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MODIFIED ENERGY FUNCTIONAL FOR THE REISSNER THEORY OF PLATES

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The solution to the plate bending problem in the Reissner theory is known to include rapidly varying functions of boundary layer type. Since the theory of plates provides exact solutions only for some exceptional cases, numerical methods, mostly based on variational principles, are of great importance. From the purely formal viewpoint, the minimum energy principle in the Reissner theory of plates seems attractive since it can be reduced to minimizing a convex functional of the first derivatives of the unknown functions. However, this functional includes variables of different asymptotic orders, which is a serious disadvantage in the numerical implementation. Actually, the solution to the boundary value problem has the property that the leading asymptotic term in the functional vanishes, whereas the unknown functions must satisfy certain constraints that should be satisfied exactly for the leading terms of their asymptotic expansions. Since, in general, this cannot be achieved by solving the problem numerically, substantial qualitative and quantitative errors are likely to occur. That is why the Kirchhoff theory of plates is preferred in practice. In the latter theory, some variables are *a priori* calculated with an error that does not tend to zero with the relative thickness of the plate. We suggest a functional that combines the advantages of the Kirchhoff functional with the possibility of finding all unknown functions with a relative error of $O(h^2)$.

Our approach is based on the fact that the overall solution combines slowly varying components that penetrate deeply into the plate domain with boundary layer terms that rapidly decay away from the boundary. The boundary layer equation admits a simple asymptotic solution containing slowly varying functions, which are determined by the equation only on the plate contour. Thus, the original functional can be expressed (with taking account of the structure of the boundary-layer part) via the slowly varying functions alone, which simplifies the numerical solution dramatically.

1. MAIN EQUATIONS OF THE REISSNER THEORY OF PLATES

We consider the problem of plate bending under a distributed transverse load $p(x, y)$. The main equations are used in the form given in [1]. Let us introduce the following characteristics of the stress-strain state of the plate: the transverse deflection w, the vector of rotation angles $\boldsymbol{\psi}$, the vector of transverse stress resultants $\mathbf{N}$, and the tensor of stress couples $\mathbf{M}$. These variables are related to the displacements and stresses in the three-dimensional elasticity theory by the formulas

$$hw = \langle \mathbf{u} \cdot \mathbf{n} \rangle, \quad h^3 \boldsymbol{\psi} = \langle \mathbf{u} z \rangle, \quad \mathbf{N} = \langle \mathbf{a} \cdot \boldsymbol{\tau} \cdot \mathbf{n} \rangle, \quad \mathbf{M} = \langle \mathbf{a} \cdot \boldsymbol{\tau} \cdot \mathbf{a} z \rangle, \quad \mathbf{a} = \mathbf{E} - \mathbf{n}\mathbf{n}, \quad \langle f \rangle = \int_{-h/2}^{h/2} f\, dz. \quad (1.1)$$

Here $\mathbf{u}$ and $\boldsymbol{\tau}$ are the displacement vector and the stress tensor of the three-dimensional theory, h is the plate thickness, $\mathbf{n}$ is the unit normal to the plate plane, and $\mathbf{E}$ is the unit tensor. The complete system of equations comprises the equilibrium equations

$$\nabla \cdot \mathbf{N} + \rho h p = 0, \quad \nabla \cdot \mathbf{M} - \mathbf{N} = 0, \quad (1.2)$$

the elasticity relations

$$\mathbf{N} = Gh\Gamma\boldsymbol{\gamma}, \quad \mathbf{M} = D[(1-\mu)\boldsymbol{\kappa} + \mu \operatorname{tr} \boldsymbol{\kappa}\mathbf{a}], \tag{1.3}$$

and the geometric relations

$$\boldsymbol{\gamma} = \nabla w + \boldsymbol{\psi}, \quad \boldsymbol{\kappa} = \frac{1}{2}(\nabla\boldsymbol{\psi} + \nabla\boldsymbol{\psi}^T). \tag{1.4}$$

Here $\boldsymbol{\gamma}$ is the vector of transverse shear deformation, $\boldsymbol{\kappa}$ is the bending-torsion tensor, $D = \frac{1}{12}Eh^3/(1-\mu^2)$ is the bending rigidity, $Gh\Gamma$ is the transverse shear rigidity, $G = \frac{1}{2}E/(1+\mu)$, Γ is the coefficient of transverse shear, μ is Poisson's ratio, and ρ is the mass density.

The kinematical boundary conditions acquire the form

$$w|_C = w^*, \quad \boldsymbol{\nu}\cdot\boldsymbol{\psi}|_C = \varphi_\nu^*, \quad \boldsymbol{\tau}\cdot\boldsymbol{\psi}|_C = \varphi_\tau^*. \tag{1.5}$$

The force boundary conditions can be represented as follows:

$$\boldsymbol{\nu}\cdot\mathbf{N}|_C = \mathbf{N}_\nu^*, \quad \boldsymbol{\nu}\cdot\mathbf{M}\cdot\boldsymbol{\nu}|_C = M_\nu^*, \quad \boldsymbol{\nu}\cdot\mathbf{M}\cdot\boldsymbol{\tau}|_C = M_\tau^*. \tag{1.6}$$

Here φ_ν^* and φ_τ^* are the angles of rotation about the tangent and the normal to the plate contour, respectively, N_ν^* is the shear force, M_ν^* is the bending moment, M_τ^* is the torque, $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are the unit normal and tangent vectors to the plate contour, respectively; the vectors $\boldsymbol{\nu}$, $\boldsymbol{\tau}$, and $\mathbf{n}$ are assumed to form a right-handed system.

Introducing the potentials Φ and F reduces the plate bending equations to the more convenient form [1]

$$D\Delta\Delta\Phi + \rho h p = 0, \quad h^2\Delta F - 12\Gamma F = 0. \tag{1.7}$$

The characteristics of the stress-strain state of the plate are expressed via the potentials by the formulas

$$\begin{aligned} w &= -\Phi + \frac{h^2\Delta\Phi}{6\Gamma(1-\mu)}, \quad \boldsymbol{\psi} = \nabla\Phi + \nabla F\times\mathbf{n}, \\ \mathbf{M} &= D\Big[(1-\mu)\nabla\nabla\Phi + \mu\Delta\Phi\mathbf{a} + \frac{1}{2}(1-\mu)(\nabla\nabla F\times\mathbf{n} - \mathbf{n}\times\nabla\nabla F)\Big], \\ \mathbf{N} &= D\nabla\Delta\Phi + Gh\Gamma\nabla F\times\mathbf{n}. \end{aligned} \tag{1.8}$$

In what follows all external loads are supposed to be slowly varying functions of the coordinates. Then the function Φ describes the solutions that penetrate into the interior of the plate domain and is a slowly varying function, that is, its derivatives in all directions have the same asymptotic order as Φ itself. The function F is a solution to an equation with a small parameter at the highest derivative [2], and, consequently, is a function of boundary layer type (that is, rapidly decays inside the domain and slowly varies along the boundary). Thus, F describes solutions decreasing away from the boundary and satisfies the asymptotic estimate $\partial F/\partial\nu \sim h^{-1}F$, $\partial F/\partial\tau \sim F$, where (ν, τ) is a local coordinate system on the boundary. Let us estimate the asymptotic order of the interior potential Φ and of the boundary layer potential F under external loads of the order $O(1)$: $\Phi \sim h^{-3}$, $F \sim h^{-1}$.

2. STATEMENT OF THE APPROXIMATE EQUATIONS AND THE BOUNDARY CONDITIONS IN THE REISSNER THEORY OF PLATES. COMPARISON WITH THE KIRCHHOFF THEORY

Let us carry out the asymptotic study of the equation for the boundary layer potential. Since the solution to the second equation in (1.7) rapidly decays (at the distance $2h$ from the plate edge it is virtually zero), the local coordinate system introduced on the plate contour is especially convenient to write out the equation provided that the contour curvature radius $R(\tau)$ is much greater than $2h$. In the local coordinate system the second equation in (1.7) reads

$$\frac{\partial^2 F}{\partial\nu^2} + \frac{1}{R}\frac{\partial F}{\partial\nu} + \frac{\partial^2 F}{\partial\tau^2} - \frac{12\Gamma}{h^2}F = 0 \quad (\nu < 0). \tag{2.1}$$

Set $\varepsilon = h/\delta$ ($\delta = \sqrt{12\Gamma}$) and $\nu = \varepsilon\eta$. In the new variables Eq. (2.1) becomes

$$\frac{\partial^2 F}{\partial\eta^2} + \frac{\varepsilon}{R}\frac{\partial F}{\partial\eta} + \frac{\varepsilon^2\partial^2 F}{\partial\tau^2} - F = 0. \tag{2.2}$$

The solution to Eq. (2.2) is sought in the form of an asymptotic series expansion in powers of ε:

$$F(\eta,\tau) = F_0(\eta,\tau) + \varepsilon F_1(\eta,\tau) + \cdots. \tag{2.3}$$

On substituting the expression (2.3) into (2.2) and on equating the coefficients of the powers of ε to zero, we obtain

$$\begin{aligned} &\frac{\partial^2 F_0}{\partial\eta^2} - F_0 = 0, \quad \frac{\partial^2 F_1}{\partial\eta^2} - F_1 = -\frac{1}{R}\frac{\partial F_0}{\partial\eta}, \\ &\frac{\partial^2 F_i}{\partial\eta^2} - F_i = -\frac{1}{R}\frac{\partial F_{i-1}}{\partial\eta} - \frac{\partial^2 F_{i-2}}{\partial\tau^2} \quad (i = 2,3,\ldots). \end{aligned} \tag{2.4}$$

The solution to Eqs. (2.4) has the form

$$\begin{aligned} &F_0(\eta,\tau) = f_0(\tau)\exp(\eta), \quad F_1(\eta,\tau) = \Big(f_1(\tau) - \frac{1}{2}f_0(\tau)\frac{\eta}{R}\Big)\exp(\eta), \\ &F_i(\eta,\tau) = f_i(\eta,\tau)\exp(\eta) \quad (i = 2,3,\ldots). \end{aligned} \tag{2.5}$$

Set $f(\tau) = f_0(\tau) + \varepsilon f_1(\tau)$; then, by Eqs. (2.3) and (2.5), we have

$$F(\eta,\tau) = \Big[f(\tau)\Big(1 - \frac{1}{2}\frac{\varepsilon\eta}{R}\Big) + O(\varepsilon^2)\Big]\exp(\eta). \tag{2.6}$$

Thus, the boundary layer potential F has the asymptotic representation

$$F(\nu,\tau) = f(\tau)\Big(1 - \frac{1}{2}\frac{\nu}{R}\Big)\exp\Big(\frac{\delta\nu}{h}\Big) \quad (\nu < 0), \tag{2.7}$$

where $f(\tau)$ is some function of the coordinate on the plate boundary and is determined by the boundary conditions. The dependence of the boundary layer potential in (2.7) on the coordinate along the normal to the plate boundary is given explicitly. The use of Eq. (2.7) for the boundary layer potential enables us to solve the plate bending problem with a relative asymptotic error $O(h^2)$. Hence, it makes little sense to retain the terms of higher order in the expressions for w, $\boldsymbol{\psi}$, $\mathbf{N}$, and $\mathbf{M}$.

Thus, the approximate solution to the plate bending problem is reduced to the integration of the first equation in (1.7) for the interior potential and to the determination of the function $f(\tau)$, which characterizes (according to (2.7)) the variation of the boundary layer potential along the plate contour. The asymptotics of the stress-strain state characteristics are

$$\begin{aligned} &w = -\Phi, \quad \boldsymbol{\psi} = \nabla\Phi - \frac{\delta}{h}f(\tau)\boldsymbol{\tau}\exp\Big(\frac{\delta}{h}\nu\Big), \\ &\mathbf{N} = D\nabla\Delta\Phi + Gh\Gamma\Big\{\Big[\Big(1 - \frac{\nu}{2R}\Big)f'(\tau) + \frac{\nu R'}{2R^2}f(\tau)\Big]\boldsymbol{\nu} - \frac{\delta}{h}\Big(1 - \frac{\nu}{2R} - \frac{h}{2R\delta}\Big)f(\tau)\boldsymbol{\tau}\Big\}\exp\Big(\frac{\delta}{h}\nu\Big), \\ &\mathbf{M} = D[(1-\mu)\nabla\nabla\Phi + \mu\Delta\Phi\mathbf{a}] \\ &\quad + \Big[D(1-\mu)\frac{\delta}{h}f'(\tau)(\boldsymbol{\nu}\boldsymbol{\nu} - \boldsymbol{\tau}\boldsymbol{\tau}) - Gh\Gamma\Big(1 - \frac{\nu}{2R} - \frac{2h}{R\delta}\Big)f(\tau)(\boldsymbol{\nu}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\nu})\Big]\exp\Big(\frac{\delta}{h}\nu\Big). \end{aligned} \tag{2.8}$$

The kinematical boundary conditions acquire the form

$$-\Phi|_C = w^*, \quad \frac{\partial\Phi}{\partial\nu}\Big|_C = \psi^*_\nu, \quad \frac{\partial\Phi}{\partial\tau}\Big|_C - \frac{\delta}{h}f(\tau) = \psi^*_\tau. \tag{2.9}$$

The boundary conditions for the forces and the moments read

$$\begin{aligned} &D\frac{\partial\Delta\Phi}{\partial\nu}\Big|_C + Gh\Gamma f'(\tau) = N^*_\nu, \\ &D\Big(\frac{\partial^2\Phi}{\partial\nu^2} + \frac{\mu}{R}\frac{\partial\Phi}{\partial\nu} + \mu\frac{\partial^2\Phi}{\partial\tau^2}\Big)\Big|_C + D(1-\mu)\frac{\delta}{h}f'(\tau) = M^*_\nu, \\ &D(1-\mu)\frac{\partial^2\Phi}{\partial\nu\partial\tau}\Big|_C - Gh\Gamma\Big(1 - \frac{2h}{R\delta}\Big)f(\tau) = M^*_\tau. \end{aligned} \tag{2.10}$$

It should be noted that, according to (2.8), the leading terms of the transversal forces and of the torques depend on the boundary layer potential. Thus, there is an error of the order $O(1)$ in the expressions given by the Kirchhoff theory for the transversal forces and torques in the vicinity of the boundary. Three types of boundary conditions, for which the leading term of the boundary layer potential is zero, are the exceptions: the kinematic boundary conditions if $\psi_\tau^* = -\partial w^*/\partial\tau$; the first and the third conditions in (2.8) and the second condition in (2.10), if $\psi_\tau^* = -\partial w^*/\partial\tau$; the first and the second conditions in (2.9) and the third condition in (2.10) if $M_\tau^* = D(1-\mu)[\partial\psi_\nu^*/\partial\tau + (1/R)\,\partial w^*/\partial\tau]$.

Thus, despite the fact that the Kirchhoff theory permits us to find the leading terms of the deflection, the angles of rotation, and the bending moments, the statement about its relative asymptotic error $O(h)$ needs to be revised since the transversal forces and the torques are evaluated at the boundary with an error in the leading terms.

We point out that the suggested approximate statement of the problem differs from the Kirchhoff theory in that it takes into account the transverse shear deformation in the vicinity of the boundary,

$$\gamma = -\frac{\delta}{h} f(\tau)\tau \exp\left(\frac{\delta\nu}{h}\right) \quad (\nu < 0), \tag{2.11}$$

which allows one to satisfy all of the three boundary conditions.

3. VARIATIONAL FORMULATION OF THE PLATE BENDING PROBLEM IN THE REISSNER THEORY

The energy of plate bending in the Reissner theory acquires the form

$$\Pi(w,\psi) = \int_{(\Delta S)} \left[\frac{1}{2}(\mathbf{M} \cdot\cdot\, \boldsymbol{\kappa} + \mathbf{N}\cdot\boldsymbol{\gamma}) - \rho h p w\right] dS - \int_C [M_\nu^*\psi_\nu + M_\tau^*\psi_\tau + N_\nu^* w]\, dC. \tag{3.1}$$

Here the tensor of moments $\mathbf{M}$, the vector of transversal forces $\mathbf{N}$, the bending-torsion tensor $\boldsymbol{\kappa}$, and the vector of transverse shear deformation $\boldsymbol{\gamma}$ are expressed via the vector of rotation angles $\boldsymbol{\psi}$ and the deflection w by formulas (1.3) and (1.4). The potential energy functional (3.1) attains the minimal value at the equilibrium configurations provided that w and $\boldsymbol{\psi}$ satisfy the kinematic boundary conditions (1.5) if they are imposed at all.

The direct use of the functional (3.1) for numerical calculations is hardly possible since it depends on functions rapidly varying in the vicinity of the boundary. The approximation of the rotation angles $\boldsymbol{\psi}$ by slowly varying functions, typical of the finite element method and some other methods, leads to substantial errors and to the necessity to refine the grid. The effective use of the functional (3.1) in numerical calculations is only possible under a quite special choice of the coordinate functions to approximate the rotation angles on the boundary. To avoid the difficulties related to this choice, one should explicitly take into account the boundary layer phenomenon.

4. MODIFIED ENERGY FUNCTIONAL IN THE REISSNER THEORY OF PLATES

To take account of the boundary layer phenomenon explicitly, we transform the functional (3.1) as follows:

i) We express the functional (3.1) via the interior potential and the boundary layer potential, thus separating the rapidly and the slowly varying functions;
ii) We transform the area integral for the boundary layer components of the functional into a contour integral by the divergence theorem;
iii) Finally, we substitute the expression (2.7) for the boundary layer potential into the functional.

This permits us to construct an energy functional in which the variation of the boundary layer potential along the normal to the contour is taken into account explicitly.

The modified energy functional is defined on the set of functions $\Phi(x,y)$ and $f(\tau)$ that satisfy the following conditions: the functions $\Phi(x,y)$ are continuous and twice continuously differentiable in the closed domain $\bar{S} = S + C$; the functions $f(\tau)$ are continuous and continuously differentiable on the curve C; both $\Phi(x,y)$ and $f(\tau)$ satisfy the kinematic boundary conditions (2.9) if they are imposed. The modified energy functional has the form

$$\Pi^*(\Phi, f) = \int_{(\Delta S)} \left\{ D\left[\frac{1}{2}(\Delta\Phi)^2 + (1-\mu)\left(\left(\frac{\partial^2\Phi}{\partial x \partial y}\right)^2 - \frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2\Phi}{\partial y^2}\right)\right] + \rho h p \Phi \right\} dS$$
$$+ \int_C \left[D(1-\mu)\frac{\delta}{h}\left(\frac{\partial\Phi}{\partial\nu}f'(\tau) + \frac{1}{R}\frac{\partial\Phi}{\partial\tau}f(\tau)\right) + Gh\Gamma\left(\frac{\delta}{2h} - \frac{1}{R}\right)f^2(\tau) \right.$$
$$\left. + N_\nu^*\Phi + M_\nu^*\frac{\partial\Phi}{\partial\nu} - M_\tau^*\left(\frac{\partial\Phi}{\partial\tau} - \frac{\delta}{h}f(\tau)\right)\right] dC. \qquad (4.1)$$

The Euler equations for the modified functional (4.1) coincide with the first equation in (1.7) and with the boundary conditions (2.10) for the forces and moments. In the equilibrium configurations, the modified functional (4.1) assumes stationary values. However, the functional is actually minimal for a majority of "nonpathological" problems.

Since the Euler equations for the functional (4.1) have been derived from the equations of the Reissner theory of plates under the condition that the terms of the relative order $O(h^2)$ are omitted, we can conclude that the functional (4.1) allows us to solve the bending problem with the relative accuracy $O(h^2)$. However, the energy functional (4.1) is not the strict asymptotic consequense of the functional (3.1), since it contains the term $Gh\Gamma R^{-1} f^2(\tau) = O(h^2)$, whereas the other $O(h^2)$ terms are not included.

5. BENDING OF A CIRCULAR PLATE BY A TORQUE UNIFORMLY DISTRIBUTED ALONG THE CONTOUR

Let us consider a plate of radius R subjected to a torque M_τ^* uniformly distributed along the boundary. The boundary conditions are given in the form

$$\boldsymbol{\nu}\cdot\mathbf{N}|_C = 0, \quad \boldsymbol{\nu}\cdot\mathbf{M}\cdot\boldsymbol{\nu}|_C = 0, \quad \boldsymbol{\nu}\cdot\mathbf{M}\cdot\boldsymbol{\tau}|_C = M_\tau^*. \qquad (5.1)$$

Since $M_\tau^* = \text{const}$, it follows that the problem is axisymmetric, that is, in the polar coordinates we have $\Phi = \Phi(r)$ and $F = F(r)$. The boundary problems for the interior potential and the boundary layer potential have the form

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2\Phi = 0, \quad \frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\Phi\Big|_{r=R} = 0, \quad \left(\frac{\partial^2}{r^2} + \frac{\mu}{r}\frac{\partial}{\partial r}\right)\Phi\Big|_{r=R} = 0, \qquad (5.2)$$

$$\left(\frac{\partial^2}{r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)F - \frac{12\Gamma}{h^2}F = 0, \quad \left(F - \frac{h^2}{6\Gamma}\frac{1}{r}\frac{\partial F}{\partial r}\right)\Big|_{r=R} = -\frac{M_\tau^*}{Gh\Gamma}. \qquad (5.3)$$

The function $\Phi \equiv 0$ is a solution to problem (5.2). The solution to problem (5.3) with the asymptotic error $O(h^2)$ is determined by the formula

$$F = -\frac{M_\tau^*}{Gh\Gamma}\frac{1 - \dfrac{1}{2}\dfrac{r-R}{R}}{1 - \dfrac{2h}{R\delta}}\exp\left(\frac{\delta}{h}(r-R)\right). \qquad (5.4)$$

Let us consider the solution to this problem with the aid of the energy functional (4.1). In the polar coordinates the functional acquires the form

$$\Pi^*(\Phi, f) = 2\pi\left\{\int_0^R D\left[\frac{1}{2}\left(\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r}\right)^2 - \frac{1-\mu}{r}\frac{\partial\Phi}{\partial r}\frac{\partial^2\Phi}{\partial r^2}\right] r\,dr + Gh\Gamma\left(\frac{\delta}{2h} - \frac{1}{R}\right)f^2 + M_\tau^*\frac{\delta}{h}f\right\}. \qquad (5.5)$$

The stationary conditions for the functional (5.5) lead to the equations

$$\Phi \equiv 0, \quad f = -\frac{M_\tau^*}{Gh\Gamma}\frac{1}{1 - \dfrac{2h}{R\delta}}. \qquad (5.6)$$

Taking into account Eq. (2.7) and the fact that the local coordinate ν introduced on the contour is equal to $r - R$, we find that the solution obtained with the aid of the functional (4.1) coincides with the exact solution.

6. BENDING OF A HINGED RECTANGULAR PLATE UNDER TRANSVERSAL LOAD

Let the plate occupy the domain $0 \le x \le a$, $0 \le y \le b$. The transversal load is $p(x,y) = p_0 \sin(\pi x/a) \times \sin(\pi y/b)$. On the plate contour the conditions of hinge support are satisfied. We point out that there are two kinds of such boundary conditions.

Problem 1.

$$w|_C = 0, \quad \boldsymbol{\nu} \cdot \mathbf{M} \cdot \boldsymbol{\nu}|_C = 0, \quad \boldsymbol{\tau} \cdot \boldsymbol{\psi}|_C = 0. \tag{6.1}$$

Problem 2.

$$w|_C = 0, \quad \boldsymbol{\nu} \cdot \mathbf{M} \cdot \boldsymbol{\nu}|_C = 0, \quad \boldsymbol{\nu} \cdot \mathbf{M} \cdot \boldsymbol{\tau}|_C = 0. \tag{6.2}$$

Both problems are considered below, and we compare the solutions.

Problem 1 has the exact solution

$$\Phi_1 = -\frac{p_0}{D} \frac{\sin(\pi x/a)\sin(\pi y/b)}{[(\pi/a)^2 + (\pi/b)^2]^2}, \quad F_1(x,y) \equiv 0. \tag{6.3}$$

Problem 2 has no known solution in a closed form. However, on representing the functions Φ and F as power series in h, the problem splits into two problems for the leading terms in the asymptotic expansion. The problem for the loading term of the interior potential coincides with Problem 1, $\Phi_2^{(0)} = \Phi_1$, whereas the problem for the boundary layer potential reads

$$\Delta F_2^{(0)} - \frac{12\,\Gamma}{h^2} F_2^{(0)} = 0, \quad F_2^{(0)}\big|_C = \frac{h^2}{6\,\Gamma} \frac{\partial^2 \Phi_2^{(0)}}{\partial \nu \partial \tau}\Big|_C. \tag{6.4}$$

It is obvious that the leading term for the boundary layer potential $F_2^{(0)}$ is not zero, and hence, the solutions to Problems 1 and 2 differ by their leading terms in the vicinity of the boundary. Let us solve Problems 1 and 2 with the aid of the functional (4.1).

Problem 1. The interior potential is sought in the form of the following series in coordinate functions:

$$\Phi(x,y) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} A_{kn} \sin\frac{\pi k x}{a} \sin\frac{\pi n y}{b}. \tag{6.5}$$

Obviously, the coordinate functions (6.5) satisfy the first condition in (2.9). The third condition in (2.9) is satisfied only if $f \equiv 0$. By substituting the expression (6.5) into Eq. (4.1), we obtain

$$\Pi^*(\Phi, f) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} A_{kn}^2 D\left[\left(\frac{\pi k}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2\right]^2 \frac{ab}{8} + A_{11} p_0 \frac{ab}{4}. \tag{6.6}$$

The stationarity condition for the functional (6.6) yields

$$A_{11} = -\frac{p_0}{D}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]^{-2}, \quad A_{kn} = 0 \quad (kn \neq 1). \tag{6.7}$$

The obtained solution (6.5), (6.7) coincides with the exact solution (6.3).

Problem 2. We search the interior potential in the form of the series (6.5). The function f need not satisfy any kinematic boundary conditions. We represent f in the form of a series in coordinate functions,

$$\begin{aligned}
y = 0: \quad & f_1(x) = \frac{C_0}{2} + \sum_{k=1}^{\infty} C_k \cos\frac{\pi k x}{a}, \\
y = b: \quad & f_2(x) = \frac{S_0}{2} + \sum_{k=1}^{\infty} S_k \cos\frac{\pi k x}{a}, \\
x = 0: \quad & f_3(y) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cos\frac{\pi n y}{b}, \\
x = a: \quad & f_4(y) = \frac{V_0}{2} + \sum_{n=1}^{\infty} V_n \cos\frac{\pi n y}{b}.
\end{aligned} \tag{6.8}$$

The substitution of the expressions (6.5) and (6.8) into Eq. (4.1) yields

$$\Pi^*(\Phi, f) = D\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\Big\{A_{kn}^2\Big[\Big(\frac{\pi k}{a}\Big)^2+\Big(\frac{\pi n}{b}\Big)^2\Big]^2\frac{ab}{8} + (1-\mu)\frac{1}{2}\frac{\delta}{h}A_{kn}\pi^2 kn\big[(C_k+(-1)^n S_k)b^{-1}-(B_n+(-1)^k V_n)a^{-1}\big]\Big\} + \frac{1}{4}\delta G\Gamma\sum_{k=0}^{\infty}\big[(C_k^2+S_k^2)a+(B_k^2+V_k^2)b\big]+\frac{1}{4}A_{11}p_0 ab. \tag{6.9}$$

The stationarity condition for the functional (6.9) yields

$$A_{11} = -\frac{p_0}{D}\Big\{\Big[\Big(\frac{\pi}{a}\Big)^2+\Big(\frac{\pi}{b}\Big)^2\Big]^2 - 8\frac{h}{\delta}(1-\mu)\pi^4(a+b)a^{-3}b^{-3}\Big\}^{-1},$$
$$B_1 = S_1 = -C_1 = -V_1 = \frac{A_{11}\pi^2 h^2}{6\,\Gamma ab}, \tag{6.10}$$
$$A_{kn} = 0 \quad (kn \neq 1), \quad C_k = S_k = B_k = V_k = 0 \quad (k \neq 1).$$

The leading term of the interior potential for the solution to Problem 2 (by the functional (4.1)), coincides with the interior potential for Problem 1. The boundary layer potential for Problem 2 is not zero, and hence, it differs from the potential for Problem 1 in the leading term.

7. BENDING OF A RECTANGULAR PLATE WITH THREE HINGED EDGES AND ONE LOADED EDGE

Consider a rectangular plate in the domain $0 \le x \le a$, $0 \le y \le b$. The hinge support conditions are imposed on the three edges,

$$w\Big|_{\substack{x=0\\ y=0,b}} = 0, \quad \boldsymbol{\tau}\cdot\boldsymbol{\psi}\Big|_{\substack{x=0\\ y=0,b}} = 0, \quad \boldsymbol{\nu}\cdot\mathbf{M}\cdot\boldsymbol{\nu}\Big|_{\substack{x=0\\ y=0,b}} = 0. \tag{7.1}$$

The load is applied on the fourth edge:

$$\boldsymbol{\nu}\cdot\mathbf{N}|_{x=a} = N_\nu^*\sin\frac{\pi y}{b}, \quad \boldsymbol{\nu}\cdot\mathbf{M}\cdot\boldsymbol{\tau}|_{x=a} = M_\tau^*\cos\frac{\pi y}{b}, \quad \boldsymbol{\nu}\cdot\mathbf{M}\cdot\boldsymbol{\nu}|_{x=a} = 0, \tag{7.2}$$

where $N_\nu^* - (\pi/b)M_\tau^* = hp$ and the quantities N_ν^*, M_τ^*, and p are of the asymptotic order $O(1)$. The leading terms of asymptotic expansions for the interior and the boundary layer potentials are determined from the solutions to the boundary value problems

$$h^2\Delta F - 12\Gamma F = 0, \quad \frac{\partial F}{\partial \nu}\Big|_{\substack{x=0\\ y=0,b}} = 0, \quad Gh\Gamma F|_{x=a} = -M_\tau^*\cos\frac{\pi y}{b}, \tag{7.3}$$

$$\Delta\Delta\Phi = 0, \quad \Phi\Big|_{\substack{x=0\\ y=0,b}} = 0, \quad \Delta\Phi\Big|_{\substack{x=0\\ y=0,b}} = 0, \tag{7.4}$$
$$\frac{\partial^3\Phi}{\partial x^3}+(2-\mu)\frac{\partial^3\Phi}{\partial x\partial y^2}\Big|_{x=a} = \frac{hp}{D}\sin\frac{\pi y}{b}, \quad \frac{\partial^2\Phi}{\partial x^2}+\mu\frac{\partial^2\Phi}{\partial y^2}\Big|_{x=a} = -(1-\mu)\frac{\partial^2 F}{\partial x\partial y}\Big|_{x=a}$$

The interior potential is of order $O(h^{-2})$ and the boundary layer potential is of order $O(h^{-1})$. By Eqs. (7.3) and (7.4), the leading term of the boundary layer potential depends only on M_τ^*, and the leading term of the interior potential depends both on p and on M_τ^*. The statement of the problem in the Kirchhoff theory yields system (7.4) with zero boundary layer potential. Obviously, the interior potential in the Kirchhoff theory is also of order $O(h^{-2})$, but it depends only on p and is independent of M_τ^*. Hence, the solution provided by the Kirchhoff theory contains an error in the leading term not only on the boundary, but also in the interior of the domain.

Let us consider the solution with the aid of the functional (4.1). The interior potential and the function f are sought in the form

$$\Phi(x,y) = u(x)\sin\frac{\pi y}{b}, \qquad x = a: \quad f(y) = S\cos\frac{\pi y}{b}. \tag{7.5}$$

According to the kinematic boundary conditions, the function f is zero on the sides $x = 0$, $y = 0$, and $y = b$, and the function $u(x)$ must satisfy the conditions

$$u(0) = 0, \quad u''(0) = 0. \tag{7.6}$$

On substituting the expressions (7.5) into Eq. (4.1), we obtain

$$\begin{aligned}\Pi^*(u(x),S) = \frac{b}{2}\Big\{D\int_0^a\Big[\frac{1}{2}\Big[u''(x) - \Big(\frac{\pi}{b}\Big)^2u(x)\Big]^2 + (1-\mu)\Big(\frac{\pi}{b}\Big)^2\big[(u'(x))^2 + u''(x)u(x)\big]\Big]\,dx \\ - D(1-\mu)\frac{\delta}{h}\frac{\pi}{b}u'(a)S + \frac{1}{2}\delta G\Gamma S^2 + hpu(a) + M_\tau^*\frac{\delta}{h}S\Big\}.\end{aligned} \tag{7.7}$$

The stationarity condition for the functional (7.7) yields

$$\begin{aligned}&u^{\mathrm{IV}}(x) - 2\Big(\frac{\pi}{b}\Big)^2u''(x) + \Big(\frac{\pi}{b}\Big)^4u(x) = 0,\\ &u'''(a) - (2-\mu)\Big(\frac{\pi}{b}\Big)^2u'(a) = \frac{hp}{D},\\ &u''(a) - \mu\Big(\frac{\pi}{b}\Big)^2u(a) = (1-\mu)\frac{\pi}{b}\frac{\delta}{h}S,\\ &S = -\frac{M_\tau^*}{Gh\Gamma} + \frac{1}{6}\frac{h^2}{\Gamma}\frac{\pi}{b}u'(a).\end{aligned} \tag{7.8}$$

Equations (7.6) and (7.8) show that the leading part of S depends on M_τ^* alone, and the leading part of $u(x)$ depends both on p and M_τ^*.

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