbol{\omega}\big). \tag{40}$$

By obtaining Eqs. (40) we used expressions for the momentum densities (33), (34) and the mass balance equation (37). Summing up both sides of Eqs. (40) we obtain the momentum balance equations for the gyrostats

$$\nabla \cdot \boldsymbol{\tau} + \rho\mathbf{f} = \rho\frac{\delta}{\delta t}\big(\mathbf{v} + B\boldsymbol{\omega}\big). \tag{41}$$

Now we formulate the equation of balance of angular momentum (12) for the carrier bodies in the control volume V:

$$\frac{d}{dt}\int_{(V)} \rho\mathbf{K}_2^{(cb)} dV = \int_{(V)} \rho\big(\mathbf{r} \times \mathbf{f} + \mathbf{r} \times \mathbf{F} + \mathbf{m}\big) dV + \int_{(S)} \big(\mathbf{r} \times \boldsymbol{\tau}_n + \boldsymbol{\mu}_n\big) dS - \int_{(S)} (\mathbf{n} \cdot \mathbf{v})\rho\mathbf{K}_2^{(cb)} dS. \tag{42}$$

Here, $\mathbf{m}$ is the mass density of external moments acting on the carrier bodies of gyrostats; $\boldsymbol{\mu}_n$ is the moment vector modelling the influence of surrounding medium on the carrier bodies of gyrostats being on surface S of control volume V. Next, we formulate the equation of balance of angular momentum (12) for the rotors in the control volume V:

$$\frac{d}{dt}\int_{(V)} \rho\mathbf{K}_2^{(rot)} dV = \int_{(V)} \rho\big(-\mathbf{r} \times \mathbf{F} + \mathbf{L}\big) dV + \int_{(S)} \mathbf{T}_n dS - \int_{(S)} (\mathbf{n} \cdot \mathbf{v})\rho\mathbf{K}_2^{(rot)} dS. \tag{43}$$

Here $\mathbf{L}$ is the mass density of external moments acting on the rotors; $\mathbf{T}_n$ is the moment vector modelling the influence of surrounding medium on the rotors of gyrostats on surface S of control volume V.

By standard reasoning, we introduce the concept of moment stress tensors $\boldsymbol{\mu}$ and $\mathbf{T}$ which are concerned with the moment vectors $\boldsymbol{\mu}_n$ and $\mathbf{T}_n$ by relations: $\boldsymbol{\mu}_n = \mathbf{n} \cdot \boldsymbol{\mu}$, $\mathbf{T}_n = \mathbf{n} \cdot \mathbf{T}$. Also by standard reasoning we derive the equations of balance of angular momentum in the local form from Eqs. (42), (43). After simple transformations, these equations can be written in the form

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \rho\mathbf{m} = \rho\frac{\delta}{\delta t}\big(\mathbf{I}_* \cdot \tilde{\boldsymbol{\omega}}\big). \tag{44}$$

$$\nabla \cdot \mathbf{T} + \rho\mathbf{L} = \rho\Big[\mathbf{v} \times B\boldsymbol{\omega} + \frac{\delta}{\delta t}\big(B\mathbf{v} + J\boldsymbol{\omega}\big)\Big]. \tag{45}$$

By obtaining Eqs. (44) and (45), we used expression for the angular momentum densities (33), (34), the equation of mass balance (37), and the equations of momentum balance (40). Equation (44) describes the rotational motion of the carrier bodies and Eq. (45) characterizes the motion of the rotors. It is easy to see that Eqs. (44), (45) do not contain the parameter ϵ.

3.4 Equation of energy balance

Now we formulate the equation of energy balance (13) for the material medium in the control volume V:

$$\frac{d}{dt}\int_{(V)} \rho(K + U) dV = \int_{(V)} \rho\big(\mathbf{f} \cdot \mathbf{v} + \mathbf{m} \cdot \tilde{\boldsymbol{\omega}} + \mathbf{L} \cdot \boldsymbol{\omega} + Q\big) dV + \\ + \int_{(S)} \big(\boldsymbol{\tau}_n \cdot \mathbf{v} + \boldsymbol{\mu}_n \cdot \tilde{\boldsymbol{\omega}} + \mathbf{T}_n \cdot \boldsymbol{\omega} + H_n\big) dS - \int_{(S)} (\mathbf{n} \cdot \mathbf{v})\,\rho(K + U) dS. \tag{46}$$

Here, U is the internal energy density per unit mass; Q and H_n are the rate of the energy supply in volume and through surface S, respectively. The rate of the energy supply through the surface can be expressed in term of energy-flux vector $\mathbf{H}$ by the formula $H_n = \mathbf{n} \cdot \mathbf{H}$.

By standard reasoning, taking into account the equation of mass balance (37), we transform the equation of energy balance (46) to the local form

$$\rho\frac{\delta}{\delta t}(K+U)=\rho\mathbf{f}\cdot\mathbf{v}+\rho\mathbf{m}\cdot\tilde{\boldsymbol{\omega}}+\rho\mathbf{L}\cdot\boldsymbol{\omega}+(\nabla\cdot\boldsymbol{\tau})\cdot\mathbf{v}+(\nabla\cdot\boldsymbol{\mu})\cdot\tilde{\boldsymbol{\omega}}+ \\ +(\nabla\cdot\mathbf{T})\cdot\boldsymbol{\omega}+\boldsymbol{\tau}^T\cdot\cdot\nabla\mathbf{v}+\boldsymbol{\mu}^T\cdot\cdot\nabla\tilde{\boldsymbol{\omega}}+\mathbf{T}^T\cdot\cdot\nabla\boldsymbol{\omega}+\nabla\cdot\mathbf{H}+\rho Q. \quad (47)$$

Here symbol "$\cdot\cdot$" has the following sense: $\mathbf{ab}\cdot\cdot\mathbf{cd}=(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$. Using expression (31) for the kinetic energy density and the balance equations (41), (44), (45), we transform the energy balance equation (47) to the form

$$\rho\frac{\delta U}{\delta t}=\boldsymbol{\tau}^T\cdot\cdot\big(\nabla\mathbf{v}+\mathbf{E}\times\tilde{\boldsymbol{\omega}}\big)+\boldsymbol{\mu}^T\cdot\cdot\nabla\tilde{\boldsymbol{\omega}}+\mathbf{T}^T\cdot\cdot\nabla\boldsymbol{\omega}+\nabla\cdot\mathbf{H}+\rho Q. \quad (48)$$

If the supply of energy of "non-mechanical nature" is ignored, i. e. the body under consideration is assumed to be isolated, then Eq. (48) takes a more simple form:

$$\rho\frac{\delta U}{\delta t}=\boldsymbol{\tau}^T\cdot\cdot\big(\nabla\mathbf{v}+\mathbf{E}\times\tilde{\boldsymbol{\omega}}\big)+\boldsymbol{\mu}^T\cdot\cdot\nabla\tilde{\boldsymbol{\omega}}+\mathbf{T}^T\cdot\cdot\nabla\boldsymbol{\omega}. \quad (49)$$

Below we consider only isolated bodies. Furthermore, we are going to derive equations of dynamics, expressions for strain tensors, and constitutive equations in the framework of the geometrically linear theory.

3.5 Linear theory of elastic medium

Let us consider the tensor $\tilde{\mathbf{P}}(\mathbf{r},t)$ (rotation tensor of carrier bodies) and the tensor $\mathbf{P}(\mathbf{r},t)$ (rotation tensor of rotors). We assume that in the reference configurations the tensors $\tilde{\mathbf{P}}(\mathbf{r},t)$ and $\mathbf{P}(\mathbf{r},t)$ are equal to the unit tensor. Therefore, upon the linearization near the reference position they take the form

$$\tilde{\mathbf{P}}(\mathbf{r},t)=\mathbf{E}+\boldsymbol{\varphi}(\mathbf{r},t)\times\mathbf{E},\qquad \mathbf{P}(\mathbf{r},t)=\mathbf{E}+\boldsymbol{\theta}(\mathbf{r},t)\times\mathbf{E}, \quad (50)$$

where $\boldsymbol{\varphi}(\mathbf{r},t),\boldsymbol{\theta}(\mathbf{r},t)$ are the rotation vector fields of carrier bodies and rotors, respectively. Kinematic relations (29) in the linear approximation are

$$\mathbf{v}=\frac{d\mathbf{u}}{dt},\qquad \tilde{\boldsymbol{\omega}}=\frac{d\boldsymbol{\varphi}}{dt},\qquad \boldsymbol{\omega}=\frac{d\boldsymbol{\theta}}{dt}. \quad (51)$$

The mass balance equation (37) in the linear approximation takes the form

$$\frac{d\rho}{dt}+\rho\,\nabla\cdot\mathbf{v}=0\qquad\Rightarrow\qquad \rho=\rho_*\left(1-\nabla\cdot\mathbf{u}\right). \quad (52)$$

Here, ρ_* is the mass density per unit volume in the reference position. Note that the mass density at the initial time instant ρ_0 may not coincide with the mass density in the reference position ρ_*. These two quantities are related to each other by the formula

$$\rho_0=\rho_*\left(1-\nabla\cdot\mathbf{u}_0\right), \quad (53)$$

and they coincide only if the material medium is not deformable at the initial time instant.

In view of the above simplifications, the equation of motion of the material continuum (41), (44) can be rewritten in the form

$$\nabla\cdot\boldsymbol{\tau}+\rho_*\mathbf{f}=\rho_*\frac{d}{dt}\big(\mathbf{v}+B\boldsymbol{\omega}\big),\qquad \nabla\cdot\boldsymbol{\mu}+\boldsymbol{\tau}_\times+\rho_*\mathbf{m}=\rho_*\frac{d}{dt}\big(\mathbf{I}_*^{(0)}\cdot\tilde{\boldsymbol{\omega}}\big), \quad (54)$$

where inertia tensor $\mathbf{I}_*^{(0)}$ is calculated in the reference configuration. The equation of motion of the rotors (45) takes the form

$$\nabla\cdot\mathbf{T}+\rho_*\mathbf{L}=\rho_*\frac{d}{dt}\big(B\mathbf{v}+J\boldsymbol{\omega}\big), \quad (55)$$

and after simple transformations the equation of energy balance (49) is written as follows:

$$\rho_* \frac{dU}{dt} = \boldsymbol{\tau}^T \cdot\cdot \frac{d\boldsymbol{\varepsilon}}{dt} + \boldsymbol{\mu}^T \cdot\cdot \frac{d\boldsymbol{\kappa}}{dt} + \mathbf{T}^T \cdot\cdot \frac{d\boldsymbol{\vartheta}}{dt}, \tag{56}$$

where the strain tensors $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$, $\boldsymbol{\vartheta}$ are introduced into consideration. These tensors are calculated by the formulas

$$\boldsymbol{\varepsilon} = \nabla \mathbf{u} + \mathbf{E} \times \boldsymbol{\varphi}, \qquad \boldsymbol{\kappa} = \nabla \boldsymbol{\varphi}, \qquad \boldsymbol{\vartheta} = \nabla \boldsymbol{\theta}. \tag{57}$$

In what follows we consider the elastic material i. e. a material whose density of internal energy and the tensors of force and moment stresses depend only on the strain tensors and do not depend on velocities. For the elastic material the Cauchy–Green relations follow from the equation of energy balance (56):

$$\boldsymbol{\tau} = \rho_* \frac{\partial U}{\partial \boldsymbol{\varepsilon}}, \qquad \boldsymbol{\mu} = \rho_* \frac{\partial U}{\partial \boldsymbol{\kappa}}, \qquad \mathbf{T} = \rho_* \frac{\partial U}{\partial \boldsymbol{\vartheta}}. \tag{58}$$

To close the system of differential equations, it is necessary to express the internal energy as a function of the strain tensors

$$\rho_* U = \rho_* U(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}, \boldsymbol{\vartheta}). \tag{59}$$

Now we consider the physically linear theory and, therefore, we represent the density of internal energy in the following form:

$$\begin{aligned} \rho_* U = \boldsymbol{\tau}_0^T \cdot\cdot \boldsymbol{\varepsilon} + \boldsymbol{\mu}_0^T \cdot\cdot \boldsymbol{\kappa} + \mathbf{T}_*^T \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*) + \frac{1}{2} \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_1 \cdot\cdot \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_2 \cdot\cdot \boldsymbol{\kappa} + \frac{1}{2} \boldsymbol{\kappa} \cdot\cdot {}^4\mathbf{C}_3 \cdot\cdot \boldsymbol{\kappa} + \\ + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_4 \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*) + \boldsymbol{\kappa} \cdot\cdot {}^4\mathbf{C}_5 \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*) + \frac{1}{2} (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*) \cdot\cdot {}^4\mathbf{C}_6 \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*). \end{aligned} \tag{60}$$

The coefficients $\boldsymbol{\tau}_0$, $\boldsymbol{\mu}_0$ and $\mathbf{T}_*$ are called the initial stresses. Coefficients of the quadratic form (60) are called the stiffness tensors. In the linear theory, the stiffness tensors do not depend on time. The only restriction imposed on the stiffness tensors is concerned with the requirement of positive definiteness of the quadratic form (60). The structure of the stiffness tensors and the values of the coefficients of elasticity are determined by the physical properties of the material medium.

After substituting the expression for the density of internal energy (60) in the Cauchy–Green relations (58), we obtain the following constitutive equations:

$$\begin{aligned} \boldsymbol{\tau}^T &= \boldsymbol{\tau}_0^T + {}^4\mathbf{C}_1 \cdot\cdot \boldsymbol{\varepsilon} + {}^4\mathbf{C}_2 \cdot\cdot \boldsymbol{\kappa} + {}^4\mathbf{C}_4 \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*), \\ \boldsymbol{\mu}^T &= \boldsymbol{\mu}_0^T + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_2 + {}^4\mathbf{C}_3 \cdot\cdot \boldsymbol{\kappa} + {}^4\mathbf{C}_5 \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*), \\ \mathbf{T}^T &= \mathbf{T}_*^T + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_4 + \boldsymbol{\kappa} \cdot\cdot {}^4\mathbf{C}_5 + {}^4\mathbf{C}_6 \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*). \end{aligned} \tag{61}$$

According to Eqs. (61), all stress tensors can depend on all strain tensors. It means, in particular, that the moment stress tensor of rotors can depend not only on their relative orientation, but also on the relative orientation and relative position of the carrier bodies.

3.6 The simplest theory of one-rotor gyrostats continuum

We consider the material continuum (see Fig. 3) that consists of one-rotor gyrostats of the kind (26)–(28). In limits of linear theory, the motion of this continuum is described by Eqs. (51), (52), (54), (55), (57), (61). Free space between the gyrostats is filled up by body-points whose structure coincides with the structure of rotors belonging to the gyroststs [see Eqs. (15), (16)]. The body-points in the space between the gyrostats are the elementary particles of a continuum which will be called the "thermal ether" in what follows. In fact, the material continuum represented in Fig. 3 is a two-component medium. We are not going to study in detail the motion of the body-points continuum ("thermal ether") and the interaction between the gyrostats continuum and the body-points continuum. We consider only the gyrostats continuum as an object under study. The interaction between the carrier bodies of the gyrostats and the interaction between rotors of the gyrostats are characterized by tensors of the force and moment stresses (61). The body-points continuum ("thermal ether") positioned in the space between gyrostats is considered to be an external factor with respect to the continuum

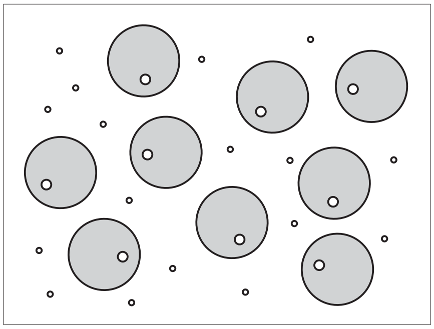

Fig. 3 Elementary volume of continuum interacting with environment

under study. That is why we will model the influence of the “thermal ether” on the gyrostats by an external moment in the equation of the rotors motion (55).

Accepting two important hypotheses, we consider a special case of the linear theory of one-rotor gyrostats continuum.

Hypothesis 1 Vector $\mathbf{L}$ (the mass density of external actions on the rotors of gyrostats) is a sum of the moment $\mathbf{L}_h$ characterizing external actions of all sorts and the moment of linear viscous damping

$$\mathbf{L}_f = -\beta(B\mathbf{v} + J\boldsymbol{\omega}). \tag{62}$$

The moment (62) characterizes the influence of the “thermal ether”. The structure of the moment is chosen in accordance with the results of solving some model problems. Discussion of these problems is beyond the scope of the paper. Now we explain the physical meaning of the moment of linear viscous damping (62). Let us remind that the one-rotor gyrostat is an approximate model of the quasi-rigid body (see Fig. 1). We suppose that the rotors of the quasi-rigid bodies interact with body-points of the “thermal ether” and this interaction is described by the elastic moments analogous to the moments characterizing the interaction of the rotors with each other. The “thermal ether” having infinite extent, it carries energy of the oscillating rotors away. The solution of modelling problems reveals that in the case of an infinite surrounding medium the dissipative moment arising due to the interaction with this medium is proportional to the proper angular momentum vector (dynamic spin).

Hypothesis 2 The moment stress tensor $\mathbf{T}$ characterizing the interactions between rotors is the spherical tensor

$$\mathbf{T} = T\mathbf{E}. \tag{63}$$

In view of assumptions (62) and (63), the equation of the rotors motion (55) takes the form

$$\nabla T - \rho_*\beta\,(B\mathbf{v} + J\boldsymbol{\omega}) + \rho_*\mathbf{L}_h = \rho_*\frac{d}{dt}\big(B\mathbf{v} + J\boldsymbol{\omega}\big), \tag{64}$$

In view of assumption (63), the last term on the right-hand side of the energy balance equation (56) can be reduced to a more simple form. By using the notation $\vartheta = \operatorname{tr}\boldsymbol{\vartheta}$, the energy balance equation (56) is written as

$$\rho_*\frac{dU}{dt} = \boldsymbol{\tau}^T\cdot\cdot\frac{d\boldsymbol{\varepsilon}}{dt} + \boldsymbol{\mu}^T\cdot\cdot\frac{d\boldsymbol{\kappa}}{dt} + T\,\frac{d\vartheta}{dt}. \tag{65}$$

The material medium under consideration being elastic, we obtain from Eq. (65), the Cauchy–Green relations of which the first and the second ones coincide with the first and the second relations of (58), respectively, and the third one has a simpler form:

$$\boldsymbol{\tau} = \rho_*\frac{\partial U}{\partial\boldsymbol{\varepsilon}}, \qquad \boldsymbol{\mu} = \rho_*\frac{\partial U}{\partial\boldsymbol{\kappa}}, \qquad T = \rho_*\frac{\partial U}{\partial\vartheta}. \tag{66}$$

According to Eq. (65), the density of internal energy is a function of arguments $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$ and ϑ. Let us construct the physically linear theory based on representation of the internal energy density in the following form:

$$\rho_* U = \boldsymbol{\tau}_0\cdot\cdot\boldsymbol{\varepsilon} + T_*\,(\vartheta - \vartheta_*) + \frac{1}{2}\,\boldsymbol{\varepsilon}\cdot\cdot{}^4\mathbf{C}_1\cdot\cdot\boldsymbol{\varepsilon} + \varUpsilon\operatorname{tr}\boldsymbol{\varepsilon}\,(\vartheta - \vartheta_*) + \frac{1}{2}K(\vartheta - \vartheta_*)^2. \tag{67}$$

Then the constitutive equations (61) take the form

$$\boldsymbol{\tau}^T = \boldsymbol{\tau}_0^T + {}^4\mathbf{C}_1 \cdot\cdot\, \boldsymbol{\varepsilon} + \Upsilon\,(\vartheta - \vartheta_*)\,\mathbf{E}, \qquad \boldsymbol{\mu} = 0, \qquad T = T_* + \Upsilon \operatorname{tr} \boldsymbol{\varepsilon} + K(\vartheta - \vartheta_*). \tag{68}$$

Thus, the simplest linear theory of the material continuum consisting of one-rotor gyrostats is described by Eqs. (51), (54), (57), (64), (68).

3.7 Temperature and entropy

Let us consider the foregoing mathematical model of elastic continuum of one-rotor gyrostats. Suppose that the model describes the behavior of the classical medium which possesses not only elastic properties but also the viscous and thermic properties. Now we can give a thermodynamic interpretation of the variables describing motion and interaction of the rotors and next we can carry out identification of parameters of the model and well-known thermodynamic constants.

Let us consider the energy balance equation (65). Conceive that Eq. (65) is the equation of energy balance for classical moment medium (medium without rotors). Then the last term on the right-hand side of Eq. (65) can be treated as thermodynamical one. The physical quantities T and ϑ acquire the meaning of temperature and volume density of entropy, respectively.

It is evident that the dimensions of the temperature and the entropy defined by formula (65) are different from the dimensions of those in classical thermodynamics of the present simple case. This problem can be solved by introduction of a normalization factor:

$$T = aT_a, \qquad \vartheta = \frac{1}{a}\,\vartheta_a. \tag{69}$$

Here, a is the normalization factor; T_a is the absolute temperature measured by a thermometer; ϑ_a is volume density of the absolute entropy. Let us introduce the similar relations for the remaining variables:

$$\boldsymbol{\theta} = \frac{1}{a}\,\boldsymbol{\theta}_a, \qquad \boldsymbol{\omega} = \frac{1}{a}\,\boldsymbol{\omega}_a, \qquad \mathbf{L}_h = a\mathbf{L}_h^a, \qquad \mathbf{L}_f = a\mathbf{L}_f^a. \tag{70}$$

Now rewriting all equations for new variables and using new parameters

$$B_a = \frac{B}{a}, \qquad J_a = \frac{J}{a^2}, \qquad \Upsilon_a = \frac{\Upsilon}{a}, \qquad K_a = \frac{K}{a^2} \tag{71}$$

we can eliminate the normalization factor a from these equations at least in the linear formulation of the problem and in some particular cases of physical nonlinearity.

4 Linear theory of thermoviscoelasticity

4.1 Hyperbolic type thermoelasticity

The classical theory of thermoelasticity is a momentless one. Therefore, considering the problem of thermoelasticity in the context of the proposed model we assume only the force interaction between carrier bodies of the gyrostats and only the force action of external factors upon them:

$$\boldsymbol{\mu} = 0, \qquad \mathbf{m} = 0. \tag{72}$$

In the static problems from the second equation of (54) under the assumption (72), it follows that $\boldsymbol{\tau}_\times = 0$. In the dynamic problems the stress tensor can be nonsymmetric in spite of assumption (72). In this case, it is necessary to take into account the dependence of the strain tensor $\boldsymbol{\varepsilon}$ on the angle of rotation of carrier bodies $\boldsymbol{\varphi}$. Thus, assumption (72) does not imply transition to the momentless theory of elasticity for carrier bodies. In addition, let us assume that $\mathbf{I}_*^{(0)} = 0$. In this case tensor $\boldsymbol{\tau}$ will be symmetric both in the static and dynamic problems and all equations concerned with rotational motions of the carrier bodies of gyrostats can be excluded.

Applying the linear theory is admissible in certain range of temperatures and entropy densities changing about some reference values T_a^* and ϑ_a^*. Let us introduce deviations of the temperature and the density of entropy from their reference values:

$$T_a = T_a^* + \tilde{T}_a, \qquad \vartheta_a = \vartheta_a^* + \tilde{\vartheta}_a. \tag{73}$$

Further derivation of the basic equations of the linear theory of elastic medium consisting of one-rotor gyrostats includes the dynamic equations (54), (64) which under notations (69), (70), (71), (73) take the form

$$\begin{gathered} \nabla\cdot\boldsymbol{\tau} + \rho_*\mathbf{f} = \rho_*\frac{d}{dt}\big(\mathbf{v} + B_a\boldsymbol{\omega}_a\big), \\ \nabla\tilde{T}_a - \rho_*\beta\,(B_a\mathbf{v} + J_a\boldsymbol{\omega}_a) + \rho_*\mathbf{L}_h^a = \rho_*\frac{d}{dt}\big(B_a\mathbf{v} + J_a\boldsymbol{\omega}_a\big), \end{gathered} \tag{74}$$

the mass balance equation (52), the kinematical and geometrical relations (51), (57) which under notations (69), (70) and condition of symmetry of the stress tensor are reduced to

$$\begin{gathered} \rho = \rho_*\,(1-\varepsilon), \qquad \mathbf{v} = \frac{d\mathbf{u}}{dt}, \qquad \boldsymbol{\omega}_a = \frac{d\boldsymbol{\theta}_a}{dt}, \\ \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \qquad \varepsilon = \operatorname{tr}\boldsymbol{\varepsilon}, \qquad \vartheta_a = \operatorname{tr}\boldsymbol{\vartheta}_a = \nabla\cdot\boldsymbol{\theta}_a, \end{gathered} \tag{75}$$

and the constitutive equations (68) which under notations (69), (70), (71), (73) and condition of symmetry of the stress tensor are written as

$$\boldsymbol{\tau} = \Big(K_{ad} - \frac{2}{3}\,G\Big)\varepsilon\mathbf{E} + 2G\,\boldsymbol{\varepsilon} + \varUpsilon_a\,\tilde{\vartheta}_a\,\mathbf{E}, \qquad \tilde{T}_a = \varUpsilon_a\,\varepsilon + K_a\,\tilde{\vartheta}_a, \tag{76}$$

where K_{ad} is the adiabatic modulus of compression (the adiabatic bulk modulus), and G is the shear modulus.

Let us compare the system of equations (74)–(76) with the classical equations of the coupled problem of thermoelasticity. The classical equations have the form, see e. g. [5], [6]

$$\begin{gathered} \nabla\cdot\boldsymbol{\tau} + \rho_*\mathbf{f} = \rho_*\frac{d^2\mathbf{u}}{dt^2}, \qquad \boldsymbol{\tau} = \Big(K_{iz} - \frac{2}{3}\,G\Big)\varepsilon\mathbf{E} + 2G\,\boldsymbol{\varepsilon} - \alpha K_{iz}\tilde{T}_a\,\mathbf{E}, \\ \mathit{\Delta}\tilde{T}_a - \frac{\rho_* c_v}{\lambda}\,\frac{d\tilde{T}_a}{dt} = \frac{\alpha K_{iz} T_a^*}{\lambda}\,\frac{d\varepsilon}{dt} - \frac{\rho_* q}{\lambda}, \qquad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \quad \varepsilon = \operatorname{tr}\boldsymbol{\varepsilon}. \end{gathered} \tag{77}$$

Here, q is the time rate of heat energy supply per unit of mass, c_v is the specific heat at constant volume, λ is the heat-conduction coefficient, K_{iz} is the isothermal modulus of compression (the isothermal bulk modulus), $\alpha = 3\alpha_*$ is the volume coefficient of thermal expansion, where α_* is the linear coefficient of thermal expansion. As a rule, values α_* for solids and values α for fluids can be found in the handbooks. The isothermal bulk modulus and the adiabatic bulk modulus are related by

$$K_{ad} = K_{iz}\frac{c_p}{c_v}, \qquad c_p - c_v = \frac{\alpha^2 K_{iz} T_a^*}{\rho_*} \qquad \Rightarrow \qquad K_{ad} = K_{iz} + \frac{\alpha^2 K_{iz}^2 T_a^*}{\rho_* c_v}, \tag{78}$$

where c_p is the specific heat at constant pressure.

Comparison of the dynamic equations. It is easy to see that the first equation in (74) and the first equation in (77) coincide under the condition $B_a = 0$. Now let us assume that $B_a = 0$. In what follows we return to this question and consider the general situation.

Comparison of the constitutive equations. For convenience of comparison, we transform the first equation in (76) eliminating the entropy from it with the help of the second equation in (76). As a result we obtain:

$$\boldsymbol{\tau} = \Big(K_{ad} - \frac{\varUpsilon_a^2}{K_a} - \frac{2}{3}\,G\Big)\varepsilon\mathbf{E} + 2G\,\boldsymbol{\varepsilon} + \frac{\varUpsilon_a}{K_a}\,\tilde{T}_a\,\mathbf{E}. \tag{79}$$

Comparing Eq. (79) with the second equation of (77) we conclude that these equations coincide if

$$\frac{\Upsilon_a^2}{K_a} = K_{ad} - K_{iz}, \qquad \frac{\Upsilon_a}{K_a} = -\alpha K_{iz}. \tag{80}$$

Upon solving Eqs. (80) under condition (78), we obtain the following expressions for constants K_a and Υ_a:

$$K_a = \frac{T_a^*}{\rho_* c_v}, \qquad \Upsilon_a = -\frac{\alpha K_{iz} T_a^*}{\rho_* c_v}. \tag{81}$$

Comparison of the heat conduction equations. Now we take the divergence of both sides of the second equation of (74) and transform the equation obtained by using Eqs. (75), (76). As a result we get

$$\Delta\tilde{T}_a - \frac{\rho_*\beta J_a}{K_a}\frac{d\tilde{T}_a}{dt} - \frac{\rho_* J_a}{K_a}\frac{d^2\tilde{T}_a}{dt^2} = \beta\rho_*\left(B_a - \frac{\Upsilon_a J_a}{K_a}\right)\frac{d\varepsilon}{dt} + \rho_*\left(B_a - \frac{\Upsilon_a J_a}{K_a}\right)\frac{d^2\varepsilon}{dt^2} - \rho_*\nabla\cdot\mathbf{L}_h^a. \tag{82}$$

Comparing the heat conduction equation (82) with the third equation in (77), we conclude that in the case of $B_a = 0$ these equations agree within the terms containing the second time derivatives if

$$\frac{\beta J_a}{K_a} = \frac{c_v}{\lambda}, \qquad \frac{\beta\rho_*\Upsilon_a J_a}{K_a} = -\frac{\alpha K_{iz} T_a^*}{\lambda}, \qquad \nabla\cdot\mathbf{L}_h^a = \frac{q}{\lambda}. \tag{83}$$

From the first equation of (83) by using Eq. (81) we obtain

$$\beta J_a = \frac{T_a^*}{\rho_*\lambda}. \tag{84}$$

In this case, the second equation in (83) is satisfied identically.

Comparison with the phonon theory. The coefficient of the second time derivative of the temperature in the heat conduction equation (82) is concerned with the velocity of propagation of the thermal wave c_r:

$$c_r^2 = \frac{K_a}{\rho_* J_a}. \tag{85}$$

Within the framework of the classical theory, the quantity c_r can not be determined. Therefore, for identification of the parameter, it is necessary to carry out the comparison with the phonon theory.

Version 1 According to the data of [7] (p. 141) and [8] (p. 19), velocity of the thermal wave propagation c_r is related to velocity of the longitudinal acoustic wave propagation c_a by

$$c_r = \frac{c_a}{\sqrt{3}}. \tag{86}$$

The velocity of the longitudinal acoustic wave propagation is calculated as follows:

$$c_a^2 = \frac{K_{ad} + 4G/3}{\rho_*}. \tag{87}$$

Taking into account the relations between K_{ad}, G and other constants (Young’s modulus E and Poisson’s ratio ν) we obtain

$$K_{ad} = \frac{E}{3(1-2\nu)}, \qquad G = \frac{E}{2(1+\nu)} \qquad \Rightarrow \qquad c_a^2 = \frac{3K_{ad}(1-\nu)}{\rho_*(1+\nu)}. \tag{88}$$

Formula (86) is valid for the lattice whose Poisson’s ratio is $\nu = 1/4$. In the case of momentless theory from Eqs. (86), (88) we have

$$c_r^2 = \frac{3K_{ad}}{5\rho_*}. \tag{89}$$

Using Eqs. (84), (85), (89) and expression for K_a (81), we obtain the following formulas for J_a and β:

$$J_a = \frac{5T_a^*}{3\rho_* c_v K_{ad}}, \qquad \beta = \frac{3c_v K_{ad}}{5\lambda}. \tag{90}$$

Version 2 According to the data of [9] (p. 215), the phonon velocity is defined by the group velocity of the longitudinal acoustic waves in a lattice:

$$c_r = c_a \qquad \Rightarrow \qquad c_r^2 = \frac{K_{ad} + 4G/3}{\rho_*}. \tag{91}$$

From Eqs. (81), (84), (85), (91) we find the following expressions for J_a and β:

$$J_a = \frac{T_a^*}{\rho_* c_v (K_{ad} + 4G/3)}, \qquad \beta = \frac{c_v (K_{ad} + 4G/3)}{\lambda}. \tag{92}$$

We emphasize the fact that expressions (81) for parameters K_a and Υ_a and expression (84) for βJ_a are always valid, while expressions (90), (92) for parameters J_a and β are valid only in the case of $B_a = 0$.

A comparison of the equations describing dynamics of the one-rotor gyrostat continuum with the classical equations of the coupled problem of thermoelasticity has shown that in the case of $B_a = 0$ these equations are only distinguished by the existence of inertial terms in the rotor dynamics equation and the absence of inertial terms in the classical heat conduction equation. Thus, in the case of $B_a = 0$, the proposed equations can be treated as the hyperbolic type equations of thermoelasticity.

4.2 Model of internal damping

There exist different macroscopic and microscopic models of internal damping. At present, however, viscoelasticity is not a well-developed science for the treatment of thermodynamical and dissipative phenomena. The point of view that internal damping is concerned with thermal effects is widespread, see e. g. [10]. When an acoustic wave propagates, the distribution of phonons is in a local thermodynamical equilibrium, i. e. the temperature changes adiabatically. Consequently, regions separated by the half-wavelength distance from one another have different temperatures and the irreversible heat flow between these regions arises as a result of heat conduction phenomena. This process causes transfer of the energy of the mechanical vibrations into heat energy. Now we do not call in question the idea about interplay of the internal damping and thermal effects. We emphasize that analysis of the experimental values of the volume (acoustic) viscosity of various substances shows that the volume viscosity is the independent substance characteristic which is not related to the heat-conduction coefficient and other thermodynamical parameters. This means that we should not consider the nature of the acoustic viscosity to be directly connected with heat conduction mechanism. Let us emphasize that discussing the internal damping we mean only the volume (acoustic) viscosity. In our opinion, the shear viscosity has an absolutely different nature and it is not discussed here.

Let us consider the energy dissipation caused by the heat conduction phenomena. It is well-known that this energy dissipation takes place only in the case when the process is not isothermal and not adiabatic. Now let us consider the energy dissipation caused by the viscosity. This energy dissipation always takes place including the case of adiabatic process. Proceeding from this fact, we assume that dissipation is caused only by viscosity and the process is adiabatic, i. e. the volume density of entropy is constant:

$$\vartheta_a = \vartheta_a^* = \text{const} \qquad \Rightarrow \qquad \tilde{\vartheta}_a = 0 \qquad \Rightarrow \qquad \tilde{T}_a = \Upsilon_a \, \varepsilon. \tag{93}$$

When the comparison of the equations describing the dynamics of one-rotor gyrostat continuum with the classical equations of thermoelasticity has been carried out we assumed that $B_a = 0$. Now we reject this restriction. We suppose that the terms containing parameter B_a are concerned with the internal damping mechanism. In order to argue in favour of this hypothesis, we consider the heat conduction equation (82). Let us transform this equation by using adiabatic condition (93). As a result, we obtain

$$\Upsilon_a \, \Delta\varepsilon - \rho_* \beta B_a \frac{d\varepsilon}{dt} - \rho_* B_a \frac{d^2\varepsilon}{dt^2} = -\rho_* \nabla \cdot \mathbf{L}_h^a. \tag{94}$$

It is easy to see that Eq. (94) contains a dissipative term. This dissipative term is in no way concerned with the heat conduction phenomena.

In order to clarify the physical meaning of the coefficients in Eq. (94), we put a stop to the discussion of the proposed model and consider the motion of a viscous fluid in which the pressure obeys the Stokes law. The liquid state (in the case of no external mass forces) is described by the following equations:

$$\nabla p = \rho_* \frac{d\mathbf{v}}{dt}, \qquad p = \eta_v \frac{d\varepsilon}{dt}, \tag{95}$$

where η_v is the volume (acoustic) viscosity. From Eqs. (95) we obtain the relation between the flow of matter $\rho_* \mathbf{v}$ and the volume strain gradient

$$\eta_v \nabla \varepsilon = \rho_* \mathbf{v}. \tag{96}$$

By taking the divergence of both sides of Eqs. (96), we obtain the self-diffusion equation which can be generalized by adding the source term $\rho_* \varPsi$ in it:

$$\eta_v \varDelta \varepsilon - \rho_* \frac{d\varepsilon}{dt} = -\rho_* \varPsi. \tag{97}$$

Comparing Eq. (94) with the self-diffusion equation (97), we find these two equations to be equivalent with the only difference that the former contains the inertial term if

$$\frac{\varUpsilon_a}{\beta B_a} = \eta_v, \qquad \frac{1}{\beta B_a} \nabla \cdot \mathbf{L}_h^a = \varPsi. \tag{98}$$

From the first equation of (98) by using the second equation of (81), we get

$$\beta B_a = -\frac{\alpha K_{iz} T_a^*}{\rho_* c_v \eta_v}. \tag{99}$$

As evident from Eq. (99), parameter B_a is negative for finite values of the volume viscosity η_v and is equal to zero when $\eta_v \to \infty$.

In order to clarify the physical meaning of the obtained result we now consider the dissipative term in Eq. (74) for the rotors dynamics

$$\rho_* \mathbf{L}_f^a = -\beta \rho_* (B_a \mathbf{v} + J_a \boldsymbol{\omega}_a). \tag{100}$$

Upon substituting expressions for parameters (84), (99) into Eq. (100) we get

$$\rho_* \mathbf{L}_f^a = \frac{\alpha K_{iz} T_a^*}{c_v \eta_v} \mathbf{v} - \frac{T_a^*}{\lambda} \boldsymbol{\omega}_a. \tag{101}$$

Let us calculate the power of the dissipative moment (101):

$$\rho_* \mathbf{L}_f^a \cdot \boldsymbol{\omega}_a = \frac{\alpha K_{iz} T_a^*}{c_v \eta_v} \mathbf{v} \cdot \boldsymbol{\omega}_a - \frac{T_a^*}{\lambda} \boldsymbol{\omega}_a \cdot \boldsymbol{\omega}_a. \tag{102}$$

The second term in expression (102) is dissipative. When the heat-conduction coefficient decreases then the dissipation increases. The first term in expression (102) determines the process which under the certain conditions can become inverse to dissipative one. In particular, in the isothermal case, the inequality $\mathbf{v} \cdot \boldsymbol{\omega}_a > 0$ is valid and, therefore, the first term in expression (102) determines process of the energy supply from the thermal ether. When the volume viscosity decreases then the energy supply in the body from the thermal ether increases.

Let us transform Eq. (102) by separating the total squares in it:

$$\rho_* \mathbf{L}_f^a \cdot \boldsymbol{\omega}_a = \frac{\lambda \alpha^2 K_{iz}^2 T_a^*}{4 \eta_v^2 c_v^2} \mathbf{v} \cdot \mathbf{v} - \frac{T_a^*}{\lambda} \left(\boldsymbol{\omega}_a - \frac{\lambda \alpha K_{iz}}{2 \eta_v c_v} \mathbf{v} \right)^2. \tag{103}$$

It is easy to see that the second term in expression (103) determines the dissipative process and the first term characterizes the process of the energy supply from the thermal ether. The first term is inversely as the square of the viscosity. Therefore, when the volume viscosity decreases then the supply of energy of the thermal ether into the body increases. The second term defining the dissipative process also depend on the volume viscosity. As a result the energy interchange between the body and the thermal ether depends on the volume viscosity in a complicated manner. Thus, the volume viscosity characterizes the natural ability of a substance to absorb the energy of the thermal ether. Will this ability be realized? It depends on other properties of the substance and external circumstances. The volume viscosity of gases is very small and therefore gases possess a good ability to absorb the energy of the thermal ether. Therefore, the gas particles are in a state of intense motion in spite of the energy dissipation caused by the heat conduction phenomena. The volume viscosity of fluids (even inviscid fluid) is much greater than the volume viscosity of gases. The volume viscosity of solids is so great that it can be considered to approach infinity. In this case, parameter B_a is negligible. Thus, the problem of thermoelasticity is admissible for solids while for fluids and gases it is important to take into account the terms dependent on the volume viscosity.

4.3 Generalized Fourier law

The classical derivation of the heat conduction equation is based on applying the Fourier law which relates heat flow and temperature gradient. In the proposed model, the counterpart of the Fourier law follows from the second equation in (74).

Let us consider the energy balance equation in integral form (46). The term $\int_{(S)} \mathbf{T}_n \cdot \boldsymbol{\omega}\, dS$ on the right-hand side of this equation expresses the power of moment acting on the rotors of the gyrostats in the control volume V produced by the rest of the continuum. Let us introduce the notation

$$h_n = \mathbf{T}_n \cdot \boldsymbol{\omega}. \tag{104}$$

A quantity h_n can be treated as the rate of supply of the rotor interaction energy through the bounding surface S of the control volume V.

By the standard reasoning we now introduce the concept of vector $\mathbf{h}$ of the rotor interaction energy flow. According to the Stokes rule, the rate of the energy supply h_n through the bounding surface is expressed in terms of the energy flow vector $\mathbf{h}$ as

$$h_n = \mathbf{n} \cdot \mathbf{h}. \tag{105}$$

Thus, in our opinion, the vector $\mathbf{h}$ defined by Eqs. (104) and (105) can be treated as the heat flow vector.

Taking into account the definition of the rotors moment stress tensor $\mathbf{M}_n = \mathbf{n} \cdot \mathbf{M}$ and using the fact that the tensor $\mathbf{M}$ is spherical we transform expression (104) as follows:

$$h_n = \mathbf{T}_n \cdot \boldsymbol{\omega} = \mathbf{n} \cdot \mathbf{T} \cdot \boldsymbol{\omega} = \mathbf{n} \cdot T\mathbf{E} \cdot \boldsymbol{\omega} = \mathbf{n} \cdot (T\boldsymbol{\omega}) = \mathbf{n} \cdot (T_a\, \boldsymbol{\omega}_a). \tag{106}$$

Upon comparing Eq. (105) with Eq. (106) in view of an arbitrary choice of a normal $\mathbf{n}$ we arrive at the conclusion:

$$\mathbf{h} = T_a\, \boldsymbol{\omega}_a. \tag{107}$$

Substituting the expression for vector $\boldsymbol{\omega}_a = \mathbf{h}/T_a$ into the second equation of (74) and carrying out some transformation within the framework of the linear model we obtain

$$\beta^{-1} \frac{d}{dt} \left(\mathbf{h} + \frac{T_a^* B_a}{J_a}\, \mathbf{v} \right) + \mathbf{h} + \frac{T_a^* B_a}{J_a}\, \mathbf{v} = \frac{T_a^*}{\rho_* \beta J_a} \big(\nabla T_a + \rho_*\, \mathbf{L}_h \big). \tag{108}$$

Substitution of expressions for parameters (84), (99) into Eq. (108) gives

$$\beta^{-1} \frac{d}{dt} \left(\mathbf{h} + \frac{\lambda\, \alpha K_{iz} T_a^*}{\eta_v c_v}\, \rho_* \mathbf{v} \right) + \mathbf{h} + \frac{\lambda\, \alpha K_{iz} T_a^*}{\eta_v c_v}\, \rho_* \mathbf{v} = \lambda \big(\nabla T_a + \rho_* \mathbf{L}_h \big). \tag{109}$$

Equation (109) is the counterpart and generalization of the Fourier law. Time derivatives in Eq. (109) being neglected we get

$$\mathbf{h} + \frac{\lambda\, \alpha K_{iz} T_a^*}{\eta_v c_v}\, \rho_* \mathbf{v} = \lambda \big(\nabla T_a + \rho_* \mathbf{L}_h \big). \tag{110}$$

We emphasize two important differences between Eq. (110) and the classical Fourier law. First, Eq. (110) contains the term characterizing the external thermal action. Second, Eq. (110) contains the term depending on the flow of matter so that the expression on the left-hand side of Eq. (110) is the generalized heat flow which consists of the true heat flow $\mathbf{h}$ and the thermo-diffusion flux. Thus, Eq. (110) not only expresses the Fourier law but also describes the Soret effect (see e. g. [11], [12]) stating that the temperature gradient causes the flow of matter.

We pay attention to the fact that the generalized heat flow (108) contains not only the heat flow and the flow of matter but also their time derivatives. The similar generalization of the transport equations can be found e.g. in [13]. The presence of the time derivatives of the flows in the transport equations reflects the fact that the body responds to appearance of the temperature gradient not instantly and possesses certain relaxation time due to inertance of the body. Coefficient β^{-1} of the derivative of the generalized heat flow is the relaxation time scale. Taking into account the fact that the generalized heat flow contains the term depending on the flow of matter it is hard to say whether the parameter β^{-1} can be identified with the heat flow relaxation time scale or whether it is a generalized characteristic of the relaxation time scales of the heat flow and the flow of matter.

4.4 Internal damping in the ideal gas

Now, we consider the problem of propagation of the acoustic waves in the ideal gas. Since the acoustic speed is much greater than “the speed of heat propagation” the terms containing derivatives with respect to space coordinates in the heat conduction equation [see the second equation of (74)] can be neglected. In addition we take into account the fact that for macroscopic scale level the inertia terms are negligible. As a result we have

$$\boldsymbol{\omega}_a \approx -\frac{B_a}{J_a}\,\mathbf{v} \quad \Rightarrow \quad \tilde{\vartheta}_a \approx -\frac{B_a}{J_a}\,\varepsilon \quad \Rightarrow \quad \tilde{T}_a \approx \left(\varUpsilon_a - \frac{B_a K_a}{J_a}\right)\varepsilon. \tag{111}$$

The second and the third equations in (111) are obtained by using Eqs. (75), (76). Substituting Eq. (111) into the first equations in (74), (76) and carrying out some reductions we get

$$\left(K_{ad} - \frac{\varUpsilon_a B_a}{J_a}\right)\varDelta\varepsilon = \rho_*\left(1 - \frac{B_a^2}{J_a}\right)\frac{d^2\varepsilon}{dt^2}. \tag{112}$$

The acoustic speed in the ideal gas is well-known to be calculated as

$$c_a^2 = \frac{K_{ad}}{\rho_*} \equiv \frac{T_a^* c_p (c_p - c_v)}{c_v}. \tag{113}$$

At the same time, according to Eq. (112), the acoustic speed in the ideal gas is determined by the formula

$$c_a^2 = \frac{K_{ad} - \varUpsilon_a B_a / J_a}{\rho_*(1 - B_a^2/J_a)}\,, \qquad \varUpsilon_a = -\frac{K_{iz}}{\rho_* c_v} \equiv -\frac{T_a^*(c_p - c_v)}{c_v}\,, \tag{114}$$

where expression for $\varUpsilon_a$ is obtained from Eqs. (81) by using the fact that for the ideal gas $\alpha = 1/T_a^*$. Upon comparing Eq. (113) with Eqs. (114), we conclude that for the ideal gas the parameter B_a is determined as follows:

$$B_a = \frac{\varUpsilon_a}{K_{ad}} \equiv -\frac{1}{\rho_* c_p}. \tag{115}$$

Now we consider a self-diffusion in the ideal gas. A diffusion process being slow, the inertia term in the first equation of (74) can be neglected. Then in the case $\mathbf{f} = 0$, the first equation in (74) takes the form $\varDelta p_e = 0$, where p_e is the elastic pressure. Then in accordance with Eqs. (76) the following relations between the temperature and the volume strain are valid:

$$\varDelta\tilde{T}_a \approx -\frac{K_a K_{iz}}{\varUpsilon_a}\,\varDelta\varepsilon, \qquad \frac{d\tilde{T}_a}{dt} \approx -\frac{K_a K_{iz}}{\varUpsilon_a}\,\frac{d\varepsilon}{dt}. \tag{116}$$

Let us neglect the inertia terms in the heat conduction equation (82) and reduce this equation by using relations (116). As a result, we obtain

$$\varDelta\varepsilon - \left[\frac{\rho_*\beta J_a}{K_a K_{iz}}\left(K_{iz} + \frac{\varUpsilon_a^2}{K_a}\right) - \frac{\rho_*\beta B_a \varUpsilon_a}{K_a K_{iz}}\right]\frac{d\varepsilon}{dt} = \frac{\rho_*\varUpsilon_a}{K_a K_{iz}}\,\nabla\cdot\mathbf{L}_h^a. \tag{117}$$

Next we substitute into Eq. (117) the expressions of the model parameters (81), (84), (99) which are valid for a substance in all aggregative states and also the relations $\alpha = 1/T_a^*$ and $K_{iz} = \rho_* T_a^*(c_p - c_v)$ which are valid for the ideal gas. Now we have

$$\varDelta\varepsilon - \left[\frac{\rho_* c_p}{\lambda} - \frac{\rho_*(c_p - c_v)}{c_v \eta_v}\right]\frac{d\varepsilon}{dt} = -\frac{\rho_*}{T_a^*}\,\nabla\cdot\mathbf{L}_h^a. \tag{118}$$

In the case of the ideal gas, the self-diffusion equation is well-known

$$D_*\varDelta\varepsilon - \frac{d\varepsilon}{dt} = \varPsi, \qquad D_* = \frac{\lambda}{\rho_* c_v}, \tag{119}$$

where D_* is the self-diffusion coefficient obtained theoretically from the molecular-kinetic theory, see e. g. [14], [15], [16]. Comparing the self-diffusion equations (118) and (119) we obtain the following value of the volume viscosity of the ideal gas:

$$\eta_v = \frac{\lambda}{c_v}. \tag{120}$$

According to the molecular-kinetic theory, the volume viscosity of the monoatomic ideal gas is equal to zero. The volume viscosity of the diatomic and polyatomic ideal gas is non-zero but much smaller than the shear viscosity. Formula (120) contradicts these facts. However, we emphasize that it is very difficult to determine the volume viscosity of a gas experimentally since the heat transfer and the mass transfer interact with each other.

Now we determine the parameter β for the ideal gas taking into account Eqs. (115), (120) and using relation (99) for βB_a which is valid for gases, however it was obtained for the viscous fluids. As a result, we obtain

$$\beta = \frac{K_{ad}}{\eta_v} \equiv \frac{c_v K_{ad}}{\lambda}. \tag{121}$$

Next we determine parameter J_a for the ideal gas using expression (121) and relation (84) for βJ_a which is valid for fluids and gases, however it was obtained for solids. As a result, we obtain

$$J_a = \frac{1}{\rho_*^2\, c_p (c_p - c_v)}. \tag{122}$$

Thus, in the case of the ideal gas as well as in the case of the solids all parameters of the model have been obtained. However, unlike for solids the parameters of the ideal gas are determined unambiguously.

4.5 Physical meaning of the relaxation time scale

In the disscussion of the generalized Fourier law the parameter β^{-1} was shown to be a relaxation time scale. However, the physical meaning of the parameter β^{-1} remains not entirely clear. The object of this section is to clarify the meaning of the parameter β^{-1} by applying the reasoning which is not concerned with the heat flow vector and the Fourier law and is based on reduction of the dynamics equation to the standard form for the classical continuum without a microstructure.

Now we consider Eqs. (74) of dynamics of the one-rotor gyrostat continuum. The first equation in (74) differs from the dynamics equation of the classical continuum by the term containing the vector $\boldsymbol{\omega}_a$. Let us reduce system (74) to eliminate the vector $\boldsymbol{\omega}_a$ from the first equation of the system. To this end, we express the derivative of $\boldsymbol{\omega}_a$ from the first equation in (74) and substitute it into the time derivative of the second equation in (74). As a result, we obtain the equation relating the absolute temperature with the quantities characterizing the stress–strain state of the continuum:

$$\frac{d}{dt}\left(\nabla\cdot\boldsymbol{\tau} + \rho_*\mathbf{f} - \rho_*\left[1 - \frac{B_a^2}{J_a}\right]\frac{d\mathbf{v}}{dt}\right) + \beta\left(\nabla\cdot\boldsymbol{\tau} + \rho_*\mathbf{f} - \rho_*\left[1 - \frac{B_a^2}{J_a}\right]\frac{d\mathbf{v}}{dt}\right) = \frac{B_a}{J_a}\,\frac{d}{dt}\left[\nabla\tilde{T}_a + \rho_*\mathbf{L}_h^a\right]. \tag{123}$$

Solving Eq. (123) for the quantity enclosed in brackets we get

$$\nabla\cdot\left[\boldsymbol{\tau} - \frac{B_a}{J_a}\int_0^t \frac{d\tilde{T}_a(\tau)}{d\tau}\, e^{-\beta(t-\tau)} d\tau\,\mathbf{E}\right] + \rho_*\left[\mathbf{f} - \frac{B_a}{J_a}\int_0^t \frac{d\mathbf{L}_h^a(\tau)}{d\tau}\, e^{-\beta(t-\tau)} d\tau\right] = \rho_*\left(1 - \frac{B_a^2}{J_a}\right)\frac{d\mathbf{v}}{dt}. \tag{124}$$

Let us introduce the notations

$$\hat{\boldsymbol{\tau}} = \left(1 - \frac{B_a^2}{J_a}\right)^{-1}\left[\boldsymbol{\tau} - \frac{B_a}{J_a}\int_0^t \frac{d\tilde{T}_a(\tau)}{d\tau}\, e^{-\beta(t-\tau)} d\tau\,\mathbf{E}\right],$$

$$\mathbf{F} = \left(1 - \frac{B_a^2}{J_a}\right)^{-1}\left[\mathbf{f} - \frac{B_a}{J_a}\int_0^t \frac{d\mathbf{L}_h^a(\tau)}{d\tau}\, e^{-\beta(t-\tau)} d\tau\right]. \tag{125}$$

Next we write down Eq. (124) upon using notations (125)

$$\nabla \cdot \hat{\boldsymbol{\tau}} + \rho_* \, \mathbf{F} = \rho_* \, \frac{d\mathbf{v}}{dt} \, , \tag{126}$$

Equation (126) can be considered as the dynamics equation of the classical continuum without a microstructure where $\hat{\boldsymbol{\tau}}$ is the stress tensor and $\mathbf{F}$ is the mass density of the external action. Due to the presence of relaxation term in the stress tensor $\hat{\boldsymbol{\tau}}$ [see Eq. (125)] we can interpret the above theory as the theory of thermoviscoelasticity.

In mathematical models of viscoelastic solids, as a rule, the relaxation terms depend on the strain rate but not on the time rate of temperature change. It is easy to show that under the adiabatic condition (93) expression (125) for $\hat{\boldsymbol{\tau}}$ takes a form which is typical for the model of viscoelastic solid:

$$\tilde{T}_a = \varUpsilon_a \, \varepsilon \quad \Rightarrow \quad \hat{\boldsymbol{\tau}} = \left(1 - \frac{B_a^2}{J_a} \right)^{-1} \left[\boldsymbol{\tau} - \frac{B_a \varUpsilon_a}{J_a} \int_0^t \frac{d\varepsilon(\tau)}{d\tau} \, e^{-\beta(t-\tau)} d\tau \, \mathbf{E} \right] . \tag{127}$$

At first sight Eq. (127) seems to allow one to interpret the parameter β^{-1} as the relaxation time of the viscous stresses. Detailed analysis reveals that it is not true. Indeed, in the case $B_a = 0$ the relaxation terms in Eqs. (125), (127) vanish and the theory of thermoviscoelasticity converts to the theory of thermoelasticity. The parameter β^{-1}, as evident from Eq. (99), is not equal to zero under both presence and absence of the viscous stresses. Therefore Eqs. (125), (127) do not give grounds to interpret parameter β^{-1} as the relaxation time of the viscous stresses.

Thus, β^{-1} can be considered either as the heat flow relaxation time or as a combination of the heat flow relaxation time and the relaxation time of the viscous stresses. In theory of rheological models, as a rule, in order to calculate the reciprocal of resultant relaxation time it is necessary to sum up the reciprocals of all relaxation times. Conforming to this rule, we suppose that β^{-1} being a combination of the relaxation times is calculated as

$$\frac{1}{\beta^{-1}} = \frac{1}{\tau_h} + \frac{1}{\tau_v}, \tag{128}$$

where τ_h is the heat flow relaxation time and τ_v is the relaxation time of the viscous stresses.

4.6 Parameters of the proposed model for solids, fluids and gases

We have obtained formulas (81), (84), (99) relating the parameters of proposed model with the known thermodynamical constants, which are valid for solids, fluids and gases. In the cases of the solid and the ideal gas we succeeded in determining all parameters of the model. For solids $B_a = 0$ and quantities J_a and β are calculated by either (90) or (92) depending on the method of determination of velocity of phonons (heat-waves) propagation. For the ideal gas quantities B_a, β and J_a are calculated correspondingly by formulas (115), (121), (122). In the case of viscous fluid we do not have enough equations to determine all parameters of the model. This problem can be solved by the assumption that the formulas relating the parameters of proposed model with known thermodynamical constants must be the same for substances in all aggregative states. For the solids two versions of determination of the parameters having been proposed. Therefore, two corresponding versions should be considered in general.

Version 1 Let us assume that the quantities J_a and β for solids are calculated by formulas (90). In order to bring to conformity the formula for parameter β in the case of solids (90) with the formula for parameter β in the case of the ideal gas (121) we rewrite the latter in the equivalent form

$$\beta = \frac{3 c_v K_{ad}}{5\lambda} + \frac{2 K_{ad}}{5 \eta_v} . \tag{129}$$

It is easy to see that when the volume viscosity η_v tends to infinity then expression (129) coincides with formula (90). Thus, formula (129) is valid for both the ideal gas and solids. Let us suppose that this formula is valid also for the viscous fluid. Then formula (129) turns out to be valid for a substance in all aggregative states.

Comparing Eq. (129) with Eq. (128) we see that it is reasonable to define the relaxation times of the heat flow and viscous stresses as

$$\tau_h = \frac{5\lambda}{3c_v K_{ad}}, \qquad \tau_v = \frac{5\eta_v}{2K_{ad}}. \tag{130}$$

We notice that Eqs. (130) are interesting only from the theoretical point of view. To carry out practical calculations we need only formula (129).

Now using Eqs. (84), (99), (129), we obtain the following expressions for parameters B_a and J_a:

$$B_a = -\frac{5\alpha T_a^* \lambda}{\rho_* c_p (2\lambda + 3c_v\eta_v)}, \qquad J_a = \frac{5T_a^* \eta_v}{\rho_* K_{ad}(2\lambda + 3c_v\eta_v)}. \tag{131}$$

It is not difficult to show that when the volume viscosity η_v tends to infinity, B_a vanishes and J_a coincides with the quantity determined by Eq. (90). In addition, we observe that when $\eta_v = \lambda/c_v$ then parameters B_a and J_a coincide with their values determined by Eqs. (115), (122). Thus we establish that expressions (131) include the formulas obtained above for solids and the ideal gas as the special cases.

Version 2 Let us assume that the quantities J_a and β for solids are calculated by formulas (92). It is easy to see that in the case of the ideal gas Eqs. (121), (122) for J_a and β are the special cases of Eqs. (92). Now we suppose that formulas (92) are valid also for the viscous fluid. Then using expression (92) for β and expression (99) for βB_a we obtain the following formula for B_a:

$$B_a = -\frac{\lambda\, \alpha K_{iz} T_a^*}{\eta_v\, \rho_* c_v^2 (K_{ad} + 4G/3)}. \tag{132}$$

It is not difficult to see that Eq. (132) vanishes in the case of solids when $\eta_v \to \infty$ and it turns into expression (115) in the case of the ideal gas when $\eta_v = \lambda/c_v$, $\alpha = 1/T_a^*$. Thus we establish that formulas (92) for J_a and β and formula (132) for B_a are valid for substances in all aggregative states.

From the formal point of view, both variants of determination of the parameters B_a, J_a and β^{-1} are equally admissible since all these formulas were deduced with the same degrees of strictness. The physical meanings of the constants obtained above are essentially different.

4.7 Thermoelastic and thermodynamic forces

The foregoing considerations reveal that upon introducing viscous stresses the first equation in (74) can be reduced to the form which is standard for the classical continuum without microstructure. Deficiency of this approach is that it can be realized only in the case of the linear theory. Now we consider an alternative method of reduction of the dynamic equation to the standard form. The advantage of this method is in the possibility to apply it in the case of the nonlinear theory.

It is well-known that an arbitrary vector can be represented in terms of the scalar and vector Helmholtz potentials. We use this representation for the dynamic term containing the vector $\boldsymbol{\omega}_a$ on the right-hand side of the first equation of system (74):

$$-\rho_* B_a \frac{d\boldsymbol{\omega}_a}{dt} = \nabla p + \nabla \times \mathbf{t}, \qquad \nabla \cdot \mathbf{t} = 0. \tag{133}$$

Here p is the scalar potential, $\mathbf{t}$ is the vector potential. Let us rewrite the first equation of (74) by using notation (133):

$$\nabla \cdot \Big(\boldsymbol{\tau} + \boldsymbol{\tau}_*\Big) + \rho_* \mathbf{f} = \rho_* \frac{d\mathbf{v}}{dt}, \qquad \boldsymbol{\tau}_* = p\,\mathbf{E} + \mathbf{E} \times \mathbf{t}. \tag{134}$$

Equation (134) is the dynamic equation of the continuum consisting of the ordinary classical particles (point masses) whose interaction is characterized by the stress tensor $\tilde{\boldsymbol{\tau}} = \boldsymbol{\tau} + \boldsymbol{\tau}_*$. At first sight, the fact that the momentum balance equation (134) contains the antisymmetric part of the stress tensor $\mathbf{E} \times \mathbf{t}$ seems to be strange. Indeed, it was a priori assumed that in the angular momentum balance equation the moment stress tensor and external distributed moment are equal to zero [see Eqs. (72)] and also the inertia tensor $\mathbf{I}_*^{(0)}$ is equal to zero. Consequently the vector invariant of the stress tensor must vanish. This contradiction can be easy eliminated by introducing the moment stress tensor $\boldsymbol{\mu}_*$ such that

$$\boldsymbol{\mu}_* = \mathbf{E} \times \boldsymbol{\mu}, \qquad \nabla \times \boldsymbol{\mu} = 2\mathbf{t}. \tag{135}$$

Taking into account the moment stress tensor (135), we write the angular momentum balance equation as

$$\nabla \cdot \boldsymbol{\mu}_* + \left(\boldsymbol{\tau}_*\right)_\times = 0. \tag{136}$$

This equation is an identity but from the formal point of view it eliminates all contradictions concerned with the presence of the antisymmetric part of the stress tensor $\tilde{\boldsymbol{\tau}}$.

As follows from the definition of stress tensor $\boldsymbol{\tau}_*$ [see Eqs. (133), (134)] in the static problems, this stress tensor vanishes. Therefore, in contrast to the tensor of thermoelastic stresses $\boldsymbol{\tau}$, we shall call tensor $\boldsymbol{\tau}_*$ the tensor of thermodynamic stresses. Quantities p and $\mathbf{t}$ will be called correspondingly the thermodynamic pressure and the thermodynamic stress vector. It is not difficult to see that thermodynamic stresses exist only when $B_a \neq 0$, i. e. in the case of gases and fluids.

From Eq. (133) upon using Eqs. (75), (76) we obtain the following differential equation for determining the thermodynamic pressure p:

$$\mathit{\Delta} p = \frac{\rho_* B_a \Upsilon_a}{K_a} \frac{d^2\varepsilon}{dt^2} - \frac{\rho_* B_a}{K_a} \frac{d^2\tilde{T}_a}{dt^2}. \tag{137}$$

From Eq. (133) by using the second equation of (74), we get the following differential equation for determining the thermodynamic stress vector $\mathbf{t}$:

$$\beta^{-1}\frac{d\mathit{\Delta}\mathbf{t}}{dt} + \mathit{\Delta}\mathbf{t} = -\frac{\rho_* B_a^2}{J_a}\left(\beta^{-1}\frac{d^2\nabla\times\mathbf{v}}{dt^2} + \frac{d\nabla\times\mathbf{v}}{dt}\right) + \frac{\rho_* B_a}{\beta J_a}\frac{d\nabla\times\mathbf{L}_h^a}{dt}\,. \tag{138}$$

Now we write down all basic equations of the coupled problem of the theory of thermoviscoelasticity for the classical momentless continuum:

$$\begin{gathered}
\nabla \cdot \tilde{\boldsymbol{\tau}} + \rho_*\mathbf{f} = \rho_*\frac{d\mathbf{v}}{dt}, \qquad \mathbf{v} = \frac{d\mathbf{u}}{dt}, \qquad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \qquad \varepsilon = \operatorname{tr}\boldsymbol{\varepsilon}, \\
\tilde{\boldsymbol{\tau}} = \left[\Big(K_{iz} - \frac{2}{3}\,G\Big)\varepsilon - \alpha K_{iz}\,\tilde{T}_a + p\right]\mathbf{E} + G\,\boldsymbol{\varepsilon} + \mathbf{E}\times\mathbf{t}, \qquad \mathit{\Delta} p = \frac{\alpha K_{iz}}{\beta\eta_v}\left[\rho_*\frac{d^2\tilde{T}_a}{dt^2} + \frac{\alpha K_{iz} T_a^*}{c_v}\,\frac{d^2\varepsilon}{dt^2}\right], \\
\mathit{\Delta}\mathbf{t} + \frac{1}{\beta}\frac{d\mathit{\Delta}\mathbf{t}}{dt} = -\frac{\lambda\,\alpha K_{iz}}{\beta c_v \eta_v}\left(\frac{\alpha K_{iz} T_a^*}{c_v\eta_v}\left[\frac{d\nabla\times\mathbf{v}}{dt} + \frac{1}{\beta}\frac{d^2\nabla\times\mathbf{v}}{dt^2}\right] + \rho_*\,\frac{d\nabla\times\mathbf{L}_h^a}{dt}\right), \\
\mathit{\Delta}\tilde{T}_a - \frac{\rho_* c_v}{\lambda}\left[\frac{d\tilde{T}_a}{dt} + \frac{1}{\beta}\frac{d^2\tilde{T}_a}{dt^2}\right] = \alpha K_{iz} T_a^*\left(\frac{1}{\lambda} - \frac{1}{c_v\eta_v}\right)\left[\frac{d\varepsilon}{dt} + \frac{1}{\beta}\frac{d^2\varepsilon}{dt^2}\right] - \rho_*\nabla\cdot\mathbf{L}_h^a,
\end{gathered} \tag{139}$$

The system (139) has been obtained by using Eqs. (75), (79), (82), (134), (137), (138) and expressions (81), (84), (99) for the parameters of the proposed model. It is easy to see that the thermodynamic stresses vanish when $\eta_v \to \infty$ and the problem of thermoviscoelasticity turns into the hyperbolic type thermoelasticity problem.

4.8 Coupled problem of thermoviscoelasticity

Above we presented the equations of the coupled problem of the momentless theory of thermoviscoelasticity written down in the form corresponding to the classical continuum. These equations are interesting from the theoretical point of view. For practical purposes, the original equations (74)–(76) are more convenient. Now we rewrite these equations upon using expressions for the parameters (81), (84), (99):

$$\begin{gathered}
\nabla \cdot \boldsymbol{\tau} + \rho_*\mathbf{f} = \rho_*\frac{d\mathbf{v}}{dt} - \beta^{-1}\frac{\alpha K_{iz} T_a^*}{c_v\eta_v}\,\frac{d\boldsymbol{\omega}_a}{dt}, \qquad \mathbf{v} = \frac{d\mathbf{u}}{dt}, \qquad \boldsymbol{\omega}_a = \frac{d\boldsymbol{\theta}_a}{dt}, \qquad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \\
\nabla\tilde{T}_a + \frac{\alpha K_{iz} T_a^*}{c_v\eta_v}\left(\mathbf{v} + \beta^{-1}\frac{d\mathbf{v}}{dt}\right) - \frac{T_a^*}{\lambda}\left(\boldsymbol{\omega}_a + \beta^{-1}\frac{d\boldsymbol{\omega}_a}{dt}\right) = -\rho_*\mathbf{L}_h^a, \\
\boldsymbol{\tau} = \left[\Big(K_{iz} - \frac{2}{3}\,G\Big)\varepsilon - \alpha K_{iz}\,\tilde{T}_a\right]\mathbf{E} + 2G\,\boldsymbol{\varepsilon}, \qquad \tilde{\vartheta}_a = \frac{\rho_* c_v}{T_a^*}\,\tilde{T}_a + \alpha K_{iz}\,\varepsilon, \qquad \tilde{\vartheta}_a = \nabla\cdot\boldsymbol{\theta}_a, \qquad \varepsilon = \operatorname{tr}\boldsymbol{\varepsilon}.
\end{gathered} \tag{140}$$

It is well-known that the classical problem of thermoelasticity can be split into two independent problems. One set of equations describes the volume thermoelastic vibrations. Another set of equations describes the shear vibrations. The proposed statement of the problem of thermoviscoelasticity possesses the same property.

1. The volume thermoviscoelastic vibrations. Now we take the divergence of both sides of the first and fourth equations of (140) and reduce obtained equations by applying the remaining equations of this system. As a result we get the closed set of equations for unknown functions ε and $\tilde{T}_a$. This equations set describing the volume thermoviscoelastic vibrations takes the form

$$\begin{gathered}\left(K_{iz}+\frac{4}{3}G\right)\mathit{\Delta}\varepsilon-\alpha K_{iz}\,\mathit{\Delta}\tilde{T}_a+\rho_*\nabla\cdot\mathbf{f}=\left(\rho_*-\frac{\alpha^2K_{iz}^2T_a^*}{\beta c_v\eta_v}\right)\frac{d^2\varepsilon}{dt^2}-\frac{\rho_*\alpha K_{iz}}{\beta\eta_v}\frac{d^2\tilde{T}_a}{dt^2},\\ \mathit{\Delta}\tilde{T}_a-\frac{\rho_*c_v}{\lambda}\left[\frac{d\tilde{T}_a}{dt}+\frac{1}{\beta}\frac{d^2\tilde{T}_a}{dt^2}\right]=\alpha K_{iz}T_a^*\left(\frac{1}{\lambda}-\frac{1}{c_v\eta_v}\right)\left[\frac{d\varepsilon}{dt}+\frac{1}{\beta}\frac{d^2\varepsilon}{dt^2}\right]-\rho_*\nabla\cdot\mathbf{L}_h^a.\end{gathered}\tag{141}$$

Let us consider the classical equations of the coupled problem of thermoelasticity (77). Taking the divergence of both sides of the dynamic equation and after some manipulations we obtain

$$\left(K_{iz}+\frac{4}{3}G\right)\mathit{\Delta}\varepsilon-\alpha K_{iz}\,\mathit{\Delta}\tilde{T}_a+\rho_*\nabla\cdot\mathbf{f}=\rho_*\frac{d^2\varepsilon}{dt^2},\qquad \mathit{\Delta}\tilde{T}_a-\frac{\rho_*c_v}{\lambda}\frac{d\tilde{T}_a}{dt}=\frac{\alpha K_{iz}T_a^*}{\lambda}\frac{d\varepsilon}{dt}-\frac{\rho_*q}{\lambda}.\tag{142}$$

Comparison of Eq. (141) describing the volume thermoviscoelastic vibrations within the framework of the proposed model with the classical equations (142) of the volume thermoelastic vibrations reveals the following facts. The dynamic equation in (141) contains the second time derivative of the temperature. Such a term is absent in the dynamic equation in (142). The heat conduction equation in (141) contains the second time derivative of the temperature and the volume strain. Such terms are absent in the heat conduction equation in (142). If $\beta^{-1}=0$, then these terms vanish. In this case, the proposed formulation of the problem differs from the classical one only by the coefficients of the volume strain in the heat conduction equations. In Eq. (141) this coefficient depends on the volume viscosity whereas in Eq. (142) it does not depend on this parameter.

2. The shear vibrations. Now we take the curl operator of both sides of the dynamics equation of (140) and eliminate $\boldsymbol{\omega}_a$ with the help of the heat conduction equation of (140). Integrating the obtained equation over time yields the following equation for the shear vibrations:

$$G\mathit{\Delta}\nabla\times\mathbf{u}+\rho_*\nabla\times\mathbf{f}=\left(\rho_*-\frac{\lambda\,\alpha^2K_{iz}^2T_a^*}{\beta c_v^2\eta_v^2}\right)\frac{d^2(\nabla\times\mathbf{u})}{dt^2}-\frac{\rho_*\lambda\,\alpha K_{iz}}{c_v\eta_v}\int_0^t\frac{d\nabla\times\mathbf{L}_h^a}{d\tau}\,e^{-\beta(t-\tau)}d\tau.\tag{143}$$

The classical equation of the shear vibrations is obtained by taking the curl operator of both sides of the first equation of (77):

$$G\mathit{\Delta}\nabla\times\mathbf{u}+\rho_*\nabla\times\mathbf{f}=\rho_*\frac{d^2(\nabla\times\mathbf{u})}{dt^2}.\tag{144}$$

It is not difficult to see that in the case of $\mathbf{L}_h^a=0$ Eqs. (143) and (144) coincide within the coefficients of the inertia terms. If $\beta^{-1}=0$ then Eqs. (143) and (144) coincide identically.

4.9 Calculation of the model parameters for some substances

Two variants of determination of the parameters of the model were proposed above. The purpose of this section is to calculate the model parameters in accordance with both variants for different substances (gases, fluids and solids) and to carry out comparison of the results.

Version 1 (based on the assertion that the propagation speed of heat-waves is $\sqrt{3}$ times smaller than the acoustic speed):

$$\begin{gathered}B_a=-\frac{5\alpha T_a^*\lambda}{\rho_*c_p(2\lambda+3c_v\eta_v)},\qquad J_a=\frac{5T_a^*\eta_v}{\rho_*K_{ad}(2\lambda+3c_v\eta_v)},\qquad K_a=\frac{T_a^*}{\rho_*c_v},\qquad \varUpsilon_a=-\frac{\alpha K_{iz}T_a^*}{\rho_*c_v},\\ \beta^{-1}=\frac{\tau_h\tau_v}{\tau_h+\tau_v},\qquad \tau_h=\frac{5\lambda}{3c_vK_{ad}},\qquad \tau_v=\frac{5\eta_v}{2K_{ad}}.\end{gathered}\tag{145}$$

Table 1 Parameters of proposed model (version 1)

Substance	$B_a \left(\frac{\mathrm{K \cdot m^2}}{\mathrm{N}}\right)$	$J_a \left(\frac{\mathrm{K^2 \cdot m^4}}{\mathrm{N^2}}\right)$	$K_a \left(\frac{\mathrm{K^2 \cdot m^2}}{\mathrm{N}}\right)$	Υ_a (K)
Ethyl alcohol (g)	$-2,26 \cdot 10^{-2}$	$3,70 \cdot 10^{-3}$	$7,16$	$-43,7$
Ethyl alcohol (f)	$-1,23 \cdot 10^{-8}$	$2,73 \cdot 10^{-13}$	$1,83 \cdot 10^{-4}$	$-1,82 \cdot 10^{2}$
Carbon bisulfide (f)	$-1,16 \cdot 10^{-7}$	$2,14 \cdot 10^{-13}$	$2,51 \cdot 10^{-4}$	$-3,28 \cdot 10^{2}$
Benzene* (f)	$-5,46 \cdot 10^{-8}$	$2,59 \cdot 10^{-13}$	$2,60 \cdot 10^{-4}$	$-3,10 \cdot 10^{2}$
Toluene* (f)	$-5,57 \cdot 10^{-8}$	$2,63 \cdot 10^{-13}$	$2,62 \cdot 10^{-4}$	$-3,13 \cdot 10^{2}$
Acetone* (f)	$-1,12 \cdot 10^{-7}$	$2,54 \cdot 10^{-13}$	$2,49 \cdot 10^{-4}$	$-3,25 \cdot 10^{2}$
Chloroform* (f)	$-9,58 \cdot 10^{-8}$	$3,50 \cdot 10^{-13}$	$2,65 \cdot 10^{-4}$	$-2,69 \cdot 10^{2}$
Tetrachloromethane* (f)	$-4,33 \cdot 10^{-7}$	$1,97 \cdot 10^{-13}$	$1,82 \cdot 10^{-3}$	$-3,80 \cdot 10^{3}$
Silver (s)	0	$1,87 \cdot 10^{-15}$	$1,24 \cdot 10^{-4}$	$-7,57 \cdot 10^{2}$
Mercury (f)	$-6,97 \cdot 10^{-8}$	$1,17 \cdot 10^{-16}$	$1,80 \cdot 10^{-4}$	$-8,44 \cdot 10^{2}$
Lead (s)	0	$1,10 \cdot 10^{-14}$	$2,07 \cdot 10^{-4}$	$-5,28 \cdot 10^{2}$
Lead (f)	$-7,91 \cdot 10^{-8}$	$2,35 \cdot 10^{-16}$	$4,01 \cdot 10^{-4}$	$-1,05 \cdot 10^{3}$

Table 2 Relaxation time scale (version 1)

Substance	τ_h (s)	τ_v (s)	β^{-1} (s)	τ_* (s)
Ethyl alcohol (g)	$1,03 \cdot 10^{-8}$	$1,55 \cdot 10^{-8}$	$6,19 \cdot 10^{-9}$	$\sim 10^{-10}$
Ethyl alcohol (f)	$1,38 \cdot 10^{-13}$	$4,58 \cdot 10^{-12}$	$1,34 \cdot 10^{-13}$	—
Carbon bisulfide (f)	$2,45 \cdot 10^{-13}$	$8,89 \cdot 10^{-13}$	$1,92 \cdot 10^{-13}$	$1,43 \cdot 10^{-13}$
Benzene* (f)	$1,29 \cdot 10^{-13}$	$1,09 \cdot 10^{-12}$	$1,15 \cdot 10^{-13}$	$1,22 \cdot 10^{-13}$
Toluene* (f)	$1,17 \cdot 10^{-13}$	$9,96 \cdot 10^{-13}$	$1,05 \cdot 10^{-13}$	$1,63 \cdot 10^{-13}$
Acetone* (f)	$1,35 \cdot 10^{-13}$	$5,98 \cdot 10^{-13}$	$1,10 \cdot 10^{-13}$	$1,36 \cdot 10^{-13}$
Chloroform* (f)	$2,43 \cdot 10^{-13}$	$1,36 \cdot 10^{-12}$	$2,06 \cdot 10^{-13}$	$1,54 \cdot 10^{-13}$
Tetrachloromethane* (f)	$1,88 \cdot 10^{-13}$	$2,68 \cdot 10^{-13}$	$1,10 \cdot 10^{-13}$	$2,15 \cdot 10^{-13}$
Silver (s)	$2,87 \cdot 10^{-11}$	∞	$2,87 \cdot 10^{-11}$	$\sim 10^{-11}$
Mercury (f)	$1,35 \cdot 10^{-11}$	$1,59 \cdot 10^{-13}$	$1,58 \cdot 10^{-13}$	—
Lead (s)	$1,48 \cdot 10^{-11}$	∞	$1,48 \cdot 10^{-11}$	$\sim 10^{-11}$
Lead (f)	$5,54 \cdot 10^{-12}$	$6,45 \cdot 10^{-14}$	$6,37 \cdot 10^{-14}$	—

Table 3 Parameters of proposed model (version 2)

Substance	$B_a \left(\frac{\mathrm{K \cdot m^2}}{\mathrm{N}}\right)$	$J_a \left(\frac{\mathrm{K^2 \cdot m^4}}{\mathrm{N^2}}\right)$	β^{-1} (s)	τ_* (s)
Ethyl alcohol (g)	$-2,26 \cdot 10^{-2}$	$3,70 \cdot 10^{-3}$	$6,19 \cdot 10^{-9}$	$\sim 10^{-10}$
Ethyl alcohol (f)	$-7,61 \cdot 10^{-9}$	$1,69 \cdot 10^{-13}$	$8,30 \cdot 10^{-14}$	—
Carbon bisulfide (f)	$-8,86 \cdot 10^{-8}$	$1,64 \cdot 10^{-13}$	$1,47 \cdot 10^{-13}$	$1,43 \cdot 10^{-13}$
Benzene* (f)	$-3,67 \cdot 10^{-8}$	$1,74 \cdot 10^{-13}$	$7,71 \cdot 10^{-14}$	$1,22 \cdot 10^{-13}$
Toluene* (f)	$-3,74 \cdot 10^{-8}$	$1,77 \cdot 10^{-13}$	$7,05 \cdot 10^{-14}$	$1,63 \cdot 10^{-13}$
Acetone* (f)	$-8,23 \cdot 10^{-8}$	$1,86 \cdot 10^{-13}$	$8,09 \cdot 10^{-14}$	$1,36 \cdot 10^{-13}$
Chloroform* (f)	$-6,78 \cdot 10^{-8}$	$2,48 \cdot 10^{-13}$	$1,46 \cdot 10^{-13}$	$1,54 \cdot 10^{-13}$
Tetrachloromethane* (f)	$-4,43 \cdot 10^{-7}$	$2,01 \cdot 10^{-13}$	$1,13 \cdot 10^{-13}$	$2,15 \cdot 10^{-13}$
Silver (s)	0	$8,21 \cdot 10^{-16}$	$1,26 \cdot 10^{-11}$	$\sim 10^{-11}$
Mercury (f)	$-3,59 \cdot 10^{-6}$	$6,03 \cdot 10^{-15}$	$8,12 \cdot 10^{-12}$	—
Lead (s)	0	$5,20 \cdot 10^{-15}$	$7,00 \cdot 10^{-12}$	$\sim 10^{-11}$
Lead (f)	$-4,13 \cdot 10^{-6}$	$1,23 \cdot 10^{-14}$	$3,33 \cdot 10^{-12}$	—

Version 2 (based on the assertion that the propagation speed of heat-waves is equal to the acoustic speed):

$$\begin{gathered} B_a = -\frac{\lambda\, \alpha K_{iz} T_a^*}{\eta_v\, \rho_* c_v^2 (K_{ad} + 4G/3)}, \qquad J_a = \frac{T_a^*}{\rho_* c_v (K_{ad} + 4G/3)}, \\ K_a = \frac{T_a^*}{\rho_* c_v}, \qquad \Upsilon_a = -\frac{\alpha K_{iz} T_a^*}{\rho_* c_v}, \qquad \beta^{-1} = \frac{\lambda}{c_v (K_{ad} + 4G/3)}. \end{gathered} \tag{146}$$

The parameter values for some substances possessing different thermal and physical properties are given in Tables 1, 2, 3. The substances list involves gases, fluids and solids. To calculate the parameters of the proposed model we used the values of the mechanical and thermodynamical constants taken from [16], [17], [18], [19], [20], [21], [22], [23]. Since there are no values of the volume viscosity in the listed handbooks we also used [24] for determination of this quantity and we took ratios of the volume viscosity to the shear viscosity.

For five fluids marked by asterisks in the Tables 1, 2, 3, we failed to find data for determination of the volume viscosity. Therefore we assumed that values of the volume viscosity of these fluids are approximately equal to values of their shear viscosity. According to the data of [24], for most of substances the volume viscosity is indeed approximately equal to the shear viscosity.

Now we proceed to the analysis of numerical results. The values of the parameters B_a, J_a, K_a and Υ_a are given in Tables 1, 3. The values of B_a and J_a calculated by formulas (145) are presented in Table 1 and the values of B_a and J_a calculated by using formulas (146) are offered in Table 3. The formulas for calculation of K_a and Υ_a are the same for both versions and therefore values of these parameters are given only in Table 1. It is obvious from the presented data that the values of parameter Υ_a are of the same order of magnitude for all considered substances. The values of parameter K_a have the same order of magnitude for fluids and solids while values of K_a for gases are three orders greater than for fluids and solids. Depending on the substance the magnitude of B_a changes within ten orders and the magnitude of J_a changes within thirteen orders. The values of B_a and J_a calculated by formulas (145) and (146) coincide exactly in the case of gases and differ less than by a factor two for all fluids with the exception of the liquid metals. In the case of the liquid metals, the values of B_a and J_a calculated by formulas (145) and (146) differ more than ten times. For solids, the values of J_a calculated by formulas (145) and (146) differ by several times.

Table 2 contains the values of β^{-1} and its components τ_h and τ_v calculated by formulas (145). Table 2 contains also the values of heat flow relaxation time τ_* obtained by other authors. Estimation of τ_* for gases is taken from [25], estimation of τ_* for metals in the solid state is taken from [26], [27]. The values of τ_* for fluids are taken from [13] where they are presented with reference to [28], [29]. It is clear from the data of Table 2 that the values of τ_h and τ_v are approximately equal in the case of all gases and fluids possessing not very great viscosity and heat conductivity. For liquid metals, the values of τ_h are two orders greater than the values of τ_v. Table 3 contains the values of β^{-1} calculated by formulas (146). For all substances with the exception of liquid metals the values of β^{-1} calculated by formulas (145) and (146) are of the same order of magnitude. The agreement between the obtained results and data of other authors (see last column of Tables 2, 3) can be considered as fair since the order of values coincides almost for all substances both in the case of calculating by formulas (145) and in the case of calculating by formula (146).

We emphasize that only calculation of the relaxation time scale β^{-1} is significant for practical purposes since β^{-1} is the only parameter in Eqs. (140) which remains indeterminate. However, from the theoretical point of view it is important to determine parameters B_a and J_a. In order that mechanical interpretation of the equations obtained above would be reasonable, the kinetic energy of proposed model must be a positive definite quadratic form. The mass density of the kinetic energy can be represented as follows:

$$K = \frac{1}{2}\,\mathbf{v}\cdot\mathbf{v} + B_a\,\mathbf{v}\cdot\boldsymbol{\omega}_a + \frac{1}{2}\,J_a\,\boldsymbol{\omega}_a\cdot\boldsymbol{\omega}_a = \frac{1}{2}\,(\mathbf{v} + B_a\,\boldsymbol{\omega}_a)^2 + \frac{1}{2}\left(J_a - B_a^2\right)\boldsymbol{\omega}_a\cdot\boldsymbol{\omega}_a. \tag{147}$$

It is clear from Eq. (147) that parameter B_a can be positive or negative but B_a and J_a should obey the condition

$$J_a - B_a^2 \geq 0. \tag{148}$$

Examination of data given in Tables 1, 3 reveals that the condition (148) holds for all substances with the exception of liquid metals. This implies that either another mechanical model should be applied to describe the thermoviscoelastic behavior of metals or the experimental values of the volume viscosity of the liquid metals should be determined more precisely. An argument is that the attenuation coefficient in vibration tests is obtained from the direct experiments and then the volume viscosity is calculated by using the theoretical formulas. These formulas are obtained from one of the models of the internal damping. The proposed model of the internal damping differs from all known models. Therefore the values of the volume viscosity appropriate for the proposed model may differ from its values presented in the handbooks.

5 Conclusion

A model of two-component continuum is suggested for the account of thermomechanical processes. A mathematical description of this model is developed in the framework of physically and geometrically linear theory.

In the future, we intend to carry out further development of the theory in two directions. The first one is concerned with consideration of nonlinear effects in the context of the same mechanical model. This is necessary for describing the behavior of substances in the states near the phase changes and heat-conduction processes under the circumstances of quickly varying and superhigh temperatures. The second direction deals with modification of the mechanical model by taking into account the additional degrees of freedom for introducing the chemical potential and a number of additional physical characteristics of the medium. This is necessary to describe the phase changes and chemical reactions and also to account for the interaction of the substance with the electromagnetic field and to describe the thermoelectric and thermomagnetic effects.

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