The Use of Moment Theory to Describe the Piezoelectric Effect in Polar and Non-Polar Materials

Elena A. Ivanova and Yaroslav E. Kolpakov

Abstract It is well known that the properties of polar and non-polar piezoelectric materials are different. For example, the polar piezoelectric materials (ferroelectrics) possess spontaneous polarization, while for non-polar materials such behavior cannot be observed. However, in the classical linear theory of piezoelectricity there is no qualitative difference between polar and non-polar materials. According to the classical theory the only difference between them consists in the fact that the piezoelectric moduli of polar materials are much greater than those of non-polar materials. The objective of our investigation is to describe piezoelectricity taking into account the qualitative peculiarities of polar and non-polar materials. Starting from the consideration of microstructure of piezoelectric materials we propose two theories of piezoelectricity based on the equations of micro-polar continuum. The first theory describes the piezoelectric effect in polar materials. This theory is based on the model of complex particle possessing a non-zero dipole moment and having seven degrees of freedom. The second theory describes the piezoelectric effect in non-polar materials. This theory is based on the model of an unit cell which has a non-zero quadrupole moment and zero dipole moment. Under certain simplifying assumptions both theories can be reduced to the classical theory of piezoelectricity.

E. A. Ivanova (✉)
St. Petersburg State Polytechnical University (SPbSPU), Politekhnicheskaya 29, St. Petersburg, Russia 195251
e-mail: elenaivanova239@post.ru

E. A. Ivanova
Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, Bolshoy pr. V. O., 61, St. Petersburg, Russia 199178

Y. E. Kolpakov
Morion Inc., Kima ave. 13a, St. Petersburg, Russia 199155
e-mail: jaroslav@morion.com.ru

H. Altenbach et al. (eds.), *Generalized Continua as Models for Materials*,
Advanced Structured Materials 22, DOI: 10.1007/978-3-642-36394-8_9,

1 Introduction

There exist many crystals having piezoelectric properties. The piezoelectric properties reveal themselves as a result of the influence of electromagnetic fields on matter. Piezoelectric materials can be divided into two classes: polar and non-polar piezoelectrics. For example, $LiGaO_2$, Li_2GeO_3, $CdTe$, $BaTiO_3$, PZT, $Pb_5Ge_3O_3$ are polar piezoelectrics, and $\alpha - HIO_3$, KH_2PO_4, TeO_2, $Bi_{12}GeO_{20}$, $Bi_{12}SiO_{20}$, $\beta - ZnS$, $\alpha - SiO_2$ are non-polar piezoelectrics. The qualitative difference of properties of polar and non-polar piezoelectric materials consists in the fact that in contrast to non-polar piezoelectrics the polar piezoelectric materials (ferroelectrics) have non-zero dipole moment unit volume, i. e. they possess spontaneous polarization. However, in the classical theory of piezoelectricity [1, 2] based on the equations of electrostatics and symmetric theory of elasticity, as well as in the improved theory of piezoelectricity [3] based on the equations of electrostatics and non-symmetric (moment) theory of elasticity, there is no qualitative difference between polar and non-polar materials. According to the classical theory the only difference between polar and non-polar materials is that the piezoelectric moduli of polar materials are much greater than those of non-polar materials. The most known approaches which allow us to take into account the electric microstructure and permanent electric polarization are developed in [4–6]. We consider the method of description of piezoelectricity which allows us to take into account the qualitative peculiarities of polar and non-polar materials. This method was proposed by P. A. Zhilin (see [7, 8]). The main ideas of the method are to consider the microstructure of piezoelectric materials and use the equations of micro-polar continuum. By another method, but also taking into account the microstructure of materials, the theories of polar piezoelectrics are constructed in [9–12]. The theory of micromorphic thermoelastic continua taking into account electromagnetic effects is considered in [13]. On the basis of this theory the different aspects of theories of micromorphic piezoelectricity, micromorphic thermopiezoelectricity and magneto-electro-elasticity are discussed in [14–17]. In the case of non-polar materials the microstructure approach to description of piezoelectric effects is also of interest because it allows us to take into account electric quadrupoles [18]. We propose two theories of piezoelectricity. The first theory describes the piezoelectric effect in polar materials, and the second one describes the piezoelectric effect in non-polar materials. We show that under certain simplifying assumptions both theories are reduced to the classical theory of piezoelectricity.

2 Polar Piezoelectric Materials

2.1 Model of the Dipole Particle

We consider the medium with particles that are neutral dipoles. The neutral dipole is a pair of charges $q^+ = q$ and $q^- = -q$ separated by a distance. The dipole can

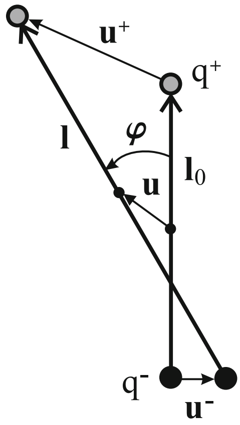

Fig. 1 The electric dipole

move and rotate in space, and also change its value, i. e. it can stretch and compress. The reference position of the dipole (see Fig. 1) is characterized by the following quantities. Radius-vectors $\mathbf{R}_0^+$ and $\mathbf{R}_0^-$ determine the positions of charges q^+ and q^- correspondingly, vector $\mathbf{l}_0$ determines the relative position of the dipole charges, and radius-vector $\mathbf{r}_0$ determines the position of dipole center. When passing to the actual position the charges q^+ and q^- move to the points determined by radius-vectors $\mathbf{R}^+$ and $\mathbf{R}^-$ correspondingly, the dipole center moves to the point determined by radius-vector $\mathbf{r}$. Vector $\mathbf{l}$ determining the relative position of the charges of dipole in the actual configuration is equal to $\mathbf{R}^+ - \mathbf{R}^-$. The quantities characterizing the displacements of the dipole center and dipole charges are determined as

$$\mathbf{u} = \mathbf{r} - \mathbf{r}_0, \qquad \mathbf{u}^+ = \mathbf{R}^+ - \mathbf{R}_0^+, \qquad \mathbf{u}^- = \mathbf{R}^- - \mathbf{R}_0^-. \tag{1}$$

Let us introduce the dipole moments in the reference and actual positions and denote them by $\mathbf{d}_0$ and $\mathbf{d}$, correspondingly:

$$\mathbf{d}_0 = q\mathbf{l}_0 = q(\mathbf{R}_0^+ - \mathbf{R}_0^-), \qquad \mathbf{d} = q\mathbf{l} = q(\mathbf{R}^+ - \mathbf{R}^-). \tag{2}$$

In addition, let us introduce the polarization vector $\mathbf{p}$ equal to change in dipole moment and the scalar quantity ξ being the relative change in absolute value of dipole moment:

$$\mathbf{p} = \mathbf{d} - \mathbf{d}_0, \qquad |\mathbf{d}| = |\mathbf{d}_0|(1 + \xi). \tag{3}$$

After simple transformations we obtain the following formula for $\mathbf{p}$:

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2, \qquad \mathbf{p}_1 = \xi\mathbf{d}_0, \qquad \mathbf{p}_2 = \boldsymbol{\varphi} \times \mathbf{d}_0, \tag{4}$$

where $\boldsymbol{\varphi}$ is the rotation vector of the dipole. Equation (4) is obtained under the assumption of smallness of rotation and extension of the dipole. This assumption is justified because we consider the linear theory.

Now we write the expression for the rate of energy change due to the influence of electric field on the dipole:

$$\dot{e} = \mathbf{F}^{+} \cdot \mathbf{v}^{+} + \mathbf{F}^{-} \cdot \mathbf{v}^{-} . \tag{5}$$

Here $\mathbf{F}^{+}$ and $\mathbf{F}^{-}$ are forces acting on the positive charge and negative charge, correspondingly; $\mathbf{v}^{+}$, $\mathbf{v}^{-}$ are the velocities of these charges. Using the known formula for force acting on a charged particle we have $\mathbf{F} = q\mathbf{E}$, where $\mathbf{E}$ is the electric field vector. Let us perform the following transformations:

$$\begin{aligned}
\dot{e} &= q^{+}\mathbf{E}(\mathbf{R}^{+}) \cdot \dot{\mathbf{u}}^{+} + q^{-}\mathbf{E}(\mathbf{R}^{-}) \cdot \dot{\mathbf{u}}^{-} \\
&= q\big(\mathbf{E}(\mathbf{R}^{+}) - \mathbf{E}(\mathbf{R}^{-})\big) \cdot \dot{\mathbf{u}} + q\mathbf{E}(\mathbf{R}^{+}) \cdot \frac{1}{2q}\dot{\mathbf{p}} + q\mathbf{E}(\mathbf{R}^{-}) \cdot \frac{1}{2q}\dot{\mathbf{p}} \\
&= \mathbf{d}_0 \cdot (\nabla\mathbf{E}) \cdot \dot{\mathbf{u}} + \mathbf{E} \cdot \dot{\mathbf{p}}.
\end{aligned}$$

Using Eq. (4) we calculate the time derivative of the polarization vector

$$\dot{\mathbf{p}} = \dot{\xi}\mathbf{d}_0 + \dot{\boldsymbol{\varphi}} \times \mathbf{d}_0. \tag{6}$$

Thus, the rate of energy change has the form

$$\dot{e} = \mathbf{d}_0 \cdot (\nabla\mathbf{E}) \cdot \dot{\mathbf{u}} + (\mathbf{d}_0 \times \mathbf{E}) \cdot \dot{\boldsymbol{\varphi}} + (\mathbf{d}_0 \cdot \mathbf{E})\, \dot{\xi}. \tag{7}$$

2.2 *Spontaneous and Piezoelectric Polarization of the Medium*

Now we introduce the density of the spontaneous polarization $\mathcal{P}^s$ of a continuous medium

$$\mathcal{P}^s = \lim_{\Delta V \to 0} \frac{\sum_{k \in \Delta V} \mathbf{d}_{0k}}{\Delta V}. \tag{8}$$

We define the density of the piezoelectric polarization $\mathcal{P}^p$ as a limit of the ratio

$$\mathcal{P}^p = \lim_{\Delta V \to 0} \frac{\sum_{k \in \Delta V} \mathbf{p}_k}{\Delta V} = \mathcal{P}^p_1 + \mathcal{P}^p_2, \tag{9}$$

where

$$\mathcal{P}^p_1 = \xi \mathcal{P}^s, \qquad \mathcal{P}^p_2 = \boldsymbol{\varphi} \times \mathcal{P}^s. \tag{10}$$

Thus, vector $\mathcal{P}^{\mathrm{p}}$ is a sum of the piezoelectric polarizations of different nature. Vector $\mathcal{P}^{\mathrm{p}}_1$ is concerned with the change in absolute value of dipole moment, and vector $\mathcal{P}^{\mathrm{p}}_2$ is concerned with the rotation of dipole moment. These vectors are mutually orthogonal.

Using Eqs. (9), (10) we write the analogue of Eq. (7) for continuous medium

$$\dot{\mathcal{E}} = \mathcal{P}^{\mathrm{s}} \cdot (\nabla \mathbf{E}) \cdot \dot{\mathbf{u}} + (\mathcal{P}^{\mathrm{s}} \times \mathbf{E}) \cdot \dot{\boldsymbol{\varphi}} + (\mathcal{P}^{\mathrm{s}} \cdot \mathbf{E})\, \dot{\xi}. \tag{11}$$

We suppose the effect of electric field to be an external action. There are two ways to calculate the power of this external action. On the one hand, the power of external actions per unit volume of continuous medium is equal to $\rho\mathbf{F} \cdot \dot{\mathbf{u}} + \rho\mathbf{L} \cdot \dot{\boldsymbol{\varphi}}$, where $\rho\mathbf{F}$ is the body force, $\rho\mathbf{L}$ is the body moment. On the other hand, the power of external actions is equal to that part of the rate of energy change $\dot{\mathcal{E}}$ which depends on the velocities $\dot{\mathbf{u}}$ and $\dot{\boldsymbol{\varphi}}$. Thus, we obtain

$$\rho\mathbf{F} \cdot \dot{\mathbf{u}} + \rho\mathbf{L} \cdot \dot{\boldsymbol{\varphi}} = \mathcal{P}^{\mathrm{s}} \cdot (\nabla \mathbf{E}) \cdot \dot{\mathbf{u}} + (\mathcal{P}^{\mathrm{s}} \times \mathbf{E}) \cdot \dot{\boldsymbol{\varphi}}. \tag{12}$$

Comparing the left-hand and the right-hand sides of Eq. (12) we conclude that the coefficient of $\dot{\mathbf{u}}$ on the right-hand side of the equation can be associated with a body force and the coefficient of $\dot{\boldsymbol{\varphi}}$ on the right-hand side of the equation can be associated with a body moment:

$$\rho\mathbf{F} = \mathcal{P}^{\mathrm{s}} \cdot \nabla \mathbf{E}, \qquad \rho\mathbf{L} = \mathcal{P}^{\mathrm{s}} \times \mathbf{E}. \tag{13}$$

Thus, the physical meaning of the first two terms on the right-hand side of Eq. (11) has been determined. The last term can be associated with the quantity $\mathcal{Q}$ characterizing the energy supply from an external source:

$$\mathcal{Q} = (\mathcal{P}^{\mathrm{s}} \cdot \mathbf{E})\dot{\xi}. \tag{14}$$

2.3 Equations of Polar Piezoelectric Medium

In view of expressions (13) for the body force and moment, the equations of motion of the polar piezoelectric medium in the linear approximation are written as

$$\nabla \cdot \boldsymbol{\tau} - \frac{1}{2}\nabla \times \mathbf{q} + \mathcal{P}^{\mathrm{s}} \cdot \nabla \mathbf{E} = \rho\ddot{\mathbf{u}}, \tag{15}$$

$$\nabla \times \mathbf{m} + \mathbf{q} + \mathcal{P}^{\mathrm{s}} \times \mathbf{E} = \rho\mathbf{J} \cdot \ddot{\boldsymbol{\varphi}}. \tag{16}$$

Here $\boldsymbol{\tau}$ is the symmetric part of stress tensor, $\mathbf{q}$ is the vector characterizing the antisymmetric part of stress tensor, $\mathbf{m}$ is the vector characterizing the antisymmetric part of moment stress tensor, ρ is the mass density in the reference configuration and $\mathbf{J}$ is the inertia tensor per unit mass.

We introduce the electric induction vector $\mathbf{D}$ by the relation

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathcal{P}^{\mathrm{p}}, \tag{17}$$

where ε_0 is the permittivity of free space. Using Eqs. (9), (10) we rewrite Eq. (17) in the form

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \xi \mathcal{P}^{\mathrm{s}} + \boldsymbol{\varphi} \times \mathcal{P}^{\mathrm{s}}. \tag{18}$$

In view of Eq. (18) the equation of electrostatics

$$\nabla \cdot \mathbf{D} = 0 \tag{19}$$

takes the form

$$\nabla \cdot \left[\varepsilon_0 \mathbf{E} + \xi \mathcal{P}^{\mathrm{s}} + \boldsymbol{\varphi} \times \mathcal{P}^{\mathrm{s}} \right] = 0. \tag{20}$$

According to Eq. (20) the Cauchy–Green relation between $\mathbf{D}$ and $\mathbf{E}$ adopted in the classical theory of piezoelectricity is unnecessary in the theory under consideration and it should be replaced by the Cauchy–Green relation between ξ and the projection of $\mathbf{E}$ on $\mathcal{P}^{\mathrm{s}}$. This is one of the essential differences between the micro-polar theory of piezoelectric medium and the classical theory of piezoelectricity.

Now we formulate the energy balance equation

$$\rho \dot{\mathscr{U}} = \boldsymbol{\tau} \cdot\cdot\, \dot{\mathbf{g}} - \mathbf{q} \cdot \dot{\boldsymbol{\theta}} - \mathbf{m} \cdot \dot{\boldsymbol{\gamma}} + \nabla \cdot \mathbf{h} + \mathscr{Q}, \tag{21}$$

where $\mathbf{h}$ is the heat flow vector, $\mathscr{Q}$ is the rate of energy supply from an external source, $\mathbf{g}$ is the strain tensor, $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$ are the strain vectors connected with the rotational degrees of freedom:

$$\mathbf{g} = \frac{1}{2} \left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}} \right), \qquad \boldsymbol{\theta} = \boldsymbol{\varphi} - \frac{1}{2} \nabla \times \mathbf{u}, \qquad \boldsymbol{\gamma} = \nabla \times \boldsymbol{\varphi}. \tag{22}$$

In order to obtain the Cauchy–Green relations we use the method developed by P. A. Zhilin [7, 8]. We represent $\boldsymbol{\tau}$, $\mathbf{q}$ and $\mathbf{m}$ in the form

$$\boldsymbol{\tau} = \boldsymbol{\tau}_e + \boldsymbol{\tau}_f, \qquad \mathbf{q} = \mathbf{q}_e + \mathbf{q}_f, \qquad \mathbf{m} = \mathbf{m}_e + \mathbf{m}_f, \tag{23}$$

where $\boldsymbol{\tau}_e$, $\mathbf{q}_e$, $\mathbf{m}_e$ are the elastic (independent of strain rate) parts of the force and moment stresses, and $\boldsymbol{\tau}_f$, $\mathbf{q}_f$ and $\mathbf{m}_f$ are the dissipative parts of these stresses. In view of Eq. (23) and the expression for the rate of energy supply (14) the energy balance equation (21) can be rewritten in the form

$$\rho\dot{\mathscr{U}} = \boldsymbol{\tau}_e \cdot\cdot \dot{\mathbf{g}} - \mathbf{q}_e \cdot \dot{\boldsymbol{\theta}} - \mathbf{m}_e \cdot \dot{\boldsymbol{\gamma}} + (\mathbf{E} \cdot \mathcal{P}^s)\,\dot{\xi} + \nabla \cdot \mathbf{h} + \boldsymbol{\tau}_f \cdot\cdot \dot{\mathbf{g}} - \mathbf{q}_f \cdot \dot{\boldsymbol{\theta}} - \mathbf{m}_f \cdot \dot{\boldsymbol{\gamma}}. \tag{24}$$

Let us introduce two scalar quantities ϑ and $\mathscr{H}$ satisfying the equation

$$\vartheta\dot{\mathscr{H}} = \nabla \cdot \mathbf{h} + \boldsymbol{\tau}_f \cdot\cdot \dot{\mathbf{g}} - \mathbf{q}_f \cdot \dot{\boldsymbol{\theta}} - \mathbf{m}_f \cdot \dot{\boldsymbol{\gamma}}, \tag{25}$$

and call them the temperature and entropy, correspondingly. The following constitutive equation can be used for the heat flow vector **h**:

$$\mathbf{h} = k\nabla\vartheta, \tag{26}$$

where k is the heat-conduction coefficient of the medium. Substituting Eq. (26) into Eq. (25) we obtain the heat conduction equation

$$k\Delta\vartheta - \vartheta\dot{\mathscr{H}} = -\boldsymbol{\tau}_f \cdot\cdot \dot{\mathbf{g}} + \mathbf{q}_f \cdot \dot{\boldsymbol{\theta}} + \mathbf{m}_f \cdot \dot{\boldsymbol{\gamma}}. \tag{27}$$

The terms on the right-hand side of Eq. (27) characterize the heat production connected with the dissipative processes.

Using Eq. (25) we rewrite the energy balance equation (21) in the form

$$\rho\dot{\mathscr{U}} = \boldsymbol{\tau}_e \cdot\cdot \dot{\mathbf{g}} - \mathbf{q}_e \cdot \dot{\boldsymbol{\theta}} - \mathbf{m}_e \cdot \dot{\boldsymbol{\gamma}} + (\mathbf{E} \cdot \mathcal{P}^s)\,\dot{\xi} + \vartheta\dot{\mathscr{H}}. \tag{28}$$

Hence $\mathscr{U} = \mathscr{U}(\mathbf{g}, \boldsymbol{\theta}, \boldsymbol{\gamma}, \mathcal{P}, \xi, \mathscr{H})$, from Eq. (28) we obtain the Cauchy–Green relations

$$\boldsymbol{\tau}_e = \frac{\partial\rho\mathscr{U}}{\partial\mathbf{g}}, \qquad \mathbf{q}_e = -\frac{\partial\rho\mathscr{U}}{\partial\boldsymbol{\theta}}, \qquad \mathbf{m}_e = -\frac{\partial\rho\mathscr{U}}{\partial\boldsymbol{\gamma}}, \qquad \mathbf{E} \cdot \mathcal{P}^s = \frac{\partial\rho\mathscr{U}}{\partial\xi}, \qquad \vartheta = \frac{\partial\rho\mathscr{U}}{\partial\mathscr{H}}. \tag{29}$$

Let us represent the internal energy as the positive defined quadratic form

$$\begin{aligned}\rho\mathscr{U} = {} & \frac{1}{2}\mathbf{g} \cdot\cdot \mathbf{C}^{(g)} \cdot\cdot \mathbf{g} + \frac{1}{2}\boldsymbol{\theta} \cdot \mathbf{C}^{(\theta)} \cdot \boldsymbol{\theta} + \frac{1}{2}\boldsymbol{\gamma} \cdot \mathbf{C}^{(\gamma)} \cdot \boldsymbol{\gamma} + \frac{1}{2}C^{(\xi)}\xi^2 \\ & + \frac{1}{2}C^{(\mathscr{H})}\mathscr{H}^2 + \boldsymbol{\theta} \cdot \mathbf{C}^{(\theta g)} \cdot\cdot \mathbf{g} + \boldsymbol{\gamma} \cdot \mathbf{C}^{(\gamma g)} \cdot\cdot \mathbf{g} + \xi\mathbf{C}^{(\xi g)} \cdot\cdot \mathbf{g} \\ & + \mathscr{H}\mathbf{C}^{(\mathscr{H} g)} \cdot\cdot \mathbf{g} + \boldsymbol{\gamma} \cdot \mathbf{C}^{(\gamma\theta)} \cdot \boldsymbol{\theta} + \xi\mathbf{C}^{(\xi\theta)} \cdot \boldsymbol{\theta} + \mathscr{H}\mathbf{C}^{(\mathscr{H}\theta)} \cdot \boldsymbol{\theta} \\ & + \xi\mathbf{C}^{(\xi\gamma)} \cdot \boldsymbol{\gamma} + \mathscr{H}\mathbf{C}^{(\mathscr{H}\gamma)} \cdot \boldsymbol{\gamma} + C^{(\xi\mathscr{H})}\xi\mathscr{H}.\end{aligned} \tag{30}$$

Substituting Eq. (30) into the Cauchy–Green relations (29) we get the constitutive equations

$$\begin{aligned}\boldsymbol{\tau}_e &= \mathbf{C}^{(g)} \cdot\cdot\, \mathbf{g} + \boldsymbol{\theta} \cdot \mathbf{C}^{(\theta g)} + \boldsymbol{\gamma} \cdot \mathbf{C}^{(\gamma g)} + \mathbf{C}^{(\xi g)}\xi + \mathbf{C}^{(\mathscr{H} g)}\mathscr{H},\\ -\mathbf{q}_e &= \mathbf{C}^{(\theta g)} \cdot\cdot\, \mathbf{g} + \mathbf{C}^{(\theta)} \cdot \boldsymbol{\theta} + \boldsymbol{\gamma} \cdot \mathbf{C}^{(\gamma\theta)} + \mathbf{C}^{(\xi\theta)}\xi + \mathbf{C}^{(\mathscr{H}\theta)}\mathscr{H},\\ -\mathbf{m}_e &= \mathbf{C}^{(\gamma g)} \cdot\cdot\, \mathbf{g} + \mathbf{C}^{(\gamma\theta)} \cdot \boldsymbol{\theta} + \mathbf{C}^{(\gamma)} \cdot \boldsymbol{\gamma} + \mathbf{C}^{(\xi\gamma)}\xi + \mathbf{C}^{(\mathscr{H}\gamma)}\mathscr{H},\\ \mathbf{E} \cdot \boldsymbol{\mathcal{P}}^s &= \mathbf{C}^{(\xi g)} \cdot\cdot\, \mathbf{g} + \mathbf{C}^{(\xi\theta)} \cdot \boldsymbol{\theta} + \mathbf{C}^{(\xi\gamma)} \cdot \boldsymbol{\gamma} + C^{(\xi)}\xi + C^{(\xi\mathscr{H})}\mathscr{H},\\ \vartheta &= \mathbf{C}^{(\mathscr{H} g)} \cdot\cdot\, \mathbf{g} + \mathbf{C}^{(\mathscr{H}\theta)} \cdot \boldsymbol{\theta} + \mathbf{C}^{(\mathscr{H}\gamma)} \cdot \boldsymbol{\gamma} + C^{(\xi\mathscr{H})}\xi + C^{(\mathscr{H})}\mathscr{H}.\end{aligned} \tag{31}$$

In order to close the set of equations (15), (16), (20), (22), (23), (27), (31) the constitutive equations for the dissipative parts of force and moment stresses $\boldsymbol{\tau}_f$, $\mathbf{q}_f$, $\mathbf{m}_f$ should be formulated.

2.4 The Simplest Theory of Polar Medium

Now we neglect the inertia of rotation and the moment interactions, i. e. we suppose that $\mathbf{J} = \mathbf{0}$ and $\mathbf{m} = \mathbf{0}$. Then the equation of the angular momentum balance (16) takes the form

$$\mathbf{q} = -\boldsymbol{\mathcal{P}}^s \times \mathbf{E}. \tag{32}$$

Substituting Eq. (32) into the equation of momentum balance (15) we obtain

$$\nabla \cdot \boldsymbol{\tau} + \frac{1}{2}\nabla \times (\boldsymbol{\mathcal{P}}^s \times \mathbf{E}) + \boldsymbol{\mathcal{P}}^s \cdot \nabla\mathbf{E} = \rho\ddot{\mathbf{u}}. \tag{33}$$

Let us neglect the dissipative and thermal effects. Then in view of Eq. (32) the constitutive equations (31) take the form

$$\begin{aligned}\boldsymbol{\tau} &= \mathbf{C}^{(g)} \cdot\cdot\, \mathbf{g} + \boldsymbol{\theta} \cdot \mathbf{C}^{(\theta g)} + \mathbf{C}^{(\xi g)}\xi,\\ \boldsymbol{\mathcal{P}}^s \times \mathbf{E} &= \mathbf{C}^{(\theta g)} \cdot\cdot\, \mathbf{g} + \mathbf{C}^{(\theta)} \cdot \boldsymbol{\theta} + \mathbf{C}^{(\xi\theta)}\xi,\\ \mathbf{E} \cdot \boldsymbol{\mathcal{P}}^s &= \mathbf{C}^{(\xi g)} \cdot\cdot\, \mathbf{g} + \mathbf{C}^{(\xi\theta)} \cdot \boldsymbol{\theta} + C^{(\xi)}\xi,\end{aligned} \tag{34}$$

where index e of tensor $\boldsymbol{\tau}$ is left out since the dissipative part of this tensor is equal to zero. In view of the relation between angles $\boldsymbol{\varphi}$ and $\boldsymbol{\theta}$ the expression (18) takes the form

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \xi\boldsymbol{\mathcal{P}}^s + \boldsymbol{\theta} \times \boldsymbol{\mathcal{P}}^s + \frac{1}{2}(\nabla \times \mathbf{u}) \times \boldsymbol{\mathcal{P}}^s \tag{35}$$

and the equation of electrostatics (20) is written as

$$\nabla \cdot \left[\varepsilon_0\mathbf{E} + \xi\boldsymbol{\mathcal{P}}^s + \boldsymbol{\theta} \times \boldsymbol{\mathcal{P}}^s + \frac{1}{2}(\nabla \times \mathbf{u}) \times \boldsymbol{\mathcal{P}}^s\right] = 0. \tag{36}$$

Thus, the set of equations (33)–(36) represents the formulation of the simplest theory of polar piezoelectric medium. Here the basic variables are the displacement vector **u**, the shear vector θ and the quantity ξ characterizing the dipole deformation.

2.5 Comparison with the Classical Theory

To compare Eqs. (33)–(36) with the equations of classical theory of piezoelectricity we should obtain the relations between $\boldsymbol{\tau}$, **D** and **g**, **E**. In order to do this we solve the system of second and third equations in (34) with respect to θ and ξ. Then we substitute the obtained expressions into the first equation in Eqs. (34) and into Eq. (35). As a result we get

$$\boldsymbol{\tau} = \mathbf{C} \cdot\cdot\, \mathbf{g} - \mathbf{E} \cdot \mathcal{M}, \qquad \mathbf{D} = \mathcal{M} \cdot\cdot\, \mathbf{g} + \boldsymbol{\epsilon} \cdot \mathbf{E} - \frac{1}{2}\, \mathcal{P}^s \times (\nabla \times \mathbf{u})\,, \tag{37}$$

where **C** is the stiffness tensor; $\mathcal{M}$ is the tensor of piezoelectric moduli; $\boldsymbol{\epsilon}$ is the permittivity tensor. Tensors **C**, $\mathcal{M}$, $\boldsymbol{\epsilon}$ can be expressed in terms of the material tensors introduced above by the sufficiently complicated formulas.

Comparison of the constitutive equations (37) and the corresponding constitutive equations of the classical theory [1, 2]

$$\boldsymbol{\tau} = \mathbf{C} \cdot\cdot\, \mathbf{g} - \mathbf{E} \cdot \mathcal{M}, \qquad \mathbf{D} = \mathcal{M} \cdot\cdot\, \mathbf{g} + \boldsymbol{\epsilon} \cdot \mathbf{E} \tag{38}$$

reveals that the constitutive equations for $\boldsymbol{\tau}$ are the same and the constitutive equations for **D** differ by the additional term which depends on the curl of displacement vector in the case of the micro-polar theory.

The equation of motion (33) differs from the classical equation of motion

$$\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{F} = \rho \ddot{\mathbf{u}}, \qquad \boldsymbol{\tau} = \boldsymbol{\tau}^T \tag{39}$$

by the presence of two terms modelling the effect of the electric field. The equation of electrostatics has the same form (19) in both theories.

3 Non-Polar Piezoelectric Materials

3.1 Model of the Unit Cell of Crystal Lattice

We consider the crystal lattice with unit cells consisting of N ions which have charges q_i (see Fig. 2). In the reference configuration the position of mass center of the cell is determined by the radius-vector **r**, the positions of ions are determined by the

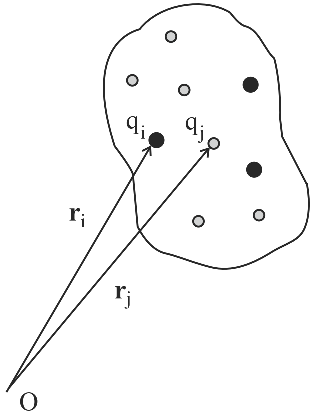

Fig. 2 Arbitrary unit cell of *crystal lattice*

radius-vectors $\mathbf{r}_i = \mathbf{r} + \mathbf{b}_i$, where the radius-vectors $\mathbf{b}_i$ determine the positions of ions relative to the mass center of the cell.

Let us introduce the electrical characteristics of the unit cell: the total charge q, the dipole moment $\mathbf{d}$ and the quadrupole moment $\mathbf{Q}$, which are calculated by the formulas

$$q = \sum_i q_i, \qquad \mathbf{d} = \sum_i q_i \mathbf{b}_i, \qquad \mathbf{q} = \frac{1}{2} \sum_i q_i \mathbf{b}_i \mathbf{b}_i. \tag{40}$$

Note that the definition of the quadrupole moment (40) is not standard. Usually another definition of quadrupole moment is introduced, namely

$$\mathbf{Q}_* = \sum_i q_i (3\mathbf{b}_i \mathbf{b}_i - \mathbf{b}_i^2 \mathbf{I}), \tag{41}$$

where $\mathbf{I}$ is the unit tensor. In crystals the total charge of the unit cell is equal to zero whereas the dipole moment $\mathbf{d}$ and the quadrupole moment $\mathbf{Q}$ can be zero or non-zero depending on the type of material. It is known that if the total charge and the dipole moment are equal to zero then the quadrupole moment does not depend on the point with respect to which it is calculated. This is true for both definitions of quadrupole moment.

To describe the kinematics of the unit sell we introduce the displacement vectors of ions $\mathbf{u}_i$. Further all displacements are supposed to be small and the following representation for $\mathbf{u}_i$ is used:

$$\mathbf{u}_i = \mathbf{u} + \boldsymbol{\varphi} \times \mathbf{b}_i + \boldsymbol{\xi}_i. \tag{42}$$

Here $\mathbf{u}$ is the displacement vector of the center of the unit cell, $\boldsymbol{\varphi}$ is the vector of small rotation of the unit cell as a rigid body, $\boldsymbol{\xi}_i$ are variables characterizing the deformation of the unit cell. We suppose the ion displacements associated with the deformation of the unit cell to be much less than the ion displacements connected with the movement of the cell as a rigid body. In other words, we assume that $|\boldsymbol{\xi}_i| \ll |\mathbf{u}|$ and $|\boldsymbol{\xi}_i| \ll |\boldsymbol{\varphi} \times \mathbf{b}_i|$. Let us introduce the polarization vector $\mathbf{p}$:

$$\mathbf{p} = \sum_i q_i(\boldsymbol{\varphi} \times \mathbf{b}_i + \boldsymbol{\xi}_i). \tag{43}$$

In view of Eq. (40) the formula (43) can be reduced to the form

$$\mathbf{p} = \mathbf{p}_r + \mathbf{p}_d, \qquad \mathbf{p}_r = \boldsymbol{\varphi} \times \mathbf{d}, \qquad \mathbf{p}_d = \sum_i q_i \boldsymbol{\xi}_i. \tag{44}$$

Here $\mathbf{p}_r$ is the polarization due to the rotation of the unit cell, and $\mathbf{p}_d$ is the polarization due to the deformation of the unit cell.

Now we write the expression for the rate of the energy change due to the work of the electric field on the ions of unit cell:

$$\dot{e} = \sum_i q_i \mathbf{E}(\mathbf{r}_i) \cdot \mathbf{v}_i, \tag{45}$$

where $\mathbf{v}_i = \dot{\mathbf{u}}_i$ is the velocity vector of the ion with the number i. Assuming that the electric field slowly varies over distances comparable with the characteristic dimensions of the unit cell we use the expansion of vector $\mathbf{E}$ in a Taylor series. Keeping the first three terms in the Taylor series we have

$$\mathbf{E}(\mathbf{r}_i) \approx \mathbf{E}(\mathbf{r}) + \mathbf{b}_i \cdot \nabla\mathbf{E} + \frac{1}{2}\mathbf{b}_i\mathbf{b}_i \cdot\cdot \nabla\nabla\mathbf{E}. \tag{46}$$

Using Eqs. (42), (46) we reduce the expression for the rate of energy change (45) to the form

$$\dot{e} \approx (\mathbf{d} \cdot \nabla\mathbf{E} + \mathbf{Q} \cdot\cdot \nabla\nabla\mathbf{E}) \cdot \dot{\mathbf{u}} + \left(\mathbf{d} \times \mathbf{E} + 2\mathbf{Q} \cdot \times\nabla\mathbf{E}\right) \cdot \dot{\boldsymbol{\varphi}} + \mathbf{E} \cdot \dot{\mathbf{p}}_d. \tag{47}$$

The formula (47) is derived in view of the fact that the total charge of the unit cell is equal to zero, and also $|\boldsymbol{\xi}_i| \ll |\mathbf{u}|$ and $|\boldsymbol{\xi}_i| \ll |\boldsymbol{\varphi} \times \mathbf{b}_i|$.

3.2 *Polarization of Continuous Medium*

Now we write the analogue of Eq. (47) for the continuous medium. In order to pass from the discrete model to the corresponding continuum model we use standard line of reasoning based on symmetry properties of the crystal lattice and the long-wave approximation. The description of the method can be found in [19]. Application of this method in the case when the rotational motion and moment interaction are taken into account is discussed in [20]. Thus, the continuous analogue of Eq. (47) is

$$\dot{\mathcal{E}} \approx (\boldsymbol{\mathcal{P}}^{s} \cdot \nabla \mathbf{E} + \boldsymbol{\mathcal{Q}} \cdot\cdot \nabla\nabla \mathbf{E}) \cdot \dot{\mathbf{u}} + \big(\boldsymbol{\mathcal{P}}^{s} \times \mathbf{E} + 2\boldsymbol{\mathcal{Q}} \cdot \times \nabla \mathbf{E}\big) \cdot \dot{\boldsymbol{\varphi}} + \mathbf{E} \cdot \dot{\boldsymbol{\mathcal{P}}}^{p}. \quad (48)$$

Here the parameters of the medium $\boldsymbol{\mathcal{P}}^{s}$ (volume density of spontaneous polarization) and $\boldsymbol{\mathcal{Q}}$ (volume density of the quadrupole moments) are

$$\boldsymbol{\mathcal{P}}^{s} = \lim_{\Delta V \to 0} \frac{\sum_{k \in \Delta V} \mathbf{d}_k}{\Delta V}, \qquad \boldsymbol{\mathcal{Q}} = \lim_{\Delta V \to 0} \frac{\sum_{k \in \Delta V} \mathbf{q}_k}{\Delta V}, \quad (49)$$

and the volume density of piezoelectric polarization $\boldsymbol{\mathcal{P}}^{p}$ being one of the basic variables is introduced by the formula

$$\boldsymbol{\mathcal{P}}^{p} = \lim_{\Delta V \to 0} \frac{\sum_{k \in \Delta V} \mathbf{p}_{dk}}{\Delta V}. \quad (50)$$

We suppose the electric field to be an external factor. The power of external actions per unit volume is equal to $\rho \mathbf{F} \cdot \dot{\mathbf{u}} + \rho \mathbf{L} \cdot \dot{\boldsymbol{\varphi}}$. The corresponding power of the electric field action is equal to that part of the rate of energy change $\dot{\mathcal{E}}$ which depends on the translational velocity $\dot{\mathbf{u}}$ and the angular velocity $\dot{\boldsymbol{\varphi}}$. Thus, we obtain

$$\rho \mathbf{F} \cdot \dot{\mathbf{u}} + \rho \mathbf{L} \cdot \dot{\boldsymbol{\varphi}} = (\boldsymbol{\mathcal{P}}^{s} \cdot \nabla \mathbf{E} + \boldsymbol{\mathcal{Q}} \cdot\cdot \nabla\nabla \mathbf{E}) \cdot \dot{\mathbf{u}} + \big(\boldsymbol{\mathcal{P}}^{s} \times \mathbf{E} + 2\boldsymbol{\mathcal{Q}} \cdot \times \nabla \mathbf{E}\big) \cdot \dot{\boldsymbol{\varphi}}. \quad (51)$$

Comparing the left-hand and the right-hand sides of Eq. (51) we conclude that the coefficient of $\dot{\mathbf{u}}$ on the right-hand side of the equation can be associated with a body force and the coefficient of $\dot{\boldsymbol{\varphi}}$ on the right-hand side of the equation can be associated with a body moment:

$$\rho \mathbf{F} = \boldsymbol{\mathcal{P}}^{s} \cdot \nabla \mathbf{E} + \boldsymbol{\mathcal{Q}} \cdot\cdot \nabla\nabla \mathbf{E}, \qquad \rho \mathbf{L} = \boldsymbol{\mathcal{P}}^{s} \times \mathbf{E} + 2\boldsymbol{\mathcal{Q}} \cdot \times \nabla \mathbf{E}. \quad (52)$$

In the case of non-polar piezoelectrics $\boldsymbol{\mathcal{P}}^{s} = \mathbf{0}$ and the expressions (52) take the simpler form

$$\rho \mathbf{F} = \boldsymbol{\mathcal{Q}} \cdot\cdot \nabla\nabla \mathbf{E}, \qquad \rho \mathbf{L} = 2\boldsymbol{\mathcal{Q}} \cdot \times \nabla \mathbf{E}. \quad (53)$$

The last term in Eq. (51) can be associated with the quantity $\mathscr{Q}$ characterizing the energy supply from an external source that cannot be expressed in terms of the power of external forces and moments:

$$\mathscr{Q} = \mathbf{E} \cdot \dot{\boldsymbol{\mathcal{P}}}^{p}. \quad (54)$$

3.3 Equations of Non-Polar Piezoelectric Medium

In view of the expressions for external force and moment (53) the equations of motion of the non-polar piezoelectric medium in the linear approximation take the form

$$\nabla\cdot\boldsymbol{\tau}-\frac{1}{2}\nabla\times\mathbf{q}+\boldsymbol{\Omega}\cdot\cdot\nabla\nabla\mathbf{E}=\rho\ddot{\mathbf{u}}, \tag{55}$$

$$\nabla\times\mathbf{m}+\mathbf{q}+2\boldsymbol{\Omega}\cdot\times\nabla\mathbf{E}=\rho\mathbf{J}\cdot\ddot{\boldsymbol{\varphi}}. \tag{56}$$

Introducing electric induction vector **D** by means of Eq. (17) we write the equation of electrostatics (19) as

$$\nabla\cdot\left[\varepsilon_0\mathbf{E}+\boldsymbol{\mathcal{P}}^{\mathrm{p}}\right]=0. \tag{57}$$

Starting from the energy balance equation in the form of Eq. (21) and using the line of reasoning similar to those which were held in the case of polar medium we obtain the heat conduction Eq. (27) and the reduced energy balance equation

$$\rho\dot{\mathscr{U}}=\boldsymbol{\tau}_e\cdot\cdot\dot{\mathbf{g}}-\mathbf{q}_e\cdot\dot{\boldsymbol{\theta}}-\mathbf{m}_e\cdot\dot{\boldsymbol{\gamma}}+\mathbf{E}\cdot\dot{\boldsymbol{\mathcal{P}}}^{\mathrm{p}}+\vartheta\dot{\mathscr{H}}. \tag{58}$$

Note that Eq. (58) is derived in view of the expression (54) for the rate of energy supply from an external source. The Cauchy–Green relations which follow from Eq. (58) are

$$\boldsymbol{\tau}_e=\frac{\partial\rho\mathscr{U}}{\partial\mathbf{g}},\quad \mathbf{q}_e=-\frac{\partial\rho\mathscr{U}}{\partial\boldsymbol{\theta}},\quad \mathbf{m}_e=-\frac{\partial\rho\mathscr{U}}{\partial\boldsymbol{\gamma}},\quad \mathbf{E}=\frac{\partial\rho\mathscr{U}}{\partial\boldsymbol{\mathcal{P}}^{\mathrm{p}}},\quad \vartheta=\frac{\partial\rho\mathscr{U}}{\partial\mathscr{H}}. \tag{59}$$

The internal energy is assumed to be the positive defined quadratic form

$$\begin{aligned}\rho\mathscr{U}=&\frac{1}{2}\mathbf{g}\cdot\cdot\mathbf{C}^{(\mathrm{g})}\cdot\cdot\mathbf{g}+\frac{1}{2}\boldsymbol{\theta}\cdot\mathbf{C}^{(\theta)}\cdot\boldsymbol{\theta}+\frac{1}{2}\boldsymbol{\gamma}\cdot\mathbf{C}^{(\gamma)}\cdot\boldsymbol{\gamma}\\&+\frac{1}{2}\boldsymbol{\mathcal{P}}^{\mathrm{p}}\cdot\mathbf{C}^{(\mathcal{P})}\cdot\boldsymbol{\mathcal{P}}^{\mathrm{p}}+\frac{1}{2}C^{(\mathscr{H})}\mathscr{H}^2+\boldsymbol{\theta}\cdot\mathbf{C}^{(\theta\mathrm{g})}\cdot\cdot\mathbf{g}+\boldsymbol{\gamma}\cdot\mathbf{C}^{(\gamma\mathrm{g})}\cdot\cdot\mathbf{g}\\&+\boldsymbol{\mathcal{P}}^{\mathrm{p}}\cdot\mathbf{C}^{(\mathcal{P}\mathrm{g})}\cdot\cdot\mathbf{g}+\mathscr{H}\mathbf{C}^{(\mathscr{H}\mathrm{g})}\cdot\cdot\mathbf{g}+\boldsymbol{\gamma}\cdot\mathbf{C}^{(\gamma\theta)}\cdot\boldsymbol{\theta}+\boldsymbol{\theta}\cdot\mathbf{C}^{(\theta\mathcal{P})}\cdot\boldsymbol{\mathcal{P}}^{\mathrm{p}}\\&+\mathscr{H}\mathbf{C}^{(\mathscr{H}\theta)}\cdot\boldsymbol{\theta}+\boldsymbol{\mathcal{P}}^{\mathrm{p}}\cdot\mathbf{C}^{(\mathcal{P}\gamma)}\cdot\boldsymbol{\gamma}+\mathscr{H}\mathbf{C}^{(\mathscr{H}\gamma)}\cdot\boldsymbol{\gamma}+\mathscr{H}\mathbf{C}^{(\mathscr{H}\mathcal{P})}\cdot\boldsymbol{\mathcal{P}}^{\mathrm{p}}.\end{aligned} \tag{60}$$

Substituting Eq. (60) into the Cauchy–Green relations (59) we get

$$\begin{aligned}\boldsymbol{\tau}_e&=\mathbf{C}^{(\mathrm{g})}\cdot\cdot\mathbf{g}+\boldsymbol{\theta}\cdot\mathbf{C}^{(\theta\mathrm{g})}+\boldsymbol{\gamma}\cdot\mathbf{C}^{(\gamma\mathrm{g})}+\boldsymbol{\mathcal{P}}^{\mathrm{p}}\cdot\mathbf{C}^{(\mathcal{P}\mathrm{g})}+\mathbf{C}^{(\mathscr{H}\mathrm{g})}\mathscr{H},\\-\mathbf{q}_e&=\mathbf{C}^{(\theta\mathrm{g})}\cdot\cdot\mathbf{g}+\mathbf{C}^{(\theta)}\cdot\boldsymbol{\theta}+\boldsymbol{\gamma}\cdot\mathbf{C}^{(\gamma\theta)}+\mathbf{C}^{(\theta\mathcal{P})}\cdot\boldsymbol{\mathcal{P}}^{\mathrm{p}}+\mathbf{C}^{(\mathscr{H}\theta)}\mathscr{H},\\-\mathbf{m}_e&=\mathbf{C}^{(\gamma\mathrm{g})}\cdot\cdot\mathbf{g}+\mathbf{C}^{(\gamma\theta)}\cdot\boldsymbol{\theta}+\mathbf{C}^{(\gamma)}\cdot\boldsymbol{\gamma}+\boldsymbol{\mathcal{P}}^{\mathrm{p}}\cdot\mathbf{C}^{(\mathcal{P}\gamma)}+\mathbf{C}^{(\mathscr{H}\gamma)}\mathscr{H},\end{aligned} \tag{61}$$

$$\mathbf{E} = \mathbf{C}^{(\mathcal{P}g)} \cdot\cdot \mathbf{g} + \theta \cdot \mathbf{C}^{(\theta\mathcal{P})} + \mathbf{C}^{(\mathcal{P}\gamma)} \cdot \gamma + \mathbf{C}^{(\mathcal{P})} \cdot \mathcal{P}^p + \mathbf{C}^{(\mathcal{H}\mathcal{P})}\mathcal{H},$$
$$\vartheta = \mathbf{C}^{(\mathcal{H}g)} \cdot\cdot \mathbf{g} + \mathbf{C}^{(\mathcal{H}\theta)} \cdot \theta + \mathbf{C}^{(\mathcal{H}\gamma)} \cdot \gamma + \mathbf{C}^{(\mathcal{H}\mathcal{P})} \cdot \mathcal{P}^p + C^{(\mathcal{H})}\mathcal{H}.$$

In order to close the set of equations (22), (23), (27), (55)–(57), (61) the constitutive equations for the dissipative parts of force and moment stresses $\boldsymbol{\tau}_f$, $\mathbf{q}_f$, $\mathbf{m}_f$ should be formulated.

3.4 Comparison with the Classical Theory

To compare the theory stated above with the classical theory of piezoelectricity we leave out the thermal effects and moment interactions and neglect the inertia of rotation. Since $\boldsymbol{\mu} = \mathbf{0}$, $\mathbf{J} = \mathbf{0}$ the angular momentum balance equation (56) takes the form

$$\mathbf{q} = -2\boldsymbol{\Omega} \cdot \times \nabla \mathbf{E}. \tag{62}$$

In view of Eq. (62) the momentum balance equation (55) is reduced to the form

$$\nabla \cdot \boldsymbol{\tau} + \boldsymbol{\Omega} \cdot \nabla\nabla \cdot \mathbf{E} = \rho \ddot{\mathbf{u}}. \tag{63}$$

In view of the foregoing assumptions the constitutive equations (61) can be rewritten as

$$\begin{aligned} \boldsymbol{\tau} &= \mathbf{C}^{(g)} \cdot\cdot \mathbf{g} + \theta \cdot \mathbf{C}^{(\theta g)} + \mathcal{P}^p \cdot \mathbf{C}^{(Pg)}, \\ -\mathbf{q} &= \mathbf{C}^{(\theta g)} \cdot\cdot \mathbf{g} + \mathbf{C}^{(\theta)} \cdot \theta + \mathbf{C}^{(\theta P)} \cdot \mathcal{P}^p, \\ \mathbf{E} &= \mathbf{C}^{(P)} \cdot \mathcal{P}^p + \mathbf{C}^{(Pg)} \cdot\cdot \mathbf{g} + \theta \cdot \mathbf{C}^{(\theta P)}, \end{aligned} \tag{64}$$

where indices e of tensor $\boldsymbol{\tau}$ and vector $\mathbf{q}$ are left out because the dissipative part of these tensors are equal to zero. Further two versions of the simplified theory are considered.

Variant 1. We suppose that the shear strain θ is equal to zero, but the corresponding part of the stress tensor determined by vector $\mathbf{q}$ is a finite quantity. Then the constitutive equations (64) take the simpler form. In view of Eq. (17) the obtained constitutive equations can be reduced to Eq. (38) where tensors $\mathbf{C}$, $\mathcal{M}$, $\boldsymbol{\epsilon}$ are expressed in terms of the material tensors introduced above. Thus the first variant of the simplified theory is the set of equations of piezoelectricity (19), (38), (63) which differs from the classical one only by the term $\boldsymbol{\Omega} \cdot \nabla\nabla \cdot \mathbf{E}$ in the equation of motion (63). Substituting the first equation in Eq. (38) into Eq. (63) we obtain

$$\nabla \cdot (\mathbf{C} \cdot\cdot \mathbf{g}) - \nabla \mathbf{E} \cdot\cdot \mathcal{M} + \boldsymbol{\Omega} \cdot \nabla\nabla \cdot \mathbf{E} = \rho \ddot{\mathbf{u}}. \tag{65}$$

In the case of long-wave processes the contribution of the term $\mathbf{\Omega} \cdot \nabla\nabla \cdot \mathbf{E}$ is small compared to the contribution of the term $\nabla\mathbf{E} \cdot\cdot\, \mathcal{M}$. However, in the case of short-wave processes the contribution of the term $\mathbf{\Omega} \cdot \nabla\nabla \cdot \mathbf{E}$ can be significant.

Variant 2. The case when $\theta \neq \mathbf{0}$ is considered. By the simple transformations in view of Eqs. (17), (62) the constitutive equations (64) are reduced to the form

$$\boldsymbol{\tau} = \mathbf{C} \cdot\cdot\, \mathbf{g} - \mathbf{E} \cdot \mathcal{M} + \mathcal{N} \cdot\cdot\, \nabla\mathbf{E}, \qquad \mathbf{D} = \mathcal{M} \cdot\cdot\, \mathbf{g} + \boldsymbol{\epsilon} \cdot \mathbf{E} - \boldsymbol{\varepsilon} \cdot\cdot\, \nabla\mathbf{E}. \tag{66}$$

Here tensors $\mathbf{C}$, $\mathcal{M}$, $\mathcal{N}$, $\boldsymbol{\epsilon}$, $\boldsymbol{\varepsilon}$ can be expressed in terms of the material tensors introduced above by the complicated formulas. It is easy to see that the constitutive equations (66) differ from the classical ones by the terms containing $\nabla\mathbf{E}$. Now it is impossible to quantify the contribution of these terms since to determine the tensors $\mathcal{N}$ and $\boldsymbol{\varepsilon}$ the physical experiments should be carried out. However, it is clear that in the case of short-wave processes the relative contribution of the terms containing $\nabla\mathbf{E}$ is greater than in the case of long-wave processes.

4 Conclusion

Above two micro-polar theories of piezoelectricity based on the continuum with internal degrees of freedom are considered. One of these theories describes the polar piezoelectric materials, and the other describes the non-polar materials. In contrast to the classical theory where the constitutive equations establish the relations between the electric field vector $\mathbf{E}$ and the electric induction vector $\mathbf{D}$, in the proposed micro-polar theories the constitutive equations relate the electric field vector $\mathbf{E}$ and the polarization vector $\mathcal{P}^{\mathrm{p}}$. It is proved that under certain simplifying assumptions the proposed theories of piezoelectricity pass into the quasi-classical ones. The quasi-classical theories differ from the classical theory of piezoelectricity by the presence of additional terms of piezoelectric nature in the equations of motion and the constitutive equations.

Acknowledgments The work was supported by the grant of RFBR N 12-01-00815-a.

References

1. Cady, W.G.: Piezoelectricity: An Introduction to the Theory and Applications of Electromechanical Phenomena in Crystals. Dover Publications, New York (1964)
2. Tiersten, H.F.: Linear Piezoelectric Plate Vibrations. Plenum Press, New York (1969)
3. Kolpakov, J.E., Zhilin, P.A.: Generalized continuum and linear theory of piezoelectric materials. In: Proceedings of XXIX Summer School-Conference "Advanced Problems in Mechanics", pp. 364–375. St. Petersburg (2002)
4. Eringen, A.C., Maugin, G.A.: Electrodynamics of Continua, vol. 1. Springer, New York (1990)

5. Maugin, G.A.: Continuum Mechanics of Electromagnetic Solids. North-Holland, Amsterdam (1988)
6. Nelson, D.F.: Electric, Optic and Acoustic Interactions in Dielectrics. Wiley-Interscience, New York (1979)
7. Zhilin, P.A.: Advanced Problems in Mechanics, vol. 2. IPME RAS, St. Petersburg (2006)
8. Zhilin, P.A.: Rational Continuum Mechanics. Politechnic University Publishing House, St. Petersburg (2012). (In Russian)
9. Treugolov, I.G.: Moment theory of electromagnetic effects in anisotropic solids. Appl. Math. Mech. **53**(6), 992–997 (1989). (In Russian)
10. Dixon, R.C., Eringen, A.C.: A dynamical theory of polar elastic dielectrics-I. Int. J. Eng. Sci. **3**, 359–377 (1965)
11. Demiray, H., Eringen, A.C.: On constitutive relations of polar elastic dielectrics. Lett. Appl. Eng. Sci. **1**, 517–527 (1973)
12. Demiray, H., Dost, S.: Diatomics elestic dielectrics with polarization gradient. Int. J. Eng. Sci. **27**, 1275–1284 (1989)
13. Eringen, A.C.: Continuum theory of micromorphic electromagnetic thermoelastic solids. Int. J. Eng. Sci. **41**, 653–665 (2003)
14. Gales, C., Ghiba, I.D., Ignyatescu, I.: Asymptotic partition of energy in micromorphic thermopiezoelectricity. J. Therm. Stresses **34**, 1241–1249 (2011)
15. Gales, C.: Spatial behavior in the electromagnetic theory of microstretch elasticity. Int. J. Solids Struct. **48**, 2755–2763 (2011)
16. Gales, C.: Some results in micromorphic piezoelectricity. Eur. J. Mech. A. Solids **31**, 37–46 (2011)
17. Gales, C.: Spatial behavior and continuous dependence results in the linear dynamic theory of magnetoelectroelasticity. J. Elast. **108**, 209–223 (2012)
18. Prechtl, A.: Defirmable bodies with electric and magnetic quadrupoles. Int. J. Eng. Sci. **18**, 665–680 (1980)
19. Born, M., Huang, K.: Dynamics Theory of Crystal Lattices. Clarendon Press, Oxford (1954)
20. Ivanova, E.A., Kirvtsov, A.M., Morozov, N.F.: Derivation of macroscopic relations of the elasticity of complex crystal lattices taking into account the moment interactions at the microlevel. Appl. Math. Mech. **71**(4), 543–561 (2007)