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COMPARATIVE ANALYSIS OF LOW-FREQUENCY FREE VIBRATIONS OF RECTANGULAR PLATES

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It is well known that in solving bending problems for plates, the Kirchhoff theory allows determining leading asymptotic terms of all variables inside the domain of the plate and yields an $O(1)$ error compared with the leading term of the expressions for shearing forces and torsional moments near the boundary. It is also known that the numerical results given by the Kirchhoff theory and the Reissner theory well agree with the conclusions drawn from asymptotic analysis. A vast variety of papers dealt with the discussion of the asymptotic transition from a Reissner-type theory to the Kirchhoff theory and with the comparison of the results predicted by the two theories, so that one cannot seemingly add anything. However, most authors restrict themselves to the consideration of clamped, hinged, or free boundaries of plates, whereas there are another five types of boundary conditions in the Reissner theory, which are also of interest at least from the theoretical viewpoint. At first sight, it seems obvious that the conclusions drawn from the analysis of the accuracy of the Kirchhoff theory for the above three types of boundary conditions are always valid. However, a more detailed insight into this matter has shown that this statement must be refined.

In the present paper, the comparison of the asymptotic and actual accuracy of the Kirchhoff theory is carried out, exemplified by problems on free vibrations of rectangular plates. By the actual accuracy we mean the relative difference of a variable calculated by the Kirchhoff theory and by the Reissner theory for a given value of the small parameter. All eight types of boundary conditions possible in the Reissner theory are considered. In six cases (including the three types of boundary conditions traditionally dealt with in the literature), there are no contradictions with the well-known facts. As to the other two types of boundary conditions, the results turn out to be surprising. It has been found that for the sliding fixing conditions at the contour ($N_\nu|_c = 0$, $\Psi_\nu|_c = 0$, and $\Psi_\tau|_c = 0$) and for the reinforced free edge conditions ($N_\nu|_c = 0$, $M_\nu|_c = 0$, and $\Psi_\tau|_c = 0$), the Kirchhoff theory results in so large real errors (even for a few first natural frequencies) that we have to conclude that the Kirchhoff theory is inapplicable for these two types of boundary conditions. This is of special interest with regard to the fact that, from the asymptotic viewpoint, the statement of a problem within the framework of the Kirchhoff theory allows one to predict the natural frequencies with an error not exceeding $O(h)$ as compared with unity for all the types of boundary conditions possible in the Reissner theory. So in the formal approach to the problem discussed it is not clear why the Kirchhoff theory provides a solution with admissible real errors under some conditions but fails to do so under another conditions. It became possible to answer this question only after the asymptotic analysis of the frequency equations had been carried out and asymptotic formulas had been obtained to estimate real errors in the natural frequencies calculated by the Kirchhoff theory for different types of boundary conditions.

In [1], an approximate statement of the problem on low-frequency vibrations of Reissner's plate was suggested, in which, as opposed to the Kirchhoff theory, the transverse shear strain near the plate edge is taken into account (the asymptotic accuracy of the statement in question is $O(h^2)$). In the present paper, the comparison of the actual accuracy of the Kirchhoff theory with that of the theory taking into account the transverse shear strain near the boundary (for all the types of boundary conditions) is exemplified by problems on natural vibrations of rectangular plates. The study accomplished shows that for the six types of boundary conditions where the Kirchhoff theory makes quite admissible errors, the allowance for the transverse shear strain significantly influences only the appearance of natural forms near the plate edge. In the two cases where the Kirchhoff theory is not applicable, namely, in the case of sliding fixing conditions at the contour ($N_\nu|_c = 0$, $\Psi_\nu|_c = 0$, and $\Psi_\tau|_c = 0$) and the reinforced free edge conditions ($N_\nu|_c = 0$, $M_\nu|_c = 0$, and $\Psi_\tau|_c = 0$), the allowance for the transverse shear strain changes the picture dramatically. It turns out that the use of the theory taking account of the transverse shear strain near the boundary allows predicting both natural frequencies and natural forms with the same real errors as for the type of boundary conditions where the Kirchhoff theory makes "normal" errors.

1. SUMMARY OF THE BASIC EQUATIONS GOVERNING FREE VIBRATIONS OF REISSNER'S PLATE

The differential equations have the form [2]

$$D\Delta\Delta\Phi + \rho h\ddot{\Phi} - \frac{\rho h^3}{12}\left(1 + \frac{2}{\Gamma(1-\mu)}\right)\Delta\ddot{\Phi} + \frac{\rho^2 h^3}{12G\Gamma}\ddot{\ddot{\Phi}} = 0, \tag{1.1}$$

$$\Delta F - \frac{12\Gamma}{h^2}F - \frac{\rho}{G}\ddot{F} = 0. \tag{1.2}$$

The variables characterizing the stress-strain state of the plate are given by

$$\begin{aligned}
\dot{w} &= -\Phi + \frac{h^2}{6\Gamma(1-\mu)}\Delta\Phi - \frac{\rho h^2}{12G\Gamma}\ddot{\Phi},\\
\Psi &= \nabla\Phi + \nabla F \times \mathbf{n},\\
\mathbf{N} &= D\nabla\Delta\Phi - \frac{\rho h^3}{12}\nabla\ddot{\Phi} + Gh\Gamma\nabla F \times \mathbf{n},\\
\mathbf{M} &= D\left[(1-\mu)\nabla\nabla\Phi + \mu\Delta\Phi\mathbf{a} + \tfrac{1}{2}(1-\mu)(\nabla\nabla F\times\mathbf{n} - \mathbf{n}\nabla\nabla F)\right],
\end{aligned} \tag{1.3}$$

where w is the deflection, Ψ is the vector of rotation angles, $\mathbf{N}$ is the vector of transverse forces, $\mathbf{M}$ is the tensor of moments, $D = \frac{1}{12}Eh^3/(1-\mu^2)$ is the bending stiffness, $Gh\Gamma$ is the shear stiffness, $G = \frac{1}{2}E/(1+\mu)$, Γ is the transverse shear coefficient, E is Young's modulus, μ is Poisson's ratio, ρ is the mass density, h is the plate thickness, $\mathbf{n}$ is the unit normal to the plate plane, $\mathbf{a} = \mathbf{E} - \mathbf{nn}$, and $\mathbf{E}$ is the identity tensor.

The boundary conditions are stated later and now we only note that the Reissner theory allows one to satisfy three conditions at the contour and, hence, eight different types of boundary conditions are possible.

2. APPROXIMATE STATEMENT OF THE PROBLEM ON LOW-FREQUENCY FREE VIBRATIONS OF REISSNER'S PLATE WHICH TAKES ACCOUNT OF THE TRANSVERSE SHEAR STRAIN NEAR THE PLATE EDGE

We consider the statement of the problem suggested in [1]. The differential equations coincide with the Kirchhoff equations

$$D\,\Delta\Delta\Phi + \rho h\ddot{\Phi} = 0. \tag{2.1}$$

The function characterizing the boundary layer has the form

$$F(\nu,\tau) = f(\tau)\left(1 - \frac{\nu}{2R}\right)\exp\left(\frac{\delta}{h}\nu\right), \qquad \delta = \sqrt{12\Gamma} \quad (\nu < 0), \tag{2.2}$$

where ν, τ is the local coordinate system introduced on the plate contour and $R(\tau)$ is the radius of curvature at a given point.

The deflection, the vector of angles, the vector of transverse forces, and the tensor of moments are given by

$$\begin{aligned}
w &= -\Phi,\\
\Psi &= \nabla\Phi - \frac{\delta}{h}f(\tau)\exp\left(\frac{\delta}{h}\nu\right)\tau,\\
\mathbf{N} &= D\,\nabla\Delta\Phi + Gh\Gamma\left\{\left[-\frac{\delta}{h}\left(1-\frac{\nu}{2R}\right)+\frac{1}{2R}\right]f(\tau)\tau + \left[\left(1-\frac{\nu}{2R}\right)f'(\tau) + \frac{\nu R'}{2R^2}f(\tau)\right]\nu\right\}\exp\left(\frac{\delta}{h}\nu\right),\\
\mathbf{M} &= D[(1-\mu)\nabla\nabla\Phi + \mu\Delta\Phi\,\mathbf{a}] + \left[D(1-\mu)\frac{\delta}{h}f'(\tau)(\nu\nu - \tau\tau) - Gh\Gamma\left(1 - \frac{\nu}{2R} - \frac{2h}{R\delta}\right)f(\tau)(\nu\tau + \tau\nu)\right]\exp\left(\frac{\delta}{h}\nu\right),
\end{aligned} \tag{2.3}$$

where ν and τ are the unit outward normal and the unit tangent vector to the plate contour (ν, τ, and $\mathbf{n}$ form a right-handed trihedral).

Note that in proceeding from the exact statement of the problem to the approximate one discussed here, the boundary conditions do not qualitatively change, since the introduction of the boundary layer function (2.2) makes it possible to satisfy all the three conditions on the contour.

3. STATEMENT OF THE PROBLEM ON FREE VIBRATIONS OF A PLATE IN THE KIRCHHOFF THEORY

The statement of the problem under consideration in the Kirchhoff theory can be obtained from system (2.1)–(2.3) provided that the boundary layer function F is assumed to be identically zero. Two difficulties are immediately encountered. The first difficulty is that the transverse forces and torsional moments near the plate boundary are determined, in the general case, with errors in the leading terms of asymptotic expansions. Obviously, this difficulty cannot be resolved within the framework of the Kirchhoff theory, since the only way out is to take account of the leading asymptotic term of the boundary layer function. The second difficulty is associated with the statement of the boundary conditions: neglecting the boundary layer function results in reducing the order of the system of differential equations for the spatial coordinates, which gives rise to the necessity to replace three conditions on the contour by two.

Let us discuss the problem of stating the boundary conditions in more detail. We pose the question: Is it possible, for all types of boundary conditions used in the Reissner theory, to restate the boundary conditions so that two of them will depend only on the function Φ characterizing the solution penetrating into the entire domain of the plate, and the third condition will be a consequence of the first two or an equation for the function F, provided that the asymptotic error is $O(h)$? The affirmative answer to this question would mean that the Kirchhoff theory allows one to determine the natural frequencies and vibrational shapes inside the plate domain with an asymptotic error of $O(h)$ for all types of boundary conditions.

Below we present the statements of boundary conditions in the Kirchhoff theory which correspond to different types of boundary conditions in the Reissner theory. The conditions for determining the leading asymptotic term of the boundary layer potential are also stated.

1. Clamping:

$$w\big|_c = 0, \quad \Psi_\nu\big|_c = 0, \quad \Psi_\tau\big|_c = 0. \tag{3.1}$$

The Kirchhoff theory:

$$\Phi\big|_c = 0, \quad \frac{\partial \Phi}{\partial \nu}\bigg|_c = 0. \tag{3.2}$$

The condition $\partial\Phi/\partial\tau\big|_c = 0$ is a consequence of the first condition in (3.4). The leading term of the boundary layer potential is zero.

2. Constrained hinged support:

$$w\big|_c = 0, \quad M_\nu\big|_c = 0, \quad \Psi_\tau\big|_c = 0. \tag{3.3}$$

The Kirchhoff theory:

$$\Phi\big|_c = 0, \quad D\left[\frac{\partial^2\Phi}{\partial\nu^2} + \frac{\mu}{R}\frac{\partial\Phi}{\partial\nu} + \mu\frac{\partial^2\Phi}{\partial\tau^2}\right]_c = 0. \tag{3.4}$$

The condition $\partial\Phi/\partial\tau\big|_c = 0$ is a consequence of the first condition in (3.4). The leading term of the boundary layer potential is zero.

3. Free hinged support:

$$w\big|_c = 0, \quad M_\nu\big|_c = 0, \quad M_\tau\big|_c = 0. \tag{3.5}$$

In the Kirchhoff theory, the boundary conditions have the form (3.4), just as in the case of constrained hinged support. However, the leading term of the boundary layer potential is nonzero:

$$f(\tau) = \frac{h^2}{6\Gamma}\frac{\partial^2\Phi}{\partial\nu\partial\tau}\bigg|_c. \tag{3.6}$$

4. Free edge:

$$N_\nu\big|_c = 0, \quad M_\nu\big|_c = 0, \quad M_\tau\big|_c = 0. \tag{3.7}$$

The Kirchhoff theory:

$$D\left[\frac{\partial^2\Phi}{\partial\nu^2} + \frac{\mu}{R}\frac{\partial\Phi}{\partial\nu} + \mu\frac{\partial^2\Phi}{\partial\tau^2}\right]_c = 0, \quad D\left[\frac{\partial\Delta\Phi}{\partial\nu} + (1-\mu)\frac{\partial^3\Phi}{\partial\nu\partial\tau^2}\right]_c = 0. \tag{3.8}$$

The leading term of the boundary layer potential is nonzero and is given by formula (3.6).

5. Sliding fixing:

$$N_\nu\big|_c = 0, \quad \Psi_\nu\big|_c = 0, \quad \Psi_\tau\big|_c = 0. \tag{3.9}$$

The Kirchhoff theory:

$$\frac{\partial \Phi}{\partial \nu}\bigg|_c = 0, \quad \frac{\partial \Phi}{\partial \tau}\bigg|_c = 0. \tag{3.10}$$

The leading term of the boundary layer potential is nonzero and is determined by the relation

$$f'(\tau) = -\frac{h^2}{6\Gamma(1-\mu)}\frac{\partial \Delta\Phi}{\partial \nu}\bigg|_c. \tag{3.11}$$

6. Weakened clamping:

$$w\big|_c = 0, \quad \Psi_\nu\big|_c = 0, \quad M_\tau\big|_c = 0. \tag{3.12}$$

In the Kirchhoff theory, the boundary conditions have the form (3.2), just as in the case of clamping. The leading term of the boundary layer potential is zero, since the third condition in (3.12) is a consequence of the first two.

7. Weakened sliding fixing:

$$N_\nu\big|_c = 0, \quad \Psi_\nu\big|_c = 0, \quad M_\tau\big|_c = 0. \tag{3.13}$$

The Kirchhoff theory:

$$\frac{\partial \Phi}{\partial \nu}\bigg|_c = 0, \quad D\left[\frac{\partial \Delta\Phi}{\partial \nu} + (1-\mu)\frac{\partial^3\Phi}{\partial\nu\partial\tau^2}\right]_c = 0. \tag{3.14}$$

The leading term of the boundary layer potential is given by (3.6), which, in view of the first condition in (3.14), becomes

$$f(\tau) = -\frac{h^2}{6\Gamma}\frac{1}{R}\frac{\partial\Phi}{\partial\tau}\bigg|_c = 0;$$

that is, the leading term of the boundary layer potential is nonzero only if the contour of the plate is curvilinear.

8. Reinforced free edge:

$$N_\nu\big|_c = 0, \quad M_\nu\big|_c = 0, \quad \Psi_\tau\big|_c = 0. \tag{3.15}$$

The Kirchhoff theory:

$$\frac{\partial \Phi}{\partial \tau}\bigg|_c = 0, \quad D\left[\frac{\partial^2\Phi}{\partial\nu^2} + \frac{\mu}{R}\frac{\partial\Phi}{\partial\nu} + \mu\frac{\partial^2\Phi}{\partial\tau^2}\right]_c = 0. \tag{3.16}$$

The leading term of the boundary layer potential is determined by (3.11).

Conclusion. The Kirchhoff theory allows one to determine the asymptotically leading terms of natural frequencies and natural forms inside the domain of the plate for all types of boundary conditions possible in the Reissner theory.

4. FREE VIBRATIONS OF A RECTANGULAR PLATE HINGED ON TWO OPPOSITE SIDES

Consider a plate occupying the domain $-a \le x \le a$, $-b \le y \le b$. The conditions of constrained hinged support are assumed to be satisfied on the sides $y = \pm b$ and an arbitrary condition, on the sides $x = \pm a$. We study vibrations symmetric about the axes $x = 0$ and $y = 0$. It can be readily shown that the natural shapes satisfying the differential equations (1.1) and (1.2) and the boundary conditions (3.3) at $y = \pm b$ are given by

$$\Phi_n(x,y) = \left[C_{1n}\cos(\lambda_{1n}x) + C_{2n}\cos(\lambda_{2n}x)\right]\cos(\mu_n y), \quad F_n(x,y) = C_{3n}\sin(\delta_n x)\sin(\mu_n y), \tag{4.1}$$

where

$$\lambda_{1n} = \sqrt{A_n - B_n}, \quad \lambda_{2n} = \sqrt{A_n + B_n}, \quad \mu_n = \frac{(2n-1)\pi}{2b}, \quad \delta_n = \sqrt{\frac{\rho\omega_n^2}{G} - \frac{12\,\Gamma}{h^2} - \mu_n^2},$$

$$A_n = \left[1 + \tfrac{1}{2}\Gamma(1-\mu)\right]\frac{\rho\omega_n^2}{2G\Gamma} - \mu_n^2, \quad B_n = \sqrt{\frac{\rho h}{D} + \left[1 - \tfrac{1}{2}\Gamma(1-\mu)\right]\frac{\rho\omega_n^2}{2G\Gamma}}.$$

Satisfying the boundary conditions at $x = \pm a$, we arrive at a system of linear homogeneous algebraic equations for the unknowns C_{1n}, C_{2n}, and C_{3n}. Equating the determinant of this system with zero yields equations for determining the natural frequencies. When solving the problem discussed, the eight types of boundary conditions have been considered and for each of them, the frequency equations have been obtained. (These equations are not written out here, because they are too cumbersome.)

It should be noted that in the Reissner theory, there exist three spectra of natural frequencies, one low-frequency spectrum ($\omega \sim h$) and two high-frequency spectra ($\omega \sim h^{-1}$). The Kirchhoff theory, just as the theory taking account of the transverse shear strain near the plate boundary [1], allows one to calculate only the frequencies pertaining to the low-frequency spectrum. Therefore, in what follows we do not consider the high-frequency vibrations.

Let us briefly dwell on specific features of the solution of the problem in question according to the theory taking account of the transverse shear strain near the plate edge [1] (in what follows, we refer to this theory as the shear theory for brevity). The natural shapes satisfying equations (2.1) and (2.2) and the boundary conditions (3.3) at $y = \pm b$ have the form

$$\begin{aligned} \Phi_n(x,y) &= \left[C_{1n}\cos(\lambda_{1n}x) + C_{2n}\cos(\lambda_{2n}x)\right]\cos(\mu_n y), \\ F_n(x,y) &= f_n(y)\left\{\exp\left[-(a-x)\frac{\delta}{h}\right] - \exp\left[-(a+x)\frac{\delta}{h}\right]\right\}, \end{aligned} \tag{4.2}$$

where

$$\lambda_{1n} = \sqrt{-\mu_n^2 - \omega_n\sqrt{\frac{\rho h}{D}}}, \quad \lambda_{2n} = \sqrt{-\mu_n^2 + \omega_n\sqrt{\frac{\rho h}{D}}}.$$

Here we have already taken account of the symmetry about the axes $x = 0$ and $y = 0$). The function $f_n(y)$ is determined by the boundary conditions at $x = \pm a$: $f_n(y) = C_{3n}\sin(\mu_n y)$.

The asymptotic analysis has shown that the frequency equations and the natural shapes calculated by the shear theory follow from the exact (provided by the Reissner theory) frequency equations and natural shapes with an asymptotic error of $O(h^2)$.

Let us discuss the solution of the problem following from the Kirchhoff theory. The natural shapes satisfying the differential equation (2.1) and conditions (3.4) of constrained hinged support at $y = \pm b$ are determined by relations (4.2), provided that the function $f_n(y)$ is assumed to be identically zero. Let us make two remarks concerning the results of solving the problem under various types of boundary conditions at $x = \pm a$.

Remark 1. If the clamping conditions (3.1), the constrained hinged support conditions (3.3), the weakened clamping conditions (3.12), or the weakened sliding fixing conditions (3.13) are set on the sides $x = \pm a$, then the leading asymptotic term of the boundary layer potential is identically zero. Hence, for these types of boundary conditions, the Kirchhoff theory permits solving the problem on natural frequencies and natural shapes with an asymptotic error of $O(h^2)$, and the solution by the shear theory completely coincides with that given by the Kirchhoff theory.

Remark 2. If the conditions of constrained hinged support (3.3), free hinged support (3.5), or reinforced free edge (3.15) are set at $x = \pm a$, then the Kirchhoff theory provides the same solution. Indeed, the conditions of free and constrained hinged support in the Kirchhoff theory coincide and have the form (3.4), and the conditions of reinforced free edge (3.16) differ from (3.4) in only one condition: $\partial\Phi/\partial\tau\big|_c = 0$ is used instead of $\Phi\big|_c = 0$, with the former being a straightforward consequence of the latter. If the conditions of clamping (3.1), sliding fixing (3.9), or weakened clamping (3.12) are set at $x = \pm a$, then the Kirchhoff theory leads to a common solution as well. The conditions of clamping and weakened clamping in the Kirchhoff theory coincide and have the form (3.2), and the conditions of sliding fixing (3.10) differ from (3.2) in only one condition: $\partial\Phi/\partial\tau\big|_c = 0$ is used instead of $\Phi\big|_c = 0$.

Thus, the eight types of boundary conditions present in the Reissner theory can be divided into four groups:

(i) clamping, sliding fixing (∗), and weakened clamping (∗);

(ii) constrained hinged support (∗), free hinged support (∗), and reinforced free edge;

(iii) free edge; and

(iv) weakened sliding fixing (∗).

The groups combine the types of boundary conditions which are indistinguishable in the Kirchhoff theory. The asterisk marks the types of boundary conditions which result in asymptotic errors of the order of $O(h^2)$.

5. FREE VIBRATIONS OF A RECTANGULAR PLATE. COMPARATIVE ANALYSIS OF THE NUMERICAL RESULTS OBTAINED BY DIFFERENT THEORIES

This study is exemplified by the problem discussed in Section 4. The calculations were performed for a plate with dimensions $a = b = 1$ m and $h = 0.1, 0.04$ m and for the constants $E = 2.1 \cdot 10^{11}$ Pa, $\mu = 0.25$, $\Gamma = 5/6$, and $\rho = 7.951 \cdot 10^3$ kg/m^3, which characterize the elastic and inertial properties of the plate material.

Table 1

No	ω_R	ω	Δ	ω_K	δ_K
			$h = 0.1$		
1	740.831	746.960	0.83	756.153	2.07
2	3613.906	3762.387	4.11	3780.612	4.61
3	3614.672	3764.217	4.14	3780.612	4.59
4	6271.550	6716.355	7.09	6805.070	8.51
5	8895.791	9807.466	10.30	9829.529	10.50
			$h = 0.04$		
1	300.561	300.990	0.14	302.461	0.63
2	1499.312	1509.547	0.68	1512.245	0.86
3	1499.373	1509.670	0.69	1512.245	0.86
4	2676.121	2708.665	1.22	2722.028	1.72
5	3860.653	3928.806	1.77	3931.812	1.84

The numerical results can be summarized as follows.

1. The first ten natural frequencies of the low-frequency spectrum are calculated. The computations are made by three different theories (the Reissner theory, the Kirchhoff theory, and the shear theory). All the type of boundary conditions possible in the Reissner theory are considered.

2. Real errors emerging in using the approximate theories are calculated (for all types of boundary conditions as well).

3. The natural forms corresponding to the first ten natural frequencies are found for the types of boundary conditions for which the leading asymptotic term of the boundary layer potential is zero.

Let us now proceed to the detailed analysis of the results obtained. The types of boundary conditions for which the leading asymptotic term of the boundary layer potential vanishes—conditions (3.1), (3.3), (3.12), and (3.13)—are of little interest and will not be discussed in what follows. Note that the calculated estimates of the real errors made by the Kirchhoff theory for these types of boundary conditions agree well with those known previously.

Tables 1–4 present numerical results for the first five natural frequencies for the cases in which the conditions of hinged support (3.5), free edge (3.7), sliding fixing (3.9), and reinforced free edge (3.15) are set at $x = \pm a$, respectively. In these cases, the leading asymptotic term of the boundary layer potential does not vanish. The following notation is used in the tables: N is the frequency number, ω_R is the frequency predicted by the Reissner theory, ω is the frequency predicted by the shear theory, Δ is the relative error of the shear theory, $\Delta = (|\omega - \omega_R|/\omega_R) \times 100\%$, ω_K is the frequency predicted by the Kirchhoff theory, and δ_K is the relative error of the Kirchhoff theory, $\delta_K = (|\omega_K - \omega_R|/\omega_R) \times 100\%$.

Tables 1 and 2 show that the errors of the Kirchhoff theory in the cases of hinged support and free edge conditions behave "normally," that is, their values, their monotonic increase with the frequency number, and overstated values of the frequencies agree with the results obtained earlier by other researchers. The allowance for the transverse shear strain near the plate edge raises the accuracy of predicting the natural frequencies insignificantly for these types of boundary conditions. However, this does not mean at all that the shear theory offers no advantages over the Kirchhoff theory in this case. Indeed, since the leading term of the boundary layer potential is nonzero, the Kirchhoff theory makes quite large errors in the shearing forces and torsional moments; it is by taking into account the transverse shear strain near the plate boundary that these errors can be eliminated.

In the cases of sliding fixing and reinforced free edge (see Tables 3 and 4), the real errors of the Kirchhoff theory cannot be considered "normal." Oppositely, their behavior is quite strange: (i) they loose the property of monotonic increase with the frequency number and (ii) for some frequencies, they become so large that we have to conclude that the Kirchhoff theory is inapplicable for these types of boundary conditions. This is remarkable, since the frequency equations obtained within the framework of the Kirchhoff theory are the leading asymptotic terms of the corresponding equations obtained by the Reissner theory! Interestingly, the errors made by the shear theory are quite "normal" (see Tables 3 and 4), i.e., the same as the errors for the other types of boundary conditions. Such results suggest to check whether there is no miscalculation. To remove doubts on this matter, we write out the frequency equations obtained by the Kirchhoff theory and by the shear theory, as well as formulas for the relative errors δ_* of the Kirchhoff theory, $\delta_* = [(\omega_K - \omega)/\omega] \times 100\%$. (We do not give the frequency equations obtained by the Reissner theory, since they are too

Table 2

No	ω_R	ω	Δ	ω_K	δ_K
		$h = 0.1$			
1	370.032	371.565	0.41	372.024	0.54
2	1385.438	1404.897	1.40	3124.050	2.79
3	3255.978	3376.258	3.69	3381.774	3.86
4	4451.880	4667.923	4.85	4713.277	5.87
5	4840.758	5102.156	5.40	5133.873	6.05
		$h = 0.04$			
1	148.626	148.748	0.08	148.810	0.12
2	565.330	569.313	0.70	569.620	0.76
3	1343.761	1351.913	0.61	1352.709	0.67
4	1863.737	1885.127	1.15	1885.311	1.16
5	2030.259	2053.365	1.14	2053.549	1.15

Table 3

No	ω_R	ω	Δ	ω_K	δ_K
		$h = 0.1$			
1	772.548	773.774	0.16	1109.024	43.55
2	2202.267	2254.363	2.37	4945.102	124.55
3	3613.446	3752.419	3.85	3915.448	8.36
4	5669.845	5985.636	5.57	7653.923	35.00
5	6104.690	6537.085	7.08	11771.931	92.83
		$h = 0.04$			
1	371.902	371.840	0.02	443.610	19.28
2	1085.428	1089.106	0.34	1978.041	82.24
3	1526.035	1535.835	0.62	1566.179	2.63
4	2653.017	2685.316	1.22	4708.772	77.49
5	2671.404	2693.713	0.84	3061.569	14.61

cumbersome. For the same reason, we compare the frequencies predicted by the Kirchhoff theory with those calculated by the shear theory rather than by the Reissner theory.)

Sliding fixing. The Kirchhoff theory yields the frequency equation

$$\lambda_{1n}\cos(\lambda_{2n}a)\sinh(\lambda_{1n}a) + \lambda_{2n}\sin(\lambda_{2n}a)\cosh(\lambda_{1n}a) = 0, \tag{5.1}$$

where

$$\lambda_{1n} = \sqrt{\omega_n\sqrt{\frac{\rho h}{D}} + \mu_n^2}, \quad \lambda_{2n} = \sqrt{\omega_n\sqrt{\frac{\rho h}{D}} - \mu_n^2}, \quad \mu_n = \frac{(2n-1)\pi}{2b}.$$

The frequency equation obtained by the shear theory has the form

$$(1-\mu)\mu_n^2\left[\lambda_{1n}\cos(\lambda_{2n}a)\sinh(\lambda_{1n}a)+\lambda_{2n}\sin(\lambda_{2n}a)\cosh(\lambda_{1n}a)\right]-4\frac{h}{\delta}\omega_n\sqrt{\frac{\rho h}{D}}\,\lambda_{1n}\lambda_{2n}\sin(\lambda_{2n}a)\sinh(\lambda_{1n}a)=0. \tag{5.2}$$

The relative error of the Kirchhoff theory is approximately given by

$$\delta_* = \frac{4}{\delta(1-\mu)}\,\frac{h}{a}\left[\frac{\mu_n^4}{(\lambda_{1n}\lambda_{2n})^2}\left(1 + \frac{\cos(\lambda_{2n}a)}{\lambda_{2n}a\sin(\lambda_{2n}a)}\right) + \frac{\mu_n^2\cos^2(\lambda_{2n}a)}{\lambda_{2n}^2\sin^2(\lambda_{2n}a)}\right]^{-1} \times 100\%. \tag{5.3}$$

Table 4

No	ω_R	ω	Δ	ω_K	δ_K
		$h = 0.1$			
1	687.816	692.719	0.71	756.153	9.94
2	2002.924	2027.900	1.25	3780.612	88.75
3	3589.084	3734.339	4.05	3780.612	5.34
4	4870.231	5229.177	5.21	9829.529	97.77
5	5665.708	5966.331	5.31	6805.070	20.11
		$h = 0.04$			
1	291.858	292.226	0.13	302.461	3.63
2	1083.650	1086.898	0.30	1512.245	39.55
3	1495.818	1505.992	0.68	1512.245	1.10
4	2266.037	2284.607	0.81	3931.812	73.51
5	2583.023	2611.645	1.11	2722.028	5.38

Table 5

No	δ_* ($h = 0.1$)	δ_* ($h = 0.04$)
1	49.18	19.67
2	223.83	89.53
3	5.49	2.20
4	32.24	212.21
5	530.52	12.90

Table 6

No	δ_* ($h = 0.1$)	δ_* ($h = 0.04$)
1	8.51	3.41
2	76.52	30.61
3	0.95	0.38
4	212.54	85.02
5	8.51	3.40

Reinforced free edge. The frequency equation given by the Kirchhoff theory reads

$$\cos(\lambda_{2n}a) = 0, \qquad \lambda_{2n} = \sqrt{\omega_n\sqrt{\frac{\rho h}{D}} - \mu_n^2}, \quad \mu_n = \frac{(2n-1)\pi}{2b}. \tag{5.4}$$

The frequency equation given by the shear theory has the form

$$(1-\mu)\mu_n^2\cos(\lambda_{2n}a)\cosh(\lambda_{1n}a) - \frac{h}{\delta}\omega_n\sqrt{\frac{\rho h}{D}}\left[\lambda_{1n}\cos(\lambda_{2n}a)\sinh(\lambda_{1n}a) + \lambda_{2n}\sin(\lambda_{2n}a)\cosh(\lambda_{1n}a)\right] = 0. \tag{5.5}$$

The relative error is

$$\delta_* = \frac{2}{\delta(1-\mu)}\frac{h}{a}\frac{\lambda_{2n}^2}{\mu_n^2} \times 100\%. \tag{5.6}$$

Tables 5 and 6 present some values of δ_* calculated by formula (5.3) for the case of sliding fixing and by formula (5.6) for the case of strengthened free edge, respectively. The comparison of δ in Tables 5 and 6 with the corresponding δ_K in Tables 3 and 4 shows quite good agreement, which rules out any miscalculation.

So why does the Kirchhoff theory turn out to be inapplicable for the two types of boundary conditions? The physical background of this phenomenon will be discussed in Section 6. From the computational viewpoint, the cause is seemingly the fact that the boundary layer potential for these types of boundary conditions turns out to be much greater than for the other types (of course, the real values of the boundary layer potential are meant rather than its asymptotic order). Below we present expressions for the natural shapes (calculated by the Reissner theory for $h = 0.1$ m) in the case of free hinged support (for which the Kirchhoff theory provides "normal" results) and in the case of sliding fixing (for which the Kirchhoff theory is inapplicable).

Free hinged support:

$$\Phi_1 = 0.0349\cos(1.55\,x)\cos(1.57\,y), \quad F_1 = 0.0002\left[e^{-31.66\,(1-x)} - e^{-31.66\,(1+x)}\right]\sin(1.57\,y),$$
$$\Phi_2 = 0.0327\cos(1.54\,x)\cos(4.71\,y), \quad F_2 = 0.0005\left[e^{-31.95\,(1-x)} - e^{-31.95\,(1+x)}\right]\sin(4.71\,y),$$
$$\Phi_3 = -0.0331\cos(4.70\,x)\cos(1.57\,y), \quad F_3 = 0.0005\left[e^{-31.64\,(1-x)} - e^{-31.64\,(1+x)}\right]\sin(1.57\,y),$$
$$\Phi_4 = -0.0314\cos(4.66\,x)\cos(4.71\,y), \quad F_4 = 0.0015\left[e^{-31.91\,(1-x)} - e^{-31.91\,(1+x)}\right]\sin(4.71\,y),$$
$$\Phi_5 = 0.0296\cos(1.53\,x)\cos(7.85\,y), \quad F_5 = 0.0009\left[e^{-32.47\,(1-x)} - e^{-32.47\,(1+x)}\right]\sin(7.85\,y).$$

Sliding fixing:

$$\Phi_1 = -0.0290\cos(1.62\,x)\cos(1.57\,y), \quad F_1 = 0.0008\left[e^{-31.66\,(1-x)} - e^{-31.66\,(1+x)}\right]\sin(1.57\,y),$$
$$\Phi_2 = 0.0295\cos(3.50\,x)\cos(1.57\,y), \quad F_2 = 0.0019\left[e^{-31.65\,(1-x)} - e^{-31.65\,(1+x)}\right]\sin(1.57\,y),$$
$$\Phi_3 = -0.0320\cos(1.53\,x)\cos(4.71\,y), \quad F_3 = 0.0014\left[e^{-31.95\,(1-x)} - e^{-31.95\,(1+x)}\right]\sin(4.71\,y),$$
$$\Phi_4 = 0.0278\cos(4.15\,x)\cos(4.71\,y), \quad F_4 = 0.0045\left[e^{-31.92\,(1-x)} - e^{-31.92\,(1+x)}\right]\sin(4.71\,y),$$
$$\Phi_5 = -0.0311\cos(6.34\,x)\cos(1.57\,y), \quad F_5 = 0.0017\left[e^{-31.61\,(1-x)} - e^{-31.61\,(1+x)}\right]\sin(1.57\,y).$$

One can see from these expressions that in the case of sliding fixing, indeed, the boundary layer potential is three times as large as that in the case of free hinged support.

Thus, it turns out that in proceeding to the Kirchhoff theory, the terms containing the boundary layer potential are discarded; in the case of sliding fixing and reinforced free edge, these terms are not small compared with the other terms. This is why inadmissibly large errors appear in the Kirchhoff theory.

6. DISCUSSION OF THE PHYSICAL MEANING OF THE RESULTS

Consider the types of boundary conditions for which the Kirchhoff theory turns out to be inapplicable, namely, the sliding fixing conditions

$$N_\nu\big|_c = 0, \quad \Psi_\nu\big|_c = 0, \quad \Psi_\tau\big|_c = 0$$

and the reinforced free edge conditions

$$N_\nu\big|_c = 0, \quad M_\nu\big|_c = 0, \quad \Psi_\tau\big|_c = 0.$$

We see from formulas (2.3) that in both cases, the only condition whose leading term depends on the boundary layer potential is the vanishing of the shearing force. Therefore, in proceeding to the Kirchhoff theory, this condition is eliminated and the boundary conditions are stated as follows: for sliding fixing, $\Psi_\nu\big|_c = 0$ and $\Psi_\tau\big|_c = 0$, and for reinforced free edge, $M_\nu\big|_c = 0$ and $\Psi_\tau\big|_c = 0$.

When discussing specific features of stating the conditions of sliding fixing and reinforced free edge in the Kirchhoff theory, we must pay attention to the following two circumstances.

The angle of rotation about the normal to the plate contour in the Kirchhoff theory is not an independent variable and is expressed via the deflection by the formula $\Psi_\tau = -\partial w/\partial\tau$. This means that the condition $\Psi_\tau\big|_c = 0$ contradicts, from the physical viewpoint, the condition $N_\nu\big|_c = 0$. Formally, the latter condition can be satisfied by taking into account the leading asymptotic term of the boundary layer potential, but actually it turns out that both the deflection and the shearing force are specified on the contour, which, generally speaking, makes no sense from the physical viewpoint.

Since in the Kirchhoff theory, the condition $\Psi_\tau\big|_c = 0$ is a consequence of the condition $w\big|_c = 0$, it turns out that the sliding fixing conditions are equivalent to the clamping conditions, and the conditions of reinforced free edge are, in turn, equivalent to the conditions of hinged support. From the physical viewpoint, such a situation is absurduous.

This allows us to draw the following conclusion. Although the transition to the Kirchhoff theory in the case of sliding fixing conditions on the contour is formally possible, the resulting statements do not reflect the physical meaning of the problem at all. Therefore, it is not surprising that the Kirchhoff theory is inapplicable for the types of boundary conditions in question.

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REFERENCES

[1] E. A. Ivanova, "Approximate Hamiltonian functionals in problems on low-frequency and high-frequency free vibrations of Reissner's plate," Izv. AN. MTT [Mechanics of Solids], No. 4, pp. 181–190, 1995.

[2] Vibrations in Engineering. Volume 1 [in Russian], Mashinostroenie, Moscow, 1978.

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