Lecture 1

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Mathematics necessary for description of continua with microstructure. Models of particles with complex structure as the base of modeling continua with microstructure.

1 The direct tensor calculus

1.1 The basic definitions

The tensor calculus is the mathematical base of continuum mechanics. There exist two approaches to the statement of tensor calculus. One of them is the coordinate tensor calculus and other one is the direct tensor calculus. In what follows we use the direct tensor calculus. Therefore a tensor is considered to be an element of the linear space which is obtained by the special multiplication of vector spaces. The formal product of two vectors $\mathbf{ab}$ is called the dyad. The tensor product is not commutative: $\mathbf{ab} \neq \mathbf{ba}$. The formal sum of several dyads

$$\mathbf{A} = \mathbf{ab} + \mathbf{cd} + \mathbf{ef} + ... \tag{1}$$

is called the second-rank tensor if

$$\begin{aligned} \mathbf{ab} + \mathbf{cd} &= \mathbf{cd} + \mathbf{ab}, \\ \mathbf{a}(\mathbf{b} + \mathbf{c}) &= \mathbf{ab} + \mathbf{ac}, \\ (\mathbf{a} + \mathbf{b})\mathbf{c} &= \mathbf{ac} + \mathbf{bc}, \\ (\alpha\mathbf{a})\mathbf{b} &= \mathbf{a}(\alpha\mathbf{b}). \end{aligned} \tag{2}$$

The following linear operations are introduced for the second-rank tensors:

$$\begin{aligned} \mathbf{A} = \mathbf{ab} + \mathbf{cd}, \qquad \mathbf{B} = \mathbf{de} + \mathbf{gh}, \\ \mathbf{S} = \mathbf{A} + \mathbf{B} = \mathbf{ab} + \mathbf{cd} + \mathbf{de} + \mathbf{gh}, \\ \alpha\mathbf{A} = (\alpha\mathbf{a})\mathbf{b} + (\alpha\mathbf{c})\mathbf{d} = \mathbf{a}(\alpha\mathbf{b}) + \mathbf{c}(\alpha\mathbf{d}). \end{aligned} \tag{3}$$

The zero second-rank tensor is defined as $\mathbf{O} = \mathbf{oa} = \mathbf{ao}$, where $\mathbf{o}$ is the zero vector.

The second-rank tensor $\mathbf{E}$ is called the unit tensor if for an arbitrary vector $\mathbf{x}$ it satisfies the following equation:

$$\mathbf{E} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{E} = \mathbf{x}. \tag{4}$$

For any non-degenerate tensor $\mathbf{A}$ there exists single inverse tensor $\mathbf{A}^{-1}$ which is defined as a solution of the equation

$$\mathbf{A}\cdot\mathbf{A}^{-1}=\mathbf{A}^{-1}\cdot\mathbf{A}=\mathbf{E}. \tag{5}$$

The formal product of three vectors $\mathbf{abc}$ is called the triad. The formal sum of several triads

$${}^{3}\mathbf{A}=\mathbf{abc}+\mathbf{def}+... \tag{6}$$

is the third-rank tensor. The tensors of rank n are introduced analogously.

1.2 The basic operations with tensors

Let us represent tensors $\mathbf{A}$ and $\mathbf{B}$ as follows: $\mathbf{A}=\sum\limits_k \mathbf{a}_k\mathbf{b}_k$, $\mathbf{B}=\sum\limits_l \mathbf{d}_l\mathbf{f}_l$.

The scalar product of tensor and vector:

$$\mathbf{A}\cdot\mathbf{c}=\sum_k \mathbf{a}_k\left(\mathbf{b}_k\cdot\mathbf{c}\right), \qquad \mathbf{c}\cdot\mathbf{A}=\sum_k\left(\mathbf{c}\cdot\mathbf{a}_k\right)\mathbf{b}_k. \tag{7}$$

The vector product of tensor and vector:

$$\mathbf{A}\times\mathbf{c}=\sum_k \mathbf{a}_k\left(\mathbf{b}_k\times\mathbf{c}\right), \qquad \mathbf{c}\times\mathbf{A}=\sum_k\left(\mathbf{c}\times\mathbf{a}_k\right)\mathbf{b}_k. \tag{8}$$

The tensor product of tensor and vector:

$$\mathbf{Ac}=\sum_k \mathbf{a}_k\mathbf{b}_k\mathbf{c}, \qquad \mathbf{cA}=\sum_k \mathbf{ca}_k\mathbf{b}_k. \tag{9}$$

The scalar product of tensors:

$$\mathbf{A}\cdot\mathbf{B}=\left(\sum_k \mathbf{a}_k\mathbf{b}_k\right)\cdot\left(\sum_l \mathbf{d}_l\mathbf{f}_l\right)=\sum_{k,l}(\mathbf{b}_k\cdot\mathbf{d}_l)\mathbf{a}_k\mathbf{f}_l. \tag{10}$$

The vector product of tensors:

$$\mathbf{A}\times\mathbf{B}=\left(\sum_k \mathbf{a}_k\mathbf{b}_k\right)\times\left(\sum_l \mathbf{d}_l\mathbf{f}_l\right)=\sum_{k,l}\mathbf{a}_k(\mathbf{b}_k\times\mathbf{d}_l)\mathbf{f}_l. \tag{11}$$

The tensor product of tensors:

$$\mathbf{AB}=\sum_{k,l}\mathbf{a}_k\mathbf{b}_k\mathbf{d}_l\mathbf{f}_l. \tag{12}$$

The double scalar product of tensors:

$$\mathbf{A}\cdot\cdot\,\mathbf{B}=\left(\sum_k \mathbf{a}_k\mathbf{b}_k\right)\cdot\cdot\left(\sum_l \mathbf{d}_l\mathbf{f}_l\right)=\sum_{k,l}(\mathbf{b}_k\cdot\mathbf{d}_l)(\mathbf{a}_k\cdot\mathbf{f}_l). \tag{13}$$

The double vector product of tensors:

$$\mathbf{A} \times\times \mathbf{B} = \left(\sum_k \mathbf{a}_k \mathbf{b}_k\right) \times\times \left(\sum_l \mathbf{d}_l \mathbf{f}_l\right) = \sum_{k,l} (\mathbf{b}_k \times \mathbf{d}_l)(\mathbf{a}_k \times \mathbf{f}_l). \tag{14}$$

The scalar-vector product of tensors:

$$\mathbf{A} \cdot\times \mathbf{B} = \left(\sum_k \mathbf{a}_k \mathbf{b}_k\right) \cdot\times \left(\sum_l \mathbf{d}_l \mathbf{f}_l\right) = \sum_{k,l} (\mathbf{b}_k \cdot \mathbf{d}_l)(\mathbf{a}_k \times \mathbf{f}_l). \tag{15}$$

The vector-scalar product of tensors:

$$\mathbf{A} \times\cdot \mathbf{B} = \left(\sum_k \mathbf{a}_k \mathbf{b}_k\right) \times\cdot \left(\sum_l \mathbf{d}_l \mathbf{f}_l\right) = \sum_{k,l} (\mathbf{a}_k \cdot \mathbf{f}_l)(\mathbf{b}_k \times \mathbf{d}_l). \tag{16}$$

The scalar $\mathrm{tr}\mathbf{A}$ calculated according to the rule

$$\mathrm{tr}\mathbf{A} = \sum_k \mathbf{a}_k \cdot \mathbf{b}_k \tag{17}$$

is called trace of tensor $\mathbf{A}$.

The vector $\mathbf{A}_\times$ calculated according to the rule

$$\mathbf{A}_\times = \sum_k \mathbf{a}_k \times \mathbf{b}_k \tag{18}$$

is called vector invariant of tensor $\mathbf{A}$.

1.3 The orthogonal tensors

The second-rank tensor $\mathbf{Q}$ satisfying the equation

$$\mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{E} \tag{19}$$

is called orthogonal tensor. The determinant of orthogonal tensor is calculated as follows:

$$\det(\mathbf{Q}^T \cdot \mathbf{Q}) = 1 \quad \Rightarrow \quad \det \mathbf{Q} = \pm 1. \tag{20}$$

The orthogonal tensor whose determinant is equal to $+1$ is called the properly orthogonal tensor or the rotation tensor.

1.4 The rotation tensor: representation and properties

Let us introduce two orthonormal bases: the starting basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and a new basis $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$. The rotation tensor $\mathbf{P}$ transferring the starting basis into the new basis can be represented in the form

$$\mathbf{P} = \mathbf{e}'_1 \mathbf{e}_1 + \mathbf{e}'_2 \mathbf{e}_2 + \mathbf{e}'_3 \mathbf{e}_3. \tag{21}$$

If we operate on a vector $\mathbf{a}$ by the rotation tensor $\mathbf{P}$ we obtain

$$\mathbf{a}' = \mathbf{P} \cdot \mathbf{a} = (\mathbf{e}'_k \mathbf{e}_k) \cdot (\mathbf{e}_m a_m) = a_k \mathbf{e}'_k, \tag{22}$$

where vector $\mathbf{a}'$ is called rotated vector. The rotated tensor $\mathbf{A}'$ is defined by the relation

$$\mathbf{A}' = \mathbf{a}'\mathbf{b}' + ... + \mathbf{c}'\mathbf{d}' = \mathbf{P} \cdot (\mathbf{ab} + ... + \mathbf{cd}) \cdot \mathbf{P}^T = \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{P}^T. \tag{23}$$

The following identities are useful:

$$(\mathbf{P} \cdot \mathbf{a}) \cdot (\mathbf{P} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}, \qquad (\mathbf{P} \cdot \mathbf{a}) \times (\mathbf{P} \cdot \mathbf{b}) = \mathbf{P} \cdot (\mathbf{a} \times \mathbf{b}). \tag{24}$$

1.5 Differentiation of tensor functions

Let $f(\mathbf{X})$ be a tensor function. If there exists such tensor $\partial f / \partial \mathbf{X} \in \mathcal{T}_{k+p}$ that for any tensor $\mathbf{A} \in \mathcal{T}_k$ the relation

$$\frac{\partial f}{\partial \mathbf{X}} \odot \mathbf{A}^T = \left. \frac{\partial f\left(\mathbf{X} + \alpha \mathbf{A}\right)}{\partial \alpha} \right|_{\alpha=0} \tag{25}$$

is valid then the tensor $\partial f / \partial \mathbf{X}$ is called the derivative of tensor function $f(\mathbf{X})$ with respect to tensor argument $\mathbf{X}$. The symbol $\odot$ denotes the operation $(\mathbf{abc}) \odot (\mathbf{def}) = (\mathbf{c} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{e})(\mathbf{a} \cdot \mathbf{f})$. The coordinate representation of the derivative of tensor function is

$$\frac{\partial f}{\partial \mathbf{X}} = \frac{\partial f_{p \ldots r}(X_{m \ldots n})}{\partial X_{m \ldots n}} \mathbf{e}_p \ldots \mathbf{e}_r \mathbf{e}_m \ldots \mathbf{e}_n \tag{26}$$

where $\mathbf{e}_k$ are the basis vectors. Notice that the representation (26) is valid only when the components of tensor $X_{m \ldots n}$ are independent.

1.6 The tensor fields

Suppose that the tensor field is given by function $\mathbf{U}(\mathbf{r})$ where $\mathbf{r} = x_k \mathbf{e}_k$ is the position vector. The differential of $\mathbf{U}(\mathbf{r})$ can be written as

$$d\mathbf{U} = d\mathbf{r} \cdot (\nabla \mathbf{U}), \qquad \nabla = \mathbf{e}_k \frac{\partial}{\partial x_k} \tag{27}$$

where ∇ is the gradient operator. The divergence of $\mathbf{U}(\mathbf{r})$ and the rotor of $\mathbf{U}(\mathbf{r})$ are determined as

$$\nabla \cdot \mathbf{U} = \mathbf{e}_k \cdot \frac{\partial \mathbf{U}}{\partial x_k}, \qquad \nabla \times \mathbf{U} = \mathbf{e}_k \times \frac{\partial \mathbf{U}}{\partial x_k}. \tag{28}$$

Let us consider a volume V bounded by the surface S with external normal vector $\mathbf{n}$. The continuously differentiable tensor field $\mathbf{U}(\mathbf{r})$ is assumed to be defined in the volume V. Then the divergence theorem is formulated as follows:

$$\int_V \nabla \cdot \mathbf{U} \, dV = \int_S \mathbf{n} \cdot \mathbf{U} \, dS. \tag{29}$$

2 Rigid bodies and dynamical structures

2.1 Kinematics of rigid bodies

In what follows we consider the models of continuum with the rotational degrees of freedom. That is why we are starting from the description of motion of rigid bodies and determination of the dynamical structures of rigid bodies.

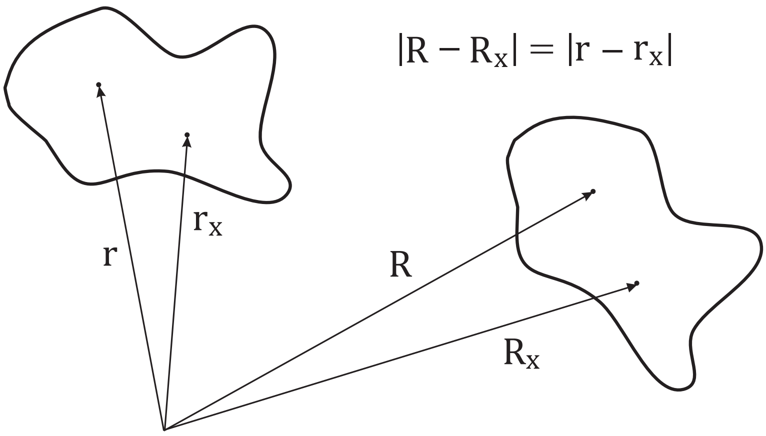

Figure 1: Kinematics of a rigid body

Definition. *If the distance between any two points of a body $\mathcal{A}$ does not change during the body motion then the body $\mathcal{A}$ is called the rigid body.*

From the definition of the rigid body it follows that the motion of the rigid body is completely determined by the position vector $\mathbf{R}_X(t)$ of an arbitrarily chosen point X which is fixed in the body, and the rotation tensor $\mathbf{P}(t)$ characterizing the rotational motion of the rigid body. The point X is called the pole.

The fundamental theorem of the rigid body kinematics is formulated as follows:

$$\begin{gathered}\mathbf{R}(t) = \mathbf{R}_X(t) + \mathbf{P}(t) \cdot (\mathbf{r} - \mathbf{r}_X)\,, \\ \mathbf{r} = \mathbf{R}(t_0), \qquad \mathbf{r}_X = \mathbf{R}_X(t_0), \qquad \mathbf{P}(t_0) = \mathbf{E},\end{gathered} \tag{30}$$

where $\mathbf{R}(t)$ is the position vector of some point of the rigid body.

The time change of the position vector is characterized by the velocity vector $\mathbf{V}(t) = \dot{\mathbf{R}}(t)$. The time change of the rotation tensor is characterized by the tensor $\dot{\mathbf{P}}(t)$. However, using of the spin tensor

$$\mathbf{S}(t) = \dot{\mathbf{P}}(t) \cdot \mathbf{P}^T(t) \tag{31}$$

is more convenient. The spin tensor is the antisymmetric tensor. Any antisymmetric tensor can be represented by means of the accompanying vector:

$$\mathbf{S}(t) = \boldsymbol{\omega}(t) \times \mathbf{E}. \tag{32}$$

Definition. *The accompanying vector $\boldsymbol{\omega}(t)$ of the spin tensor $\mathbf{S}(t)$ is called the angular velocity vector.*

The angular velocity vector satisfies the equation by Poisson:

$$\dot{\mathbf{P}}(t) = \boldsymbol{\omega}(t) \times \mathbf{P}(t). \tag{33}$$

2.2 Dynamical structures of rigid bodies

Kinetic energy, momentum and angular momentum are called the Dynamical structures of a body. Now we determine this quantities for the rigid body.

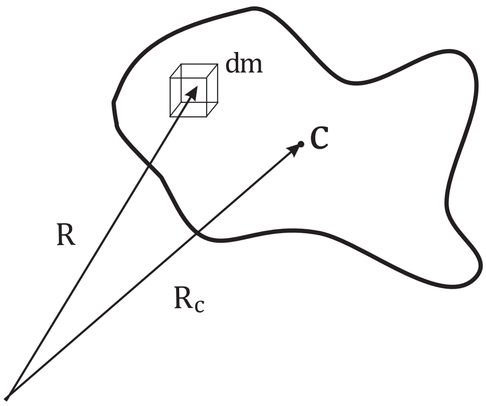

Figure 2: Dynamical structures of a rigid body

The elementary mass dm whose position is determined by the position vector $\mathbf{R}(t)$ is assumed to be a point mass. Therefore the kinetic energy $K(\mathcal{A})$ of the rigid body, its momentum $\mathbf{K}_1(\mathcal{A})$ and its angular momentum $\mathbf{K}_2^Q(\mathcal{A})$ calculated with respect to the the point Q have the following form

$$K(\mathcal{A}) = \frac{1}{2}\int_{(m)} \mathbf{V}(t)\cdot\mathbf{V}(t)\,dm, \qquad \mathbf{K}_1(\mathcal{A}) = \int_{(m)} \mathbf{V}(t)\,dm,$$
$$\mathbf{K}_2^Q(\mathcal{A}) = \int_{(m)} \big(\mathbf{R}(t)-\mathbf{R}_Q\big)\times\mathbf{V}(t)\,dm. \tag{34}$$

After transformations we obtain

$$K(\mathcal{A}) = \frac{1}{2}m\,\mathbf{V}_X\cdot\mathbf{V}_X + \mathbf{V}_X\cdot m\mathbf{B}_X\cdot\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{C}_X\cdot\boldsymbol{\omega},$$
$$\mathbf{K}_1(\mathcal{A}) = m\,\mathbf{V}_X + m\mathbf{B}_X\cdot\boldsymbol{\omega}, \tag{35}$$
$$\mathbf{K}_2^Q(\mathcal{A}) = (\mathbf{R}(t)-\mathbf{R}_Q)\times\mathbf{K}_1(\mathcal{A}) + \mathbf{V}_X\cdot m\mathbf{B}_X + \mathbf{C}_X\cdot\boldsymbol{\omega}.$$

Here m is the mass of body $\mathcal{A}$; $m\mathbf{B}_X$ and $\mathbf{C}_X$ are the inertia tensors of body $\mathcal{A}$ determined by the formulas

$$\mathbf{B}_X = \big[\mathbf{R}_X(t)-\mathbf{R}_C(t)\big]\times\mathbf{E} = \mathbf{P}(t)\cdot\big[(\mathbf{r}_X-\mathbf{r}_C)\times\mathbf{E}\big]\cdot\mathbf{P}^T(t),$$
$$\mathbf{C}_X = \mathbf{P}(t)\cdot\int_{(m)}\Big[(\mathbf{r}-\mathbf{r}_X)^2\,\mathbf{E}-(\mathbf{r}-\mathbf{r}_X)\,(\mathbf{r}-\mathbf{r}_X)\Big]dm\cdot\mathbf{P}^T(t). \tag{36}$$

where vector $\mathbf{R}_C(t)$ determines the actual position of the mass center of body $\mathcal{A}$, and $\mathbf{r}_C = \mathbf{R}_C(t_0)$.

Tensor $m\mathbf{B}_X$ is antisymmetric one and its value is determined by the mass of a body and the radius-vector that extends from a pole to the mass center. The pole coinciding with the mass center, tensor $m\mathbf{B}_X$ is equal to zero.

3 A body-point as the base model in the continuum mechanics

3.1 Dynamical structures of a body-point

Constructing a model of continuum we will use a body-point as the base material object. The body-point, unlike a point mass, undergoes to not only translational but also rotational motions. The body-point is the material object occupying zero volume in space. Position of a body-point is considered to be determined if the position vector $\mathbf{R}(t)$ and the rotation tensor $\mathbf{P}(t)$ are assigned.

Definition. The kinetic energy of a body-point is a quadratic form of its translational and angular velocities:

$$K = \frac{1}{2} m\, \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot m\mathbf{B} \cdot \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\omega} \cdot m\mathbf{J} \cdot \boldsymbol{\omega}. \tag{37}$$

Here the second-rank tensors $m\mathbf{B}$, $m\mathbf{J}$ are the inertia tensors of a body-point and m is the mass of a body-point respectively. The inertia tensors are frame-indifferent characteristics of a body-point, therefore they should depend on rotation tensor $\mathbf{P}(t)$ as

$$m\mathbf{B}(t) = \mathbf{P}(t) \cdot m\mathbf{B}_0 \cdot \mathbf{P}^T(t), \qquad m\mathbf{J}(t) = \mathbf{P}(t) \cdot m\mathbf{J}_0 \cdot \mathbf{P}^T(t), \tag{38}$$

where $m\mathbf{B}_0$, $m\mathbf{J}_0$ are the inertia tensors at the reference position, i. e. for those values t_0 at which $\mathbf{P}(t_0) = \mathbf{E}$.

Definition. The momentum of a body-point is the linear form of its translational and angular velocities:

$$\mathbf{K}_1 = \frac{\partial K}{\partial \mathbf{v}} = m\, \mathbf{v} + m\mathbf{B} \cdot \boldsymbol{\omega}. \tag{39}$$

Definition. The proper angular momentum (dynamic spin) of a body-point is the linear form of its translational and angular velocities:

$$\mathbf{K}_2 = \frac{\partial K}{\partial \boldsymbol{\omega}} = \mathbf{v} \cdot m\mathbf{B} + m\mathbf{J} \cdot \boldsymbol{\omega}. \tag{40}$$

Definition. The angular momentum of a body-point calculated with respect to fixed reference point Q is defined by the following formula:

$$\mathbf{K}_2^Q = (\mathbf{R} - \mathbf{R}_Q) \times \frac{\partial K}{\partial \mathbf{v}} + \frac{\partial K}{\partial \boldsymbol{\omega}}. \tag{41}$$

The first term on the right-hand side of Eq. (41) is the moment of momentum and the second one is the dynamic spin.

3.2 A point mass and the infinitesimal rigid body in continuum mechanics

In the momentless theories of continua (such as the classical theories of elasticity, viscoelasticity and plasticity) the elementary volume of a continuum is considered to be point mass. In the moment theories of continua (such as the rod theory, the shell theory, the 3D Cosserat continuum, etc.) the elementary volume of a continuum is considered to be infinitesimal rigid body. Thus inertia tensors in the continuum mechanics have the same structure as the inertia tensors of macroscopic rigid bodies.

The theory of rectilinear beams and curvilinear rods. In the case of rectilinear beam the mass center of a cross-section is on the middle line. Therefore vector $\mathbf{R}(s)$ characterizing the position of the point of rod determines the position of the cross-section mass center. Hence the inertia tensor $\mathbf{B}$ is equal to zero. In the case of curvilinear rod the mass center of a cross-section is situated not on the middle line. Then the inertia tensor $\mathbf{B}$ is not equal to zero and has the form: $\mathbf{B} = \big[\mathbf{R}(s) - \mathbf{R}_C(s)\big] \times \mathbf{E}$.

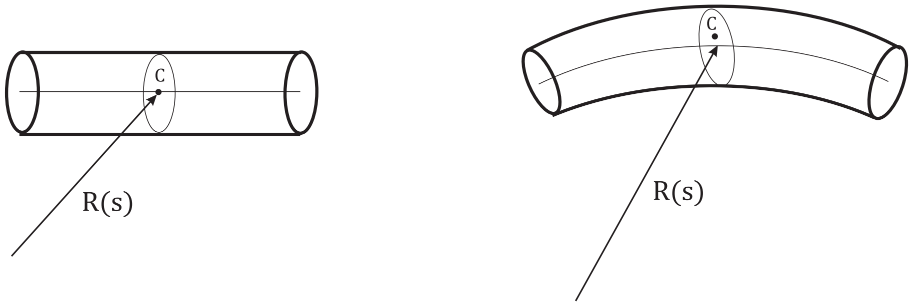

Figure 3: Rectilinear beam and curvilinear rod

The theory of plates and shells. In the case of plate the mass center of a filament is on the middle plane. Hence $\mathbf{B} = 0$. In the case of shell the mass center of a filament is situated on the middle surface. Then $\mathbf{B}$ is not equal to zero and has the form: $\mathbf{B} = \big[\mathbf{R}(x_1, x_2) - \mathbf{R}_C(x_1, x_2)\big] \times \mathbf{E}$.

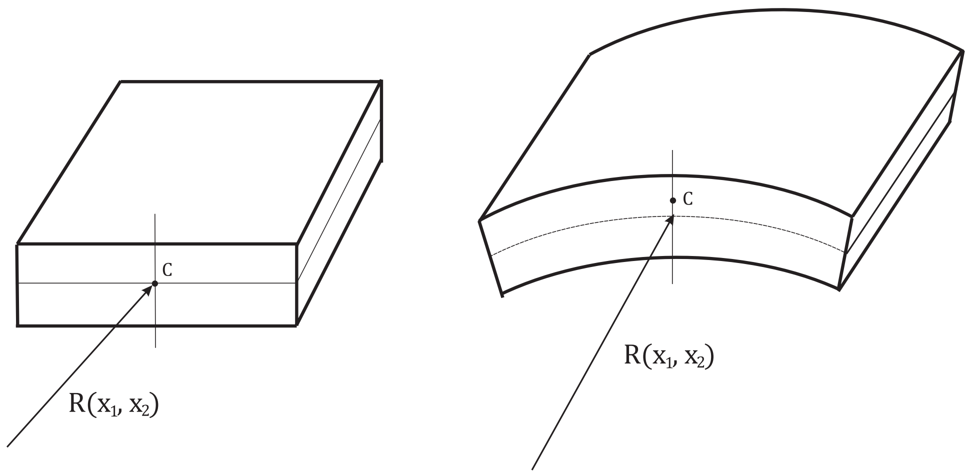

Figure 4: Plate and shell

4 Body-point of a general kind

4.1 A body-point different from the infinitesimal rigid body

Let us consider a body-point whose inertia tensors are the spherical part of tensors and the kinetic energy has the form

$$K = m\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \hat{B}\,\mathbf{v}\cdot\boldsymbol{\omega} + \frac{1}{2}\hat{J}\,\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right). \tag{42}$$

Here m is the mass of a body-point, $\hat{B}$ and $\hat{J}$ are the moments of inertia. The momentum and the proper angular momentum of a body-point are

$$\mathbf{K}_1 = m\,(\mathbf{v} + \hat{B}\,\boldsymbol{\omega}), \qquad \mathbf{K}_2 = m\,(\hat{B}\,\mathbf{v} + \hat{J}\,\boldsymbol{\omega}). \tag{43}$$

It is important to notice that the body-point (42), (43) differs from the infinitesimal rigid body by the additional parameter $\hat{B}$ which equals to zero in the case of rigid body. For the first time the body-point (42), (43) has been introduced by P. A. Zhilin.

4.2 Ground of the model of body-point of a general kind

We consider the material system (see Fig. 5) consisting of the frame and N rigid bodies attached to the frame by means of elastic springs. For simplicity we suppose that all bodies can move only in the line of axis x and rotate only on axis x.

We introduce following notations: m, J, x, φ are the mass, the moment of inertia, the displacement and the angle of rotation of the frame; m_i, J_i are the mass and the moment of inertia of rigid body number i; x_i, φ_i are the displacement and the angle of rotation of rigid body number i relative to the frame.

The springs are considered to be elastic helical lines whose property consists in the fact that when twisting in one direction they become longer and when twisting in the opposite direction they shorten. Conformably, when stretching and pressing the springs become twisted in different directions. We suppose that the internal energy U_i of spring number i as well as the force F_i and the twisting moment M_i modeling the influence of spring number i on the frame take the form:

$$\begin{gathered} U_i = U_i(x_i + \chi\varphi_i), \\ F_i = \frac{\partial U_i}{\partial x_i}, \qquad M_i = \frac{\partial U_i}{\partial \varphi_i}, \end{gathered} \tag{44}$$

where χ is the coefficient, characterizing the difference of the elastic spring under consideration from analogous spring possessing the axial symmetry. Objects similar to considered spring are usually called chiral objects. Therefore we call χ by coefficient of chirality.

As evident from Eq. (44), the force and the twisting moment can be represented by means of derivative of the internal energy with respect to its argument $x_i + \chi\varphi_i$.

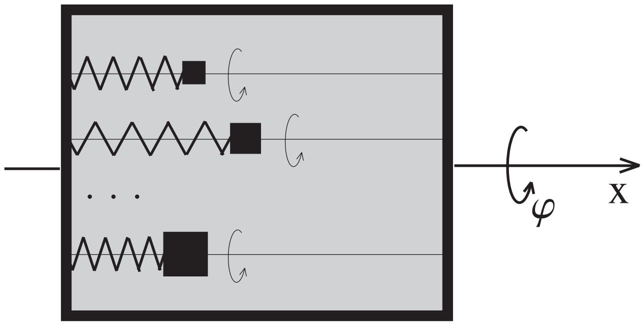

Figure 5: Particle possessing the internal structure

(In what follows we denote this derivative by stroke.) As a result the simple relation between M_i and F_i can be brought to light:

$$F_i = U_i', \qquad M_i = \chi U_i' \qquad \Rightarrow \qquad M_i = \chi F_i. \tag{45}$$

The equations of motion of the frame have the form:

$$m\ddot{x} = F + \sum_{i=1}^{N} F_i, \qquad J\ddot{\varphi} = M + \sum_{i=1}^{N} M_i, \tag{46}$$

where F and M are the external force and the external twisting moment acting on the frame. The equations of motion of the rigid bodies are:

$$m_i(x + x_i)^{\cdot\cdot} = -F_i, \qquad J_i(\varphi + \varphi_i)^{\cdot\cdot} = -M_i, \qquad i = \overline{1, N}. \tag{47}$$

Analysis of Eqs. (46), (47) shows that the translational and rotational motion of the frame are interdependent. If there is no external moment acting on the frame and the initial angular velocity is equal to zero, then because of the internal dynamics of the system the frame become to rotate. If there is no external force acting on the frame and the initial velocity is equal to zero, then because of the internal dynamics of the system the frame become to move.

Example. We consider the motion of the system represented in Fig. 5 under the action of the external force and twisting moment being linear time functions:

$$F = A_F t, \qquad M = A_M t, \qquad A_F = \text{const}, \qquad A_M = \text{const}. \tag{48}$$

Taking into account (48) we write the equation of the frame motion (46) in the form:

$$m\ddot{x} = A_F t + \sum_{i=1}^{N} F_i, \qquad J\ddot{\varphi} = A_M t + \sum_{i=1}^{N} M_i. \tag{49}$$

For simplicity we suppose that all rigid bodies have the same masses $m_i = m_*/N$ and the same moments of inertia $J_i = J_*/N$. Moreover, we suppose that the internal energies of the springs U_i are the quadratic forms of deformations and all springs have stiffness equal c. In that case:

$$U_i = c\,(x_i + \chi\varphi_i)^2 \qquad \Rightarrow$$
$$\Rightarrow \qquad F_i = c\,(x_i + \chi\varphi_i), \qquad M_i = \chi c\,(x_i + \chi\varphi_i). \tag{50}$$

Taking into account (50) we rewrite the equations of the motion of the rigid bodies (47) in the form:

$$\begin{aligned} &\frac{m_*}{N}\,(x+x_i)^{\cdot\cdot} = -c\,(x_i+\chi\varphi_i),\\ &\frac{J_*}{N}\,(\varphi+\varphi_i)^{\cdot\cdot} = -\chi c\,(x_i+\chi\varphi_i). \end{aligned} \tag{51}$$

From Eq. (51) we obtain:

$$(x_i+\chi\varphi_i)^{\cdot\cdot} + k_*^2\,(x_i+\chi\varphi_i) = -(x+\chi\varphi)^{\cdot\cdot}, \qquad k_*^2 = Nc\left(\frac{1}{m_*}+\frac{\chi^2}{J_*}\right). \tag{52}$$

By using Eqs. (50) for the forces F_i and the moments M_i the equations of motion of the frame (49) can be reduced to the equivalent system including the equation

$$m\ddot{x} - \frac{J}{\chi}\,\ddot{\varphi} = A_F t - \frac{A_M t}{\chi} \tag{53}$$

and the equation

$$(x+\chi\varphi)^{\cdot\cdot} = \frac{A_F t}{m} + \frac{\chi A_M t}{J} + \frac{\tilde{k}^2}{N}\sum_{i=1}^{N}(x_i+\chi\varphi_i). \tag{54}$$

From Eqs. (52), (54) we obtain the equation in $x+\chi\varphi$:

$$(x+\chi\varphi)^{\cdot\cdot\cdot\cdot} + k^2(x+\chi\varphi)^{\cdot\cdot} = k_*^2\left(\frac{A_F t}{m}+\frac{\chi A_M t}{J}\right). \tag{55}$$

Solving Eq. (55) we get the following expression for variable $(x+\chi\varphi)^{\cdot\cdot}$

$$(x+\chi\varphi)^{\cdot\cdot} = (x+\chi\varphi)\Big|_{t=0}\cos(kt) + \frac{1}{k}(x+\chi\varphi)^{\cdot}\Big|_{t=0}\sin(kt) + \frac{k_*^2}{k^2}\left(\frac{A_F t}{m}+\frac{\chi A_M t}{J}\right). \tag{56}$$

Now we suppose that the oscillation period is much smaller than an observing time on the motion process. In that case the characteristics of the motion averaged over a period is interested for us:

$$\bar{x}(t) = \frac{k}{2\pi}\int_{t-\pi/k}^{t+\pi/k} x(\tau)d\tau, \qquad \bar{\varphi}(t) = \frac{k}{2\pi}\int_{t-\pi/k}^{t+\pi/k} \varphi(\tau)d\tau. \tag{57}$$

By averaging over a period Eqs. (53), (56) we obtain:

$$\begin{aligned} &m\ddot{\bar{x}} - \frac{J}{\chi}\,\ddot{\bar{\varphi}} = A_F t - \frac{A_M t}{\chi},\\ &\ddot{\bar{x}} + \chi\ddot{\bar{\varphi}} = \frac{k_*^2}{k^2}\left(\frac{A_F t}{m}+\frac{\chi A_M t}{J}\right). \end{aligned} \tag{58}$$

Now we transform the system (58) to the following form:

$$
\begin{aligned}
&m\left(1+\frac{Jk^2}{\chi^2 m k_*^2}\right)\left(1+\frac{J}{\chi^2 m}\right)^{-1}\ddot{x}+\frac{J\tilde{k}^2}{\chi k_*^2}\left(1+\frac{J}{\chi^2 m}\right)^{-1}\ddot{\varphi}=A_F t,\\
&\frac{\chi m\tilde{k}^2}{k_*^2}\left(1+\frac{J}{\chi^2 m}\right)^{-1}\ddot{x}+J\left(1+\frac{Jk^2}{\chi^2 m k_*^2}\right)\left(1+\frac{J}{\chi^2 m}\right)^{-1}\ddot{\varphi}=A_M t.
\end{aligned}
\tag{59}
$$

Let us suppose that the mass and the moment of inertia of the frame are related by the formula

$$
J=\chi^2 m. \tag{60}
$$

We introduce following notations:

$$
\hat{m}=\frac{m}{2}\left(1+\frac{k^2}{k_*^2}\right),\qquad \hat{B}=\frac{\chi m\tilde{k}^2}{2k_*^2},\qquad \hat{J}=\frac{J}{2}\left(1+\frac{k^2}{k_*^2}\right). \tag{61}
$$

Taking into account Eqs. (60), (61) we rewrite the system (59) in the form:

$$
\hat{m}\ddot{x}+\hat{B}\ddot{\varphi}=A_F t,\qquad \hat{B}\ddot{x}+\hat{J}\ddot{\varphi}=A_M t. \tag{62}
$$

When the comparison of Eq. (62) describing the behavior of the average over a period characteristics of the motion with the starting Eq. (49) is carried out we see that the influence of the internal structure of the system on the motion of the frame can be taken into account both by means of the internal forces and moments and with the aid of the additional inertial parameters ensuring the interplay of the translational and rotational motions.

4.3 Definition of the body-point of a general kind

The body-point of a general kind is defined by its kinetic energy which is a quadratic form of translational and angular velocities of the body-point:

$$
K=\frac{1}{2}\mathbf{v}\cdot m\mathbf{A}\cdot\mathbf{v}+\mathbf{v}\cdot m\mathbf{B}\cdot\boldsymbol{\omega}+\frac{1}{2}\boldsymbol{\omega}\cdot m\mathbf{J}\cdot\boldsymbol{\omega}. \tag{63}
$$

Here the second-rank tensors $m\mathbf{A}$, $m\mathbf{B}$, $m\mathbf{J}$ are the inertia tensors of a body-point which depend on the rotation tensor $\mathbf{P}(t)$ as

$$
\begin{aligned}
m\mathbf{A}(t)&=\mathbf{P}(t)\cdot m\mathbf{A}_0\cdot\mathbf{P}^T(t), \qquad \mathbf{A}_0=\mathbf{A}_0^T,\\
m\mathbf{B}(t)&=\mathbf{P}(t)\cdot m\mathbf{B}_0\cdot\mathbf{P}^T(t), \qquad \mathbf{B}_0 \text{ is arbitrary tensor},\\
m\mathbf{J}(t)&=\mathbf{P}(t)\cdot m\mathbf{J}_0\cdot\mathbf{P}^T(t), \qquad \mathbf{J}_0=\mathbf{J}_0^T.
\end{aligned}
\tag{64}
$$

Here $m\mathbf{A}_0$, $m\mathbf{B}_0$, $m\mathbf{J}_0$ are the inertia tensors at the reference position.

Constructing model of continuum we can use the body-point of a general kind as the base material object. The body-points of a general kind possess the additional inertia characteristics as compared to the infinitesimal rigid body. Therefore the continual models based on using body-points of a general kind will possess some additional properties as compared to the known continual models.

5 Particles with internal degrees of freedom

5.1 One-rotor gyrostat

The one-rotor gyrostat is a complex object which consists of the carrier body and the rotor (see Fig. 6). The rotor can rotate independently of rotation of the carrier body, but it can not translate relative to the carrier body.

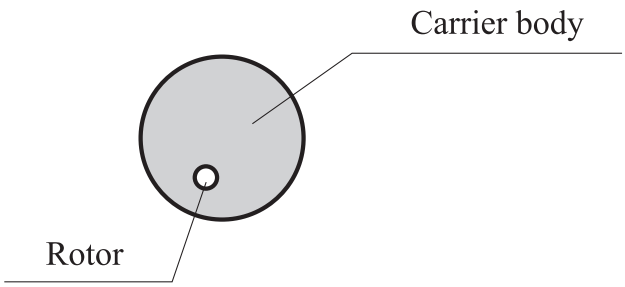

Figure 6: One-rotor gyrostat

5.2 Multi-rotor gyrostat

The multi-rotor gyrostat is the complex object (see Fig. 7) consisting of the carrier body and the rotors inside of the carrier body.

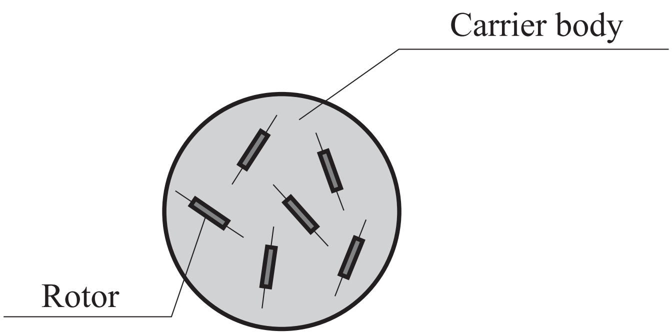

Figure 7: Multi-rotor gyrostat

5.3 Complex particles in continuum mechanics

It is the presence of additional rotational degrees of freedom and, accordingly, additional inertia and elastic characteristics which can be interpreted as the non-mechanical constants that distinguish the models based on the complex particles among other continual models. If the mathematical description of some continual model can be reduced to the known physical equations (for example, the heat conduction equation, the electrodynamics equations, etc.) then the continual model can be considered as the mechanical model of the corresponding physical process.