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APPROXIMATE HAMILTON FUNCTIONALS FOR THE PROBLEMS OF LOW-FREQUENCY AND HIGH-FREQUENCY FREE VIBRATIONS OF THE REISSNER PLATE

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It is known that the inertia of rotation and transverse shear deformations must be taken into account in some problems of forced vibrations of plates, in particular, for the plate vibrations under impact and other rapidly time-varying loads. Therefore, the solution to the problem of eigenfrequencies and modes of plate vibrations for the Reissner theory is of great importance. Since the exact solution is possible only in some exceptional cases, the solution to this problem is obtained, as a rule, by numerical methods. Most numerical methods are based on the use of variational principles. The Hamilton functional for the Reissner theory of plates includes functions rapidly varying with respect to the space coordinates, which is a serious disadvantage in numerical procedures. The aim of this paper is to formulate approximate Hamilton functionals for the Reissner theory of plates so as to eliminate all functions rapidly varying with respect to the space coordinates.

We perform asymptotic study of free vibrations of a plate. We show that the functions that describe the stress-strain state of the plate for the low-frequency and high-frequency vibrations, have quite a different character of varying with respect to the space coordinates. However, the solutions include rapidly varying functions for low-frequency free vibrations as well as for high-frequency ones: for the low-frequency case they are functions of boundary-layer type, and for the high-frequency case these functions deeply penetrate into the plate domain. Two Hamilton functionals independent of the rapidly varying functions are suggested below: one of them is introduced to solve the problem of low-frequency free vibrations, and the other is formulated for high-frequency free vibrations. The asymptotic accuracy of the functionals is the following: the low-frequency functional permits us to obtain the eigenfrequencies and modes with a relative error $O(h^2)$; the low-frequency functional gives a relative error $O(h^4)$ for the eigenfrequencies and permits us to obtain the leading terms of the vibration modes.

The problem of low-frequency free vibrations was studied by many authors. The new point here is the variational statement of the problem. It should be noticed that the formulation of the Hamilton functional is similar to the formulation of energy functional for the problem of static bending of a Reissner plate.[1] Therefore, we restrict the description of the low-frequency problem to the main results only.

The problem of high-frequency free vibrations of plate has been studied insufficiently. In the paper [1] the equations of free plate vibrations are obtained on the basis of three-dimensional elasticity by a variational-asymptotic method for various frequency spectra. In contrast to [1], our attempt revises the problem of high-frequency vibrations so as to make it more convenient for numerical implementation and to cover all possible types of boundary conditions. We also note that the given equations for high-frequency vibrations differ from the equations in [1].

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[1] The results of this problem study will soon be published jointly with P. A. Zhilin.

1. MAIN EQUATIONS OF THE REISSNER THEORY OF PLATES

Let us consider the problem of free vibrations of a plate with taking account of the inertia of rotation, and of the transverse shear deformation. The equations that describe the free vibrations of a plate acquire the form [2]

$$D\Delta\Delta\Phi + \rho h\ddot{\Phi} - \frac{\rho h^3}{12}\left(1 + \frac{2}{\Gamma(1-\mu)}\right)\Delta\ddot{\Phi} + \frac{\rho^2 h^3}{12\,G\Gamma}\ddot{\ddot{\Phi}} = 0, \tag{1.1}$$

$$\Delta F - \frac{12\,\Gamma}{h^2}F - \frac{\rho}{G}\ddot{F} = 0. \tag{1.2}$$

The deflection w, the vector ψ of rotation angles, the vector $\mathbf{N}$ of shear forces, and the moment tensor $\mathbf{M}$ are given by the formulas

$$\begin{aligned} &w = -\Phi + \frac{h^2}{6\,\Gamma(1-\mu)}\Delta\Phi - \frac{\rho h^2}{12\,G\Gamma}\ddot{\Phi}, \quad \psi = \nabla\Phi + \nabla F \times \mathbf{n}, \\ &\mathbf{N} = D\nabla\Delta\Phi - \frac{\rho h^3}{12}\Delta\ddot{\Phi} + Gh\Gamma\nabla F \times \mathbf{n}, \\ &\mathbf{M} = D[(1-\mu)\nabla\nabla\Phi + \mu\Delta\Phi\mathbf{a} + \tfrac{1}{2}(1-\mu)(\nabla\nabla F \times \mathbf{n} - \mathbf{n} \times \nabla\nabla F)]. \end{aligned} \tag{1.3}$$

Here h is the plate thickness, $D = \frac{1}{12}Eh^2/(1-\mu^2)$ is the bending stiffness, $Gh\Gamma$ is the shear stiffness, $G = \frac{1}{2}E\Gamma/(1+\mu)$ is the coefficient of transverse shear, μ is Poisson's ratio, ρ is the mass density, $\mathbf{n}$ is the vector of the unit normal to the plate plane, and $\mathbf{a} = \mathbf{E} - \mathbf{nn}$, where $\mathbf{E}$ is the unit tensor.

The deflection, the vector of rotation angles, the vector of shearing forces and the moment tensor are related to the displacements and stresses as follows in three-dimensional elasticity [3]:

$$hw = \langle \mathbf{u}\cdot\mathbf{n}\rangle, \quad h^3\psi = \langle \mathbf{u}z\rangle, \quad \mathbf{N} = \langle \mathbf{a}\cdot\boldsymbol{\tau}\cdot\mathbf{n}\rangle, \quad \mathbf{M} = \langle \mathbf{a}\cdot\boldsymbol{\tau}\cdot\mathbf{a}z\rangle, \quad \langle f\rangle = \int_{-h/2}^{h/2} f\,dz, \tag{1.4}$$

where $\mathbf{u}$ and $\boldsymbol{\tau}$ are the displacement vector and the stress tensor in the three-dimensional theory.

The kinematic boundary conditions acquire the form

$$w|_c = 0, \quad \boldsymbol{\nu}\cdot\psi|_c = 0, \quad \boldsymbol{\tau}\cdot\psi|_c = 0. \tag{1.5}$$

The force boundary conditions can be written as follows:

$$\boldsymbol{\nu}\cdot\mathbf{N}|_c = 0, \quad \boldsymbol{\nu}\cdot\mathbf{M}\cdot\boldsymbol{\nu}|_c = 0, \quad \boldsymbol{\nu}\cdot\mathbf{M}\cdot\boldsymbol{\tau}|_c = 0. \tag{1.6}$$

Here $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are the unit outward normal vector and the unit tangent vector to the plate contour, respectively; the vectors $\boldsymbol{\nu}$, $\boldsymbol{\tau}$, and $\mathbf{n}$, are assumed to form a right-handed system.

It is known [4] that in the Reissner theory of plates there are three spectra of eigenfrequencies, which satisfy the following asymptotic estimates:

$$\omega_{(1)i}^2 = h^2\omega_{2,i}^{(1)} + h^3\omega_{3,i}^{(1)} + \cdots, \quad \omega_{(2)i}^2 = \frac{12G\Gamma}{\rho h^2} + \omega_{0,i}^{(2)} + \cdots, \quad \omega_{(3)i}^2 = \frac{12G\Gamma}{\rho h^2} + \omega_{0,i}^{(3)} + \cdots, \tag{1.7}$$

where the first spectrum in Eqs. (1.7) describes the low frequency bending vibrations, whereas the second and the third spectra in (1.7) characterize the high frequency shear and bending vibrations.

2. FREE LOW-FREQUENCY VIBRATIONS

For the free low-frequency vibrations, the functions Φ and F have the same properties as for the static bending of the plate. Here Φ is a slowly varying function of the spatial coordinates and describes solutions that penetrate into the entire plate domain. Modulo $O(h^2)$, Eq. (1.1) coincides with the equation for deflection in the Kirchhoff theory

$$D\Delta\Delta\Phi + \rho h\ddot{\Phi} = 0. \tag{2.1}$$

Furthermore, F is a function of boundary-layer type; it rapidly decays away from the boundary and is slowly varying along the plate contour. It is proved[2] that modulo $O(h^2)$, the boundary layer potential F can be represented as follows:

$$F(\nu,\tau) = f(\tau)\Big(1 - \frac{\nu}{2R}\Big)\exp\Big(\frac{\delta}{h}\nu\Big), \quad \delta = \sqrt{12\,\Gamma} \quad (\nu < 0). \tag{2.2}$$

Here (ν, τ) is the local coordinate system introduced on the plate contour; R is the curvature radius of the contour at the considered point. Note that formula (2.2) holds when $h/R \ll 1$. In the case of free low-frequency vibrations, for the functions F and Φ we have the asymptotic relation $F \sim h^2\Phi$.

The variables that describe the stress-strain state of the plate for the free low-frequency vibrations have the same asymptotic expressions as for the static bending of the plate:

$$\begin{aligned} &w = -\Phi, \quad \boldsymbol{\psi} = \nabla\Phi - \frac{\delta}{h} f(\tau)\exp\Big(\frac{\delta}{h}\nu\Big)\boldsymbol{\tau}, \\ &\mathbf{N} = D\nabla\Delta\Phi + Gh\Gamma\Big\{\Big[-\frac{\delta}{h}\Big(1 - \frac{\nu}{2R}\Big) + \frac{1}{2R}\Big]f(\tau)\boldsymbol{\tau} + \Big[\Big(1 - \frac{\nu}{2R}\Big)f'(\tau) + \frac{\nu R'}{2R^2}f(\tau)\Big]\boldsymbol{\nu}\Big\}\exp\Big(\frac{\delta}{h}\nu\Big), \\ &\mathbf{M} = D[(1-\mu)\nabla\nabla\Phi + \mu\Delta\Phi\mathbf{a}] + \Big[D(1-\mu)\frac{\delta}{h}f'(\tau)(\boldsymbol{\nu}\boldsymbol{\nu} - \boldsymbol{\tau}\boldsymbol{\tau}) \\ &\qquad - Gh\Gamma\Big(1 - \frac{\nu}{2R} - \frac{2h}{R\delta}\Big)f(\tau)(\boldsymbol{\nu}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\nu})\Big]\exp\Big(\frac{\delta}{h}\nu\Big). \end{aligned} \tag{2.3}$$

The kinematic boundary conditions become

$$-\Phi|_c = 0, \quad \frac{\partial\Phi}{\partial\nu}\Big|_c = 0, \quad \frac{\partial\Phi}{\partial\tau}\Big|_c - \frac{\delta}{h}f(\tau) = 0. \tag{2.4}$$

The force boundary conditions acquire the form

$$\begin{aligned} &D\frac{\partial\Delta\Phi}{\partial\nu}\Big|_c + Gh\Gamma f'(\tau) = 0, \\ &D\Big(\frac{\partial^2\Phi}{\partial\nu^2} + \frac{\mu}{R}\frac{\partial\Phi}{\partial\nu} + \mu\frac{\partial^2\Phi}{\partial\tau^2}\Big)\Big|_c + D(1-\mu)\frac{\delta}{h}f'(\tau) = 0, \\ &D(1-\mu)\frac{\partial^2\Phi}{\partial\nu\partial\tau}\Big|_c - Gh\Gamma\Big(1 - \frac{2h}{R\delta}\Big)f(\tau) = 0. \end{aligned} \tag{2.5}$$

The physical meaning of the boundary conditions (2.4) and (2.5) is the same as that of the conditions (1.5) and (1.6), respectively.

Equations (2.1) and (2.2) in conjunction with the boundary conditions (2.4) and (2.5) give an approximate statement of the problem of free low-frequency vibrations, which permits us to find the eigenfrequencies and all physical characteristics of the stress-strain state modulo $O(h^2)$.

3. FREE HIGH-FREQUENCY VIBRATIONS

For the free high-frequency vibrations, the functions Φ and F substantially differ from those for the low-frequency vibrations or static bending. The function F varies slowly with respect to the spatial coordinates and is not of boundary-layer type. This is due to the fact that the leading terms in the first and in the second components in Eq. (1.2) cancel each other. For the approximate statement of the problem, Eq. (1.2) remains the same. A very important feature of the high-frequency vibrations is that the asymptotic orders of the functions F and Φ are the same: $F \sim \Phi$. The penetrating potential Φ for the high-frequency vibrations has quite a different structure than in the low-frequency case; namely, along with functions, slowly varying with respect to the spatial coordinates, it includes a rapidly varying function, which was lacking in the preceding cases. Let us denote this rapidly varying function by φ, and retain the notation Φ for the slowly varying component as well as for the penetrating potential itself. This seems convenient, since there is an

[2] see the previous footnote.

asymptotic relation $\varphi \sim h^2\Phi$. The function φ seemingly need not be taken into account since it is relatively small. However, this is not the case, and the function φ may exert influence on the leading terms of some characteristics of the stress-strain state, which can depend not only on the penetrating potential, but also on its derivatives of order ≤ 3. Let us proceed from Eq. (1.1) to the approximate equations for the components Φ and φ. We suppose that for the functions Φ, and φ the following asymptotic estimates hold

$$\frac{\partial\Phi}{\partial x} \sim \frac{\partial\Phi}{\partial y} \sim \Phi, \quad \frac{\partial\varphi}{\partial x} \sim \frac{\partial\varphi}{\partial y} \sim \frac{\varphi}{h}, \quad \ddot{\Phi} = \Big(-\frac{12G\Gamma}{\rho h^2} + O(1)\Big)\dot{\Phi}, \quad \ddot{\varphi} = \Big(-\frac{12G\Gamma}{\rho h^2} + O(1)\Big)\varphi.$$

On substituting the expression for the penetrating potential $\Phi + \varphi$ into Eq. (1.1) and by retaining only the leading terms, we obtain

$$\begin{gathered} GhA(\Phi) + DB(\varphi) = 0, \\ A(\Phi) = \Big(\Gamma + \frac{2}{1-\mu}\Big)\Delta\Phi - \frac{12\,\Gamma}{h^2}\Phi - \frac{\rho}{G}\ddot{\Phi}, \\ B(\varphi) = \Delta\Big[\Delta\varphi + \frac{12}{h^2}\Big(1 + \frac{1}{2}\Gamma(1-\mu)\Big)\varphi\Big]. \end{gathered} \tag{3.1}$$

Note that Eq. (3.1) contains an obvious contradiction. On the one hand, $A(\Phi)$ is a slowly varying function, since it depends on Φ, and $B(\varphi)$ is a rapidly varying function, since it depends on φ. On the other hand, according to Eq. (3.1), the functions $A(\Phi)$ and $B(\varphi)$ are proportional. Thus, $A(\Phi)$ and $B(\varphi)$ are slowly varying and rapidly varying simultaneously, that is, they are both zero. Hence, Eq. (3.1) represents the two equations

$$A(\Phi) = 0, \quad B(\varphi) = 0. \tag{3.2}$$

The first equation in (3.2) has the form

$$\Big(\Gamma + \frac{2}{1-\mu}\Big)\Delta\Phi - \frac{12\,\Gamma}{h^2}\Phi - \frac{\rho}{G}\ddot{\Phi} = 0. \tag{3.3}$$

Equation (3.3) permits us to find the leading term of the slowly varying part of the penetrating potential. It should be noted that Eq. (3.3) for Φ coincides with Eq. (1.2) for F to within a constant coefficient of the first summand, and the behavior of Φ is similar to that of F in the case of high-frequency vibrations.

The second equation in (3.2) is

$$\Delta z(\varphi) = 0, \quad z(\varphi) = \Delta\varphi + \frac{12}{h^2}\Big(1 + \frac{1}{2}\Gamma(1-\mu)\Big)\varphi. \tag{3.4}$$

The solution of the equation $\Delta z = 0$ is a slowly varying function. Since $z(\varphi)$ varies rapidly, all solution except for zero are excluded: $z \equiv 0$. Then the equation for φ acquires the form

$$\Delta\varphi + \frac{12}{h^2}\Big(1 + \frac{1}{2}\Gamma(1-\mu)\Big)\varphi = 0. \tag{3.5}$$

Equation (3.5) allows one to find the leading term of the rapidly varying part of the penetrating potential. Note that in contrast to the low-frequency vibrations, the rapidly varying function for the high-frequency vibrations is not of boundary-layer type, but penetrates into the entire plate domain.

For the high-frequency vibrations, the characteristics of stress-strain state of the plate have the following asymptotic representations:

$$\begin{gathered} w = -\frac{h^2}{12}\Delta\Phi - \Big(1 + \frac{2}{\Gamma(1-\mu)}\Big)\varphi, \quad \boldsymbol{\psi} = \nabla\Phi + \nabla F \times \mathbf{n}, \quad \mathbf{N} = Gh\Gamma(\nabla\Phi + \nabla F \times \mathbf{n}), \\ \mathbf{M} = D[(1-\mu)\nabla\nabla(\Phi+\varphi) + \mu\Delta(\Phi+\varphi)\mathbf{a} + \tfrac{1}{2}(1-\mu)(\nabla\nabla F \times \mathbf{n} - \mathbf{n} \times \nabla\nabla F)]. \end{gathered} \tag{3.6}$$

Before proceeding to the statement of the boundary conditions, let us focus our attention on the following important property of φ. Consider a boundary condition depending on φ, for example, $w|_c = 0$. Obviously,

φ cannot vary rapidly on the boundary, since the other components in the boundary condition vary slowly. A similar conclusion holds for any boundary condition that depends on φ. In addition, φ is generally nonzero on the plate contour, since otherwise it would obviously be zero in the entire plate domain. The most natural conclusion follows: on the plate boundary, the function φ loses the property of being rapidly varying and becomes a slowly varying function along the contour. We point out that φ varies slowly only on the plate boundary; it rapidly varies in every direction in the interior of the domain arbitrarily close to the boundary. Thus, the tangent derivative of φ on the plate contour has the same asymptotic order as φ, $\partial\varphi/\partial\tau|_c \sim \varphi$.

The kinematic boundary conditions acquire the form

$$\frac{h^2}{12}\Delta\Phi - \left(1 + \frac{2}{\Gamma(1-\mu)}\right)\varphi\bigg|_c = 0, \quad \frac{\partial\Phi}{\partial\nu} + \frac{\partial F}{\partial\tau}\bigg|_c = 0, \quad \frac{\partial\Phi}{\partial\tau} - \frac{\partial F}{\partial\nu}\bigg|_c = 0. \tag{3.7}$$

The force boundary conditions are written as follows:

$$\begin{aligned} &Gh\Gamma\left(\frac{\partial\Phi}{\partial\nu} + \frac{\partial F}{\partial\tau}\right)\bigg|_c = 0, \\ &D(1-\mu)\left[\frac{\partial^2 F}{\partial\nu\partial\tau} - \left(\frac{1}{R}\frac{\partial\Phi}{\partial\nu} + \frac{\partial^2\Phi}{\partial\tau^2}\right)\right] + D\left[\Delta\Phi - \frac{12}{h^2}\left(1 + \frac{1}{2}\Gamma(1-\mu)\right)\varphi\right]\bigg|_c = 0, \\ &D(1-\mu)\left[\frac{\partial^2\Phi}{\partial\nu\partial\tau} + \left(\frac{1}{R}\frac{\partial F}{\partial\nu} + \frac{\partial^2 F}{\partial\tau^2}\right)\right] - \left(\frac{\rho h^3}{12}\ddot{F} + Gh\Gamma F\right)\bigg|_c = 0. \end{aligned} \tag{3.8}$$

The physical meaning of the boundary conditions (3.7) and (3.8) is the same as that of conditions (1.5) and (1.6), respectively.

Equations (1.2), (3.3), and (3.5) supplemented with the boundary conditions (3.7) and (3.8) form an approximate statement of the problem of free high-frequency vibrations, which permits us to find the leading terms of the characteristics of the stress-strain state and the eigenfrequencies with a relative error $O(h^4)$. Recall that the main terms of the eigenfrequencies are equivalent to $\sqrt{12G\Gamma/(\rho h^2)}$.

As was noted in the preceding, φ is a rapidly varying function and penetrates into the entire plate domain. Hence, the cited statement for the high-frequency vibrations is practically invalid in numerical implementations. Is it possible to revise this problem without taking account of φ? Before answering this question, let us note the following: the vector $\boldsymbol{\psi}$ of rotation angles and the vector $\mathbf{N}$ of shearing forces, which do not depend on φ (see Eqs. (3.6)), are two orders of magnitude larger than the deflection w and the moment tensor $\mathbf{M}$, respectively. Thus allows one to claim, that the stress-strain state is mainly characterized by the vector of rotation angles and the vector of shear forces, whereas the deflection and the moment tensor play a less important role. Therefore, the statement of this problem without the function φ is consistent in principle.

Thus, let us eliminate Eq. (3.5) from the system of equations for the high-frequency vibrations. Then the order of the system with respect to the spatial derivatives is reduced from six to four. Hence, the three boundary conditions in the original statement must be replaced by two boundary conditions independent of φ. One of the original three boundary conditions is the third condition in (3.7) or in (3.8). Since these conditions do not depend on φ, they are retained.

Two other conditions are replaced by one condition according to the following rule:

- the second condition in (3.7) and the first condition in (3.8) are equivalent, and if they are given simultaneously, there is no need to choose one of them (note that the function φ is identically zero for these boundary conditions);
- if the first and the second conditions in (3.7) are given, the second condition must be retained, since it is independent of φ (whereas the first condition depends on φ);
- if the first and the second conditions in (3.8) are given, the first condition must be retained, since it is independent of φ (whereas the second condition depends on φ);
- since the first condition in (3.7) and the second condition in (3.8) depend on φ, they can be replaced by the combination

$$D(1-\mu)\left[\frac{\partial^2 F}{\partial\nu\partial\tau} - \left(\frac{1}{R}\frac{\partial\Phi}{\partial\nu} + \frac{\partial^2\Phi}{\partial\tau^2}\right)\right] + \left(\frac{\rho h^3}{12}\ddot{\Phi} + Gh\Gamma\Phi\right)\bigg|_c = 0. \tag{3.9}$$

The physical meaning of the boundary conditions without the function φ is the following. The third condition in (3.7) means that the angle $\boldsymbol{\tau} \cdot \boldsymbol{\psi}$ of rotation about the normal to the plate contour is zero. The second condition in (3.7) (and the equivalent first condition in (3.8)) indicates that the angle $\boldsymbol{\nu} \cdot \boldsymbol{\psi}$ of rotation about the tangent to the plate contour is zero. The third condition in (3.8) means that the torque $\boldsymbol{\nu} \cdot \mathbf{M} \cdot \boldsymbol{\tau}$ is zero. Condition (3.9) indicates that the reduced bending moment $\boldsymbol{\nu} \cdot \mathbf{M} \cdot \boldsymbol{\nu} - Gh\Gamma w$ is zero.

4. ORTHOGONALITY RELATIONS

The performed study of the approximate equations and boundary conditions for the low-frequency and high-frequency vibrations yields orthogonality relations for the vibration modes. The low-frequency modes satisfy the relation

$$\int_{(\Delta S)} \rho h \Phi_i \Phi_j \, dS = 0 \quad (i \neq j). \tag{4.1}$$

The high-frequency modes satisfy the orthogonality relation

$$\int_{(\Delta S)} \frac{\rho h^3}{12} \left[\left(\frac{\partial \Phi_i}{\partial x} + \frac{\partial F_i}{\partial y} \right) \left(\frac{\partial \Phi_j}{\partial x} + \frac{\partial F_j}{\partial y} \right) + \left(\frac{\partial \Phi_i}{\partial y} - \frac{\partial F_i}{\partial x} \right) \left(\frac{\partial \Phi_j}{\partial y} - \frac{\partial F_j}{\partial x} \right) \right] dS = 0 \quad (i \neq j). \tag{4.2}$$

Two modes, of which the ith mode corresponds to low eigenfrequency and the jth mode to the high eigenfrequency, satisfy the orthogonality relation

$$\int_{(\Delta S)} \frac{\rho h^3}{12} \left[\Phi_i \Delta \Phi_j + \frac{\partial \Phi_i}{\partial x} \left(\frac{\partial \Phi_j}{\partial x} + \frac{\partial F_j}{\partial y} \right) + \frac{\partial \Phi_i}{\partial y} \left(\frac{\partial \Phi_j}{\partial y} - \frac{\partial F_j}{\partial x} \right) \right] dS = 0. \tag{4.3}$$

5. VARIATIONAL STATEMENTS OF THE PROBLEMS OF FREE LOW-FREQUENCY AND HIGH-FREQUENCY VIBRATIONS

The structure of the Hamilton functional for the problem of the low-frequency vibrations substantially differs from that of the Hamilton functionals known in the plate theory. The difference manifests itself in some components, which are not represented in the form of integrals over the plate area, but rather as integrals over the plate contour. Just this way to proceed from the integral of the boundary layer potential over the plate area to the integral over the plate contour, with the subsequent substitution of Eq. (2.2), permits us to eliminate the rapidly varying functions from the low-frequency Hamilton functional. It should be noted, that this low-frequency Hamilton functional has a structure similar to that of the potential energy functional for the static bending of plate. Therefore, we do not discuss in details the basic ideas of the construction. The Hamilton functional for the problem of high-frequency vibrations corresponds to the problem with eliminated function φ, rapidly varying with respect to the space coordinates. The structure of the high-frequency functional is typical of the Hamilton functionals. The functionals presented in the following are not the strict asymptotic consequence of the Hamilton functional for the Reissner plate theory, since some components of the order $O(h^2)$ are retained there, whereas the others are omitted.

The low-frequency Hamilton functional is defined on the set of functions $\Phi(x, y)$ and $f(\tau)$ that satisfy the following conditions:

- the functions $\Phi(x, y)$ are continuous together with their derivatives of order ≤ 2 in the closed domain $\bar{S} = S + C$;
- the functions $f(\tau)$ are continuously differentiable along C;
- the functions $\Phi(x, y)$ and $f(\tau)$ satisfy the kinematic boundary conditions (2.4), if they are imposed:

$$\int_{t_1}^{t_2} (K - \Pi) \, dt = \int_{t_1}^{t_2} \left\{ \int_{(\Delta S)} \left\{ \frac{1}{2} \rho h \dot{\Phi}^2 - D \left[\frac{1}{2} (\Delta \Phi)^2 + (1 - \mu) \left[\left(\frac{\partial^2 \Phi}{\partial x \partial y} \right)^2 - \frac{\partial^2 \Phi}{\partial x^2} \frac{\partial^2 \Phi}{\partial y^2} \right] \right] \right\} dS \right.$$
$$\left. - \int_C \left\{ D(1 - \mu) \frac{\delta}{h} \left[\frac{\partial \Phi}{\partial \nu} f'(\tau) + \frac{1}{R} \frac{\partial \Phi}{\partial \tau} f(\tau) \right] + Gh\Gamma \left(\frac{\delta}{2h} \frac{1}{R} \right) f^2(\tau) \right\} dC \right\} dt. \tag{5.1}$$

The Euler equations for the functional (5.1) coincide with the differential equation (2.1) for the penetrating potential and with the force boundary conditions (2.5).

The high-frequency Hamilton functional is determined on the set of functions $\Phi(x,y)$ and $F(x,y)$ that satisfy the conditions:

- the functions $\Phi(x,y)$ and $F(x,y)$ are continuous together with their derivatives of order ≤ 2 in the closed domain $\bar{S} = S + C$;
- the functions $\Phi(x,y)$ and $F(x,y)$ satisfy the kinematic boundary conditions (the second and the third conditions in (3.7)), if they are imposed:

$$\int_{t_1}^{t_2} (K - \Pi)\, dt = \int_{t_1}^{t_2} \int_{(\Delta S)} \left\{ \frac{1}{2} \frac{\rho h^3}{12} \left[\left(\frac{\partial \dot{\Phi}}{\partial x} + \frac{\partial \dot{F}}{\partial y} \right)^2 + \left(\frac{\partial \dot{\Phi}}{\partial y} - \frac{\partial \dot{F}}{\partial x} \right)^2 \right] \right.$$
$$- \frac{1}{2} Gh\Gamma \left[\left(\frac{\partial \Phi}{\partial x} + \frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial \Phi}{\partial y} - \frac{\partial F}{\partial x} \right)^2 \right]$$
$$- \frac{1}{2} D \left[\left(1 + \frac{1}{2} \Gamma (1-\mu) \right) (\Delta \Phi)^2 + \frac{1}{2} (1-\mu)(\Delta F)^2 \right]$$
$$- D(1-\mu) \left[\left(\frac{\partial^2 \Phi}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 F}{\partial x \partial y} \right)^2 - \frac{\partial^2 \Phi}{\partial x^2} \frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 F}{\partial y^2} \right.$$
$$\left.\left. + \frac{\partial^2 \Phi}{\partial x \partial y} \left(\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x^2} \right) + \frac{\partial^2 F}{\partial x \partial y} \left(\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} \right) \right] \right\} dS\, dt. \tag{5.2}$$

The Euler equations for the functional (5.2) are the differential equations (1.2) and (3.3) and the boundary conditions (3.9) with the third condition in (3.8).

6. DISCUSSION OF THE PHYSICAL MEANING OF OBTAINED RESULTS

From the physical viewpoint, it seems obvious that the high-frequency vibrations are produced by the shear phenomena. The presented asymptotic estimates confirm this assertion. Indeed, for the low-frequency vibrations the vector $\boldsymbol{\psi} = -\nabla w + \boldsymbol{\gamma}$ of rotation angles is actually determined by the deflection w, and the vector $\boldsymbol{\gamma}$ of transverse shear deformation represents the unimportant refinements: $\boldsymbol{\gamma} \sim h\boldsymbol{\psi}$ in the vicinity of the boundary and $\boldsymbol{\gamma} \sim h^2\boldsymbol{\psi}$ inside the domain. For the high-frequency vibrations the situation is quite opposite: the vector $\boldsymbol{\psi}$ of rotation angles practically coincides with the vector of transverse shear deformation $\boldsymbol{\gamma}$, and the deflection w adds unimportant refinements: $\boldsymbol{\psi} \sim \boldsymbol{\gamma}$, $\nabla w \sim h\boldsymbol{\psi}$. Since the nature of the high-frequency shear and high-frequency bending vibrations is the same (in particular, this is confirmed by the coincidence of leading eigenfrequency terms in the shear and bending spectra), the equations for both vibrations naturally seem to be similar. This is just the case in the suggested statement of problem for the high-frequency vibrations: Eq. (3.3) for the bending vibrations practically coincides with Eq. (1.2) for the shear vibrations.

Thus, if we assume similarity of the bending and shear vibrations, then the order of the system with respect to the space coordinates is reduced. This means that yet another function is not included. Since we already have the equations for the shear and bending vibrations, the equation for this function is likely to be the static equation. In the presented statement, Eq. (3.5) for φ just does not contain the time derivatives. From the physical considerations, it is obvious that this function characterizes the bending phenomena and has no importance in the problem of high-frequency vibrations. This assertion is supported by the fact that this function rapidly varies with respect to the space coordinates, as we can see from Eq. (3.5), and penetrates into the entire plate domain. Would this function be of primary importance for high-frequency vibrations, we should adopt that the plate theory does not apply to this problem. Thus, the possibility of stating the problem without rapidly varying functions seems to be natural from the physical viewpoint. The boundary conditions in the proposed statement are also worth saying a few words. First, it must be noted that there are four possible boundary conditions, and two of them are kinematic: $\psi_\nu|_c = 0$ (the angle of rotation about the tangent) and $\psi_\tau|_c = 0$ (the angle of rotation about the normal); the other two are the force conditions: $M^*_\nu|_c = 0$ (reduced bending moment) and $M_\tau|_c = 0$ (torque).

Note that the meaning of the kinematic and the force conditions slightly change in proceeding from the original statement to the approximate statement in the Reissner theory. For the approximate statement, the conditions $\psi_\nu|_c = 0$ and $N_\nu|_c = 0$ are equivalent, since $\mathbf{N} = Gh\Gamma\boldsymbol{\psi}$, and the force condition $N_\nu|_c = 0$ becomes kinematic. The kinematic condition $w|_c = 0$ disappears in the approximate statement, and the deflection appears in the condition $M^*_\nu|_c = 0$, since $M^*_\nu = M_\nu - Gh\Gamma w$; that is, it becomes part of the force boundary condition. A certain symmetry for the problem of high-frequency vibrations is not restricted by the similarity of Eqs. (1.2) and (3.3). The similarity also occurs for the boundary conditions. In fact, the condition $\psi_\tau|_c = 0$ yields $\psi_\nu|_c = 0$, and the condition $M_\tau|_c = 0$ yields $M_\nu^*|_c = 0$ by the substitution $\Phi \to F$, $F \to -\Phi$. Such symmetry also supports the cited statement (rather aesthetically than physically).

7. RESULTS OF THE TEST SOLUTIONS

Since we have not succeeded in proving the presented formulas in the general case, the question of reliability naturally arises. For this case the only argument in favor of our equations is the solution to test problems. Exact solutions to test problems were obtained for the free vibrations of a circular plate under all possible boundary conditions. It is shown, that the solutions for the approximate statements are asymptotic consequences of the solution by the exact theory (the eigenfrequencies and the modes were compared). The exact solutions were also obtained for the rectangular plate with two opposite sides simply supported, and with all possible boundary conditions on the other sides. It is shown that the frequency equations for the approximate statements are asymptotic consequences of the frequency equations for the exact theory. In addition, for the rectangular plates we estimate the fist ten eigenfrequencies in the high-frequency spectra for all kinds of boundary conditions, and their modes for some boundary conditions. The dimensionless thicknesses of plate were $h = 0.04$ and $h = 0.1$. The comparison of the results for the exact and the approximate theory shows that the accuracy is rather large for all kinds of boundary conditions.

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