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ON ONE APPROACH TO SOLVING THE DARBOUX PROBLEM

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The Darboux problem is one of the determination of the quantities characterizing the orientation of a rigid body by the prescribed time history of the angular velocity vector and the initial orientation of the body. This problem has been drawing attention of researchers for long. Darboux, who initiated these investigations, has reduced this problem to the solution of a Riccati equation for a complex variable [1]. Despite the apparent simplicity and attractiveness, the equation obtained in the work by Darboux virtually is not utilized for solving specific problems. In practice, preference is usually given to methods based on the direct integration of Poisson's kinematic equations. No solution of the Darboux problem for an arbitrary time history of the angular velocity has been known. However, these are a number of special cases for which the exact analytical solution of this problem can be constructed. The integrability conditions for the Darboux problem were investigated, for example, in [2–4].

It is worthwhile to note also the work [5] proving the possibility of solving the Darboux problem in closed form if the time history of the angular velocity is approximated by a trigonometric series. In the present paper, an alternative approach to the solution of the Darboux problem is suggested. This approach is based on the concepts of the tensor of rotation and right and left angular velocities. This approach makes it possible to identity the Darboux problem with one of the determination of a vector function by known absolute values of its derivatives. This is of considerable interest, since a number of problems in mechanics lead to the determination of a vector function by the absolute values of its derivatives.

The approach suggested reduces the Darboux problem to solving a third-order linear differential equation with respect to a real variable. The classical approach based on the Riccati equation leads to a fourth-order linear differential equation for a real variable.

This work is a development of the ideas of [6].

1. TENSOR OF ROTATION. LEFT AND RIGHT VECTORS OF ANGULAR VELOCITY

In this section, we briefly outline some familiar results which will be used later in this paper. For more detailed information about the tensor of rotation and related concepts, see, for example, [7, 8].

DEFINITION. *A tensor of rotation is a proper orthogonal tensor which satisfies the relations*

$$\mathbf{P}\cdot\mathbf{P}^{\mathrm{T}}=\mathbf{P}^{\mathrm{T}}\cdot\mathbf{P}=\mathbf{E},\quad \det P=+1. \tag{1.1}$$

The identities

$$(\mathbf{P}\cdot\mathbf{a})\cdot(\mathbf{P}\cdot\mathbf{b})=\mathbf{a}\cdot\mathbf{b},\quad (\mathbf{P}\cdot\mathbf{a})\times(\mathbf{P}\cdot\mathbf{b})=\mathbf{P}\cdot(\mathbf{a}\times\mathbf{b}) \tag{1.2}$$

are valid for any rotation tensor **P** and any vectors **a** and **b**.

THEOREM (REPRESENTATION OF THE TENSOR OF ROTATION). *Let* $\mathbf{d}_1$, $\mathbf{d}_2$, *and* $\mathbf{d}_3$ *be arbitrary linearly independent vectors and let* $\mathbf{D}_1$, $\mathbf{D}_2$, *and* $\mathbf{D}_3$ *be the vectors related to the former three vectors by a tensor of rotation* **P** *in accordance with the relation* $\mathbf{D}_i=\mathbf{P}\cdot\mathbf{d}_i$. *Then the tensor of rotation* **P** *can represented in the form*

$$\mathbf{P}=\mathbf{D}_1\mathbf{d}^1+\mathbf{D}_2\mathbf{d}^2+\mathbf{D}_3\mathbf{d}^3, \tag{1.3}$$

where $\mathbf{d}^1$, $\mathbf{d}^2$, *and* $\mathbf{d}^3$ *are the unit vectors of the basis which is dual to that defined by the vectors* $\mathbf{d}_1$, $\mathbf{d}_2$, *and* $\mathbf{d}_3$. *The vectors* $\mathbf{d}^1$, $\mathbf{d}^2$, *and* $\mathbf{d}^3$ *are defined by*

$$\mathbf{d}^1 = \frac{\mathbf{d}_2 \times \mathbf{d}_3}{\mathbf{d}_1 \cdot (\mathbf{d}_2 \times \mathbf{d}_3)}, \quad \mathbf{d}^2 = \frac{\mathbf{d}_3 \times \mathbf{d}_1}{\mathbf{d}_2 \cdot (\mathbf{d}_3 \times \mathbf{d}_1)}, \quad \mathbf{d}^3 = \frac{\mathbf{d}_1 \times \mathbf{d}_2}{\mathbf{d}_3 \cdot (\mathbf{d}_1 \times \mathbf{d}_2)}. \tag{1.4}$$

The proof of this theorem is based on the definition of the dual basis in terms of the relations $\mathbf{d}_i \cdot \mathbf{d}^j = \delta_i^j$, where δ_i^j is the Kronecker delta, and the representation of the unit tensor in the form $\mathbf{E} = \mathbf{D}_1\mathbf{D}^1 + \mathbf{D}_2\mathbf{D}^2 + \mathbf{D}_3\mathbf{D}^3$.

DEFINITION. *If* $\mathbf{P}(t)$ *is a rotation tensor, the tensors* $\mathbf{S}(t)$ *and* $\mathbf{S}_r(t)$ *defined by the expressions*

$$\mathbf{S}(t) = \dot{\mathbf{P}}(t) \cdot \mathbf{P}^{\mathrm{T}}(t), \quad \mathbf{S}_r(t) = \mathbf{P}^{\mathrm{T}}(t) \cdot \dot{\mathbf{P}}(t) \tag{1.5}$$

are called the right and left spin tensors corresponding to $\mathbf{P}(t)$, *respectively.*

DEFINITION. *The concomitant vectors,* $\boldsymbol{\omega}$ *and* $\boldsymbol{\Omega}$, *of the left and right spin tensors defined by the relations*

$$\mathbf{S}(t) = \boldsymbol{\omega}(t) \times \mathbf{E}, \quad \mathbf{S}_r(t) = \boldsymbol{\Omega}(t) \times \mathbf{E} \tag{1.6}$$

are called the left and right angular velocities, respectively.

The left and right angular velocities are related by

$$\boldsymbol{\omega}(t) = \mathbf{P}(t) \cdot \boldsymbol{\Omega}(t), \quad \boldsymbol{\Omega}(t) = \mathbf{P}^{\mathrm{T}}(t) \cdot \boldsymbol{\omega}(t). \tag{1.7}$$

As applied to the motion of a rigid body, the left angular velocity $\boldsymbol{\omega}$ is usually referred to as the *true angular velocity* and the right angular velocity $\boldsymbol{\Omega}$ as the *angular velocity in the body.* Given a tensor of rotation $\mathbf{P}$, one can calculate the left and right angular velocities using the expressions

$$\boldsymbol{\omega}(t) = -\frac{1}{2}(\dot{\mathbf{P}} \cdot \mathbf{P}^{\mathrm{T}})_{\times}, \quad \boldsymbol{\Omega}(t) = -\frac{1}{2}(\mathbf{P}^{\mathrm{T}} \cdot \dot{\mathbf{P}})_{\times}. \tag{1.8}$$

The inverse problem, i.e., the problem of determining the rotation tensor by a known angular velocity, is substantially more complicated. This problem is known as the Darboux problem. It can be stated in two forms, depending on which of the angular velocities (left or right) is given.

The left Darboux problem is the initial value problem of the form

$$\dot{\mathbf{P}}(t) = \boldsymbol{\omega}(t) \times \mathbf{P}(t), \quad \dot{\mathbf{P}}(0) = \mathbf{P}_0. \tag{1.9}$$

The right Darboux problem has the form

$$\dot{\mathbf{P}}(t) = \mathbf{P}(t) \times \boldsymbol{\Omega}(t), \quad \mathbf{P}(0) = \mathbf{P}_0. \tag{1.10}$$

Let the left angular velocity, $\boldsymbol{\omega}$, be known. By solving the Darboux problem of (1.9) one can calculate the rotation tensor $\mathbf{P}$ and then, by using the second relation of (1.8), determine the right angular velocity $\boldsymbol{\Omega}$. An alternative approach to the determination of the tensor $\mathbf{P}$ and vector $\boldsymbol{\Omega}$ is also possible. This approach involves the determination of the right angular velocity $\boldsymbol{\Omega}$ by the given left angular velocity $\boldsymbol{\omega}$ by solving a certain differential equation. Knowing both the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$, one can restore the rotation tensor $\mathbf{P}$ without additional integration. This approach applies also to the case where the right angular velocity is known, while the rotation tensor $\mathbf{P}$ and the left angular velocity are to the determined.

The left and right Darboux problems are equivalent mathematically. For this reason, without loss of generality one can consider the solution of one of these problems. As a rule, it is the right Darboux problem that is treated in the literature. We will give below new statements of both problems and consider two example dynamical problems one of which is reduced to the right Darboux problem and the other to the left one.

2. REPRESENTATION OF THE ROTATION TENSOR IN TERMS OF THE LEFT AND RIGHT ANGULAR VELOCITIES

The representation of the rotation tensor in terms of the angular velocities relies on the facts and considerations listed below.

1. According to the theorem of the representation of a rotation tensor in the form of Eqs. (1.3) and (1.4), to determine the rotation tensor $\mathbf{P}$, it suffices to know three pairs of vectors $\mathbf{D}_i$ and $\mathbf{d}_i$ related by $\mathbf{D}_i = \mathbf{P}\cdot\mathbf{d}_i$. The vectors $\mathbf{d}_i$ must be linearly independent.
2. One can take the left and right angular velocities to be the vectors of one of the pairs $(\mathbf{D}_i, \mathbf{d}_i)$, since, in accordance with (1.7), these vectors are related by $\boldsymbol{\omega} = \mathbf{P}\cdot\boldsymbol{\Omega}$.
3. If one can find two more pairs of vectors $(\mathbf{f}_i(\boldsymbol{\omega}), \boldsymbol{\varphi}_i(\boldsymbol{\Omega}))$ such that (i) the vectors $\boldsymbol{\Omega}$, $\boldsymbol{\varphi}_1(\boldsymbol{\Omega})$, and $\boldsymbol{\varphi}_2(\boldsymbol{\Omega})$ are linearly independent and (ii) the relations $\mathbf{f}_i(\boldsymbol{\omega}) = \mathbf{P}\cdot\boldsymbol{\varphi}_i(\boldsymbol{\Omega})$ are satisfied, then one can represent the rotation tensor in the form of Eqs. (1.3) and (1.4) in terms of these pairs of vectors.

There exist infinitely many relations of the form $\mathbf{f}_i(\boldsymbol{\omega}) = \mathbf{P}\cdot\boldsymbol{\varphi}_i(\boldsymbol{\Omega})$. We will use the simplest ones,

$$\dot{\boldsymbol{\omega}} = \mathbf{P}\cdot\dot{\boldsymbol{\Omega}}, \quad \boldsymbol{\omega}\times\dot{\boldsymbol{\omega}} = \mathbf{P}\cdot(\boldsymbol{\Omega}\times\dot{\boldsymbol{\Omega}}).$$

The proof of this relations is based on Eqs. (1.7) and (1.10) and the identities of (1.2). We have

$$\dot{\boldsymbol{\omega}} = \frac{d}{dt}(\mathbf{P}\cdot\boldsymbol{\Omega}) = \dot{\mathbf{P}}\cdot\boldsymbol{\Omega} + \mathbf{P}\cdot\dot{\boldsymbol{\Omega}} = (\mathbf{P}\times\boldsymbol{\Omega})\cdot\boldsymbol{\Omega} + \mathbf{P}\cdot\dot{\boldsymbol{\Omega}} = \mathbf{P}\cdot\dot{\boldsymbol{\Omega}},$$
$$\boldsymbol{\omega}\times\dot{\boldsymbol{\omega}} = (\mathbf{P}\cdot\boldsymbol{\Omega})\times(\mathbf{P}\cdot\dot{\boldsymbol{\Omega}}) = \mathbf{P}\cdot(\boldsymbol{\Omega}\times\dot{\boldsymbol{\Omega}}).$$

One can readily show that the vectors $\boldsymbol{\Omega}$, $\dot{\boldsymbol{\Omega}}$, and $\boldsymbol{\Omega}\times\dot{\boldsymbol{\Omega}}$ are linearly independent, with the exception of the trivial case where the vector $\boldsymbol{\Omega}$ is constant in direction. Thus, we have found three pairs of vectors—$(\boldsymbol{\omega}, \boldsymbol{\Omega})$, $(\dot{\boldsymbol{\omega}}, \dot{\boldsymbol{\Omega}})$, and $(\boldsymbol{\omega}\times\dot{\boldsymbol{\omega}}, \boldsymbol{\Omega}\times\dot{\boldsymbol{\Omega}})$—satisfying the conditions of the theorem of Section 1. The vectors in these pairs are related by

$$\boldsymbol{\omega} = \mathbf{P}\cdot\boldsymbol{\Omega}, \quad \dot{\boldsymbol{\omega}} = \mathbf{P}\cdot\dot{\boldsymbol{\omega}}, \quad \boldsymbol{\omega}\times\dot{\boldsymbol{\omega}} = \mathbf{P}\cdot(\boldsymbol{\Omega}\times\dot{\boldsymbol{\Omega}}). \tag{2.1}$$

Hence, the rotation tensor can be represented in the form

$$\mathbf{P} = \mathbf{D}_1\mathbf{d}^1 + \mathbf{D}_2\mathbf{d}^2 + \mathbf{D}_3\mathbf{d}^3, \quad \mathbf{D}_1 = \boldsymbol{\omega}, \quad \mathbf{D}_2 = \dot{\boldsymbol{\omega}}, \quad \mathbf{D}_3 = \boldsymbol{\omega}\times\dot{\boldsymbol{\omega}},$$
$$\mathbf{d}^1 = \frac{(\dot{\boldsymbol{\Omega}}\cdot\dot{\boldsymbol{\Omega}})\boldsymbol{\Omega} - \frac{1}{2}\frac{d}{dt}(\Omega^2)\dot{\boldsymbol{\Omega}}}{(\boldsymbol{\Omega}\times\dot{\boldsymbol{\Omega}})^2}, \quad \mathbf{d}^2 = \frac{\Omega^2\dot{\boldsymbol{\Omega}} - \frac{1}{2}\frac{d}{dt}(\Omega^2)\boldsymbol{\Omega}}{(\boldsymbol{\Omega}\times\dot{\boldsymbol{\Omega}})^2}, \quad \mathbf{d}^3 = \frac{(\boldsymbol{\Omega}\times\dot{\boldsymbol{\Omega}})}{(\boldsymbol{\Omega}\times\dot{\boldsymbol{\Omega}})^2}. \tag{2.2}$$

By simple transformations one can reduce expression (2.2) for the rotation tensor to the form symmetric with respect to the angular velocities $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$,

$$\mathbf{P} = A\boldsymbol{\omega}\boldsymbol{\Omega} + B\frac{d}{dt}(\boldsymbol{\omega}\boldsymbol{\Omega}) + C\dot{\boldsymbol{\omega}}\dot{\boldsymbol{\Omega}} + D(\boldsymbol{\omega}\times\dot{\boldsymbol{\omega}})(\boldsymbol{\omega}\times\dot{\boldsymbol{\Omega}}). \tag{2.3}$$

The scalar coefficients A, B, C, and D in (2.3) are defined by

$$A = \dot{\boldsymbol{\Omega}}\cdot\dot{\boldsymbol{\Omega}}D = \dot{\boldsymbol{\omega}}\cdot\dot{\boldsymbol{\omega}}D, \quad B = -\frac{1}{2}\frac{d}{dt}(\Omega^2)D = -\frac{1}{2}\frac{d}{dt}(\omega^2)D,$$
$$C = \Omega^2 D = \omega^2 D, \quad D = \frac{1}{(\boldsymbol{\Omega}\times\dot{\boldsymbol{\Omega}})^2} = \frac{1}{(\boldsymbol{\omega}\times\dot{\boldsymbol{\omega}})^2}. \tag{2.4}$$

The Darboux problem in the traditional formulation is one of the determination of the rotation tensor by one of the angular velocities (left or right). To solve this problem one should integrate the corresponding system of differential equations. We have shown above that if both angular velocities are known, then one can determine the rotation tensor in accordance with the relations of (2.3) and (2.4), additional integration being not required. In view of this, one can represent the Darboux problem in the following formulations:

LEFT DARBOUX PROBLEM. *Given the left angular velocity, determine the right angular velocity.*

RIGHT DARBOUX PROBLEM. *Given the right angular velocity, determine the left angular velocity.*

These formulations form a basis for the new approach to solving the Darboux problem which is developed in the present paper.

3. SCALAR EQUATIONS RELATING THE LEFT AND RIGHT ANGULAR VELOCITIES

This section plays an auxiliary role. Some scalar relations necessary for the subsequent constructions are presented here. The derivation of these relations is based on Eqs. (1.9), (1.10), and (2.1).

Equations (2.1) imply the scalar relations

$$\omega^2 = \Omega^2, \quad \dot{\omega} \cdot \dot{\omega} = \dot{\mathbf{\Omega}} \cdot \dot{\mathbf{\Omega}}, \quad (\omega \times \dot{\omega})^2 = (\mathbf{\Omega} \times \dot{\mathbf{\Omega}})^2. \tag{3.1}$$

To obtain scalar relations containing the second derivatives of the angular velocities, we differentiate the second equation of (2.1). By transforming the result of this differentiation with reference to Eqs. (1.9) and (1.10) and also the third equation of (2.1), we arrive at the vector equations

$$\ddot{\omega} = \mathbf{P} \cdot (\ddot{\mathbf{\Omega}} + \mathbf{\Omega} \times \dot{\mathbf{\Omega}}), \quad \ddot{\omega} - \omega \times \dot{\omega} = \mathbf{P} \cdot \ddot{\mathbf{\Omega}}. \tag{3.2}$$

Equations (2.1) and (3.2) imply the scalar relations

$$\ddot{\omega} \cdot \ddot{\omega} = (\ddot{\mathbf{\Omega}} + \mathbf{\Omega} \times \dot{\mathbf{\Omega}})^2, \quad \ddot{\omega} \cdot (\omega \times \dot{\omega}) = \ddot{\mathbf{\Omega}} \cdot (\mathbf{\Omega} \times \dot{\mathbf{\Omega}}) + (\mathbf{\Omega} \times \dot{\mathbf{\Omega}})^2, \tag{3.3}$$

$$\ddot{\mathbf{\Omega}} \cdot \ddot{\mathbf{\Omega}} = (\ddot{\omega} - \omega \times \dot{\omega})^2, \quad \ddot{\mathbf{\Omega}} \cdot (\mathbf{\Omega} \times \dot{\mathbf{\Omega}}) = \ddot{\omega} \cdot (\omega \times \dot{\omega}) - (\omega \times \dot{\omega})^2, \tag{3.4}$$

$$(\omega \times \dot{\omega}) \cdot (\dot{\omega} \times \ddot{\omega}) = (\mathbf{\Omega} \times \dot{\mathbf{\Omega}}) \cdot (\dot{\mathbf{\Omega}} \times \ddot{\mathbf{\Omega}}). \tag{3.5}$$

When deriving the relations of (3.1)–(3.5) we used the identities of (1.2).

Remark. The scalar relations obtained permit one to identify the Darboux problem with one of the determination of a vector function by the absolute values of the function itself and its first and second derivatives.

Let the left angular velocity vector, ω, be known. Then, using the first two relations of (3.1) and the first relation of (3.4), one can determine the absolute values of the derivatives of the right angular velocity vector $\mathbf{\Omega}$,

$$|\mathbf{\Omega}| = \varphi_1(t), \quad |\dot{\mathbf{\Omega}}| = \varphi_2(t), \quad |\ddot{\mathbf{\Omega}}| = \varphi_3(t). \tag{3.6}$$

Hence, the left Darboux problem can be stated as one of the determination of the right angular velocity, given the absolute values (3.6) of its derivatives.

In a similar way, if the right angular velocity $\mathbf{\Omega}$ is known, then using the first two relations of (3.1) and the first relation of (3.3), one can determine the absolute values of the derivatives of the left angular velocity,

$$|\omega| = f_1(t), \quad |\dot{\omega}| = f_2(t), \quad |\ddot{\omega}| = f_3(t). \tag{3.7}$$

Accordingly, the right Darboux problem can be stated as the problem of determining the left angular velocity ω, given the absolute values (3.7) of its derivatives.

The identification of the Darboux problem with that of the determination of a vector function by the absolute values of its derivatives is of interest, since a number of problems of mechanics is reduced to the latter problem. For example, a problem of this sort is one of the determination of the position vector of a particle, $\mathbf{R}$, if the length of this vector, as well as the magnitude of the particle velocity and acceleration are prescribed as functions of time, i.e.,

$$|\mathbf{R}(t)| = r(t), \quad |\dot{\mathbf{R}}(t)| = v(t), \quad |\ddot{\mathbf{R}}(t)| = w(t). \tag{3.8}$$

4. DETERMINATION OF A VECTOR FUNCTION BY ITS SCALAR CHARACTERISTICS

Consider an arbitrary vector function $\mathbf{R}(t)$. We assume that this vector is not constant in direction. In this case, the vectors

$$\mathbf{e}_1 = \mathbf{R}(t), \quad \mathbf{e}_2 = \dot{\mathbf{R}}(t), \quad \mathbf{e}_3 = \mathbf{R}(t) \times \dot{\mathbf{R}}(t) \tag{4.1}$$

are linearly independent and, hence, form a basis. We represent the vector $\ddot{\mathbf{R}}$ in the basis of (4.1),

$$\ddot{\mathbf{R}}(t) = -a(t)\dot{\mathbf{R}}(t) - b(t)\mathbf{R}(t) + c(t)\mathbf{R}(t) \times \dot{\mathbf{R}}(t). \tag{4.2}$$

The functions a, b, and c, can be expressed as

$$a(t) = \ddot{\mathbf{R}} \cdot \mathbf{e}^2 = -\frac{d}{dt}(\ln|\mathbf{R} \times \dot{\mathbf{R}}|), \quad b(t) = -\ddot{\mathbf{R}} \cdot \mathbf{e}^1 = \frac{(\mathbf{R} \times \dot{\mathbf{R}}) \cdot (\dot{\mathbf{R}} \times \ddot{\mathbf{R}})}{(\mathbf{R} \times \dot{\mathbf{R}})^2}, \quad c(t) = \ddot{\mathbf{R}} \cdot \mathbf{e}^3 = \frac{\ddot{\mathbf{R}} \cdot (\mathbf{R} \times \dot{\mathbf{R}})}{(\mathbf{R} \times \dot{\mathbf{R}})^2}. \tag{4.3}$$

If the scalar functions $a(t), b(t)$, and $c(t)$ are known, then Eq. (4.2) is a nonlinear differential equation with variable coefficients for the vector function $\mathbf{R}(t)$. If $c(t) \equiv 0$, then this equation is linear. If $c(t) \neq 0$, then Eq. (4.2) can be reduced to a third-order linear equation for the vector $\mathbf{R}$. To this end, one should (i) differentiate Eq. (4.2), (ii) multiply this equation vectorially by $\mathbf{R}$, and (iii) add the resulting equations and Eq. (4.2) together having first multiplied these by appropriate scalar coefficients. The resulting equation has the form

$$\dddot{\mathbf{R}}(t) + A(t)\ddot{\mathbf{R}}(t) + B(t)\dot{\mathbf{R}}(t) + C(t)\mathbf{R}(t) = \mathbf{0}, \tag{4.4}$$

where

$$A(t) = 2a(t) - \frac{d}{dt}\ln c(t), \quad B(t) = \dot{a}(t) + a(t)\left[a(t) - \frac{d}{dt}\ln c(t)\right] + b(t) + c^2(t)R^2(t),$$
$$C(t) = \dot{b}(t) + b(t)\left[a(t) - \frac{d}{dt}\ln c(t)\right] - \frac{1}{2}c^2(t)\frac{d}{dt}[\mathbf{R}^2(t)]. \tag{4.5}$$

Thus, if the scalar functions (4.3) for a vector $\mathbf{R}$ are known, then this vector can be identified by solving the second-order nonlinear differential equation (4.2) or the third-order linear differential equation (4.4).

Remark. Equations (4.2) and (4.4) permit one to determine the unknown vector function $\mathbf{R}(t)$ by prescribed absolute values of its derivatives.

To prove this proposition, it suffices to express the scalar coefficients (4.3) in terms of the functions $|\mathbf{R}|$, $|\dot{\mathbf{R}}|$, and $|\ddot{\mathbf{R}}|$. These expressions have the form

$$a(t) = -\frac{1}{2}\frac{d}{dt}\left\{\ln\left[R^2v^2 - \frac{1}{4}\left(\frac{d}{dt}(R^2)\right)^2\right]\right\},$$
$$b(t) = \left[\frac{1}{4}\frac{d}{dt}(R^2)\frac{d}{dt}(v^2) - \frac{1}{2}\frac{d^2}{dt^2}(R^2)v^2 + v^4\right]\left\{R^2v^2 - \frac{1}{4}\left[\frac{d}{dt}(R^2)\right]^2\right\}^{-1}$$
$$c^2(t) = \left\{w^2 + \frac{1}{2}a(t)\frac{d}{dt}(v^2) + b(t)\left[\frac{1}{2}\frac{d^2}{dt^2}(R^2) - v^2\right]\right\}\left\{R^2v^2 - \frac{1}{4}\left[\frac{d}{dt}(R^2)\right]^2\right\}^{-1},$$
$$v^2 = \dot{\mathbf{R}}\cdot\dot{\mathbf{R}}, \quad w^2 = \ddot{\mathbf{R}}\cdot\ddot{\mathbf{R}}. \tag{4.6}$$

5. VECTOR STATEMENTS OF THE DARBOUX PROBLEM

In this section, we treat the Darboux problem as a problem of determining one of the angular velocities, provided that the other angular velocity is known. We present below two statements for the left Darboux problem and two similar statements for the right Darboux problem. One of these statements is based on a nonlinear second-order equation of the type of (4.2), while the other uses a linear third-order equation of the type of (4.4). These statements apply only to the cases where the angular velocities are not constant in direction.

Left Darboux problem (determination of the right angular velocity by the known left angular velocity).

Statement 1. The initial value problem for determining the right angular velocity has the form

$$\ddot{\mathbf{\Omega}} + a(\omega)\dot{\mathbf{\Omega}} + b(\omega)\mathbf{\Omega} = c(\omega)\mathbf{\Omega}\times\dot{\mathbf{\Omega}}, \tag{5.1}$$

$$\mathbf{\Omega}(0) = \mathbf{P}_0^{\mathrm{T}}\cdot\omega_0, \quad \dot{\mathbf{\Omega}}(0) = \mathbf{P}_0^{\mathrm{T}}\cdot\dot{\omega}_0. \tag{5.2}$$

The coefficients $a(\omega)$, $b(\omega)$, and $c(\omega)$ are calculated in accordance with relations (4.3) and the scalar relations (3.1), (3.4), and (3.5) between the angular velocities; we have

$$a(\omega) = -\frac{d}{dt}(\ln|\omega\times\dot{\omega}|), \quad b(\omega) = \frac{(\omega\times\dot{\omega})\cdot(\dot{\omega}\times\ddot{\omega})}{(\omega\times\dot{\omega})^2}, \quad c(\omega) = \frac{\ddot{\omega}\cdot(\omega\times\dot{\omega})}{(\omega\times\dot{\omega})^2} - 1. \tag{5.3}$$

Statement 2 follows from Statement 1. The differential equation has the form

$$\dddot{\mathbf{\Omega}} + A(\omega)\ddot{\mathbf{\Omega}} + B(\omega)\dot{\mathbf{\Omega}} + C(\omega)\mathbf{\Omega} = \mathbf{0}. \tag{5.4}$$

The differential equation (5.1) at $t = 0$ and the initial conditions (5.2) are used to form the initial conditions for Eq. (5.4). The coefficients $A(\omega)$, $B(\omega)$, and $C(\omega)$ are related to $a(\omega)$, $b(\omega)$, and $c(\omega)$ by the relations of (4.5).

Right Darboux problem (determination of the left angular velocity by the right angular velocity).

Statement 1. The initial value problem for the left angular velocity reads

$$\ddot{\omega} + a(\Omega)\dot{\omega} + b(\Omega)\omega = c(\Omega)\omega \times \dot{\omega}, \tag{5.5}$$

$$\omega(0) = \mathbf{P}_0 \cdot \boldsymbol{\Omega}_0, \quad \dot{\omega}(0) = \mathbf{P}_0 \cdot \dot{\boldsymbol{\Omega}}_0. \tag{5.6}$$

The coefficients $a(\Omega)$, $b(\Omega)$, and $c(\Omega)$ are calculated in accordance with relations (4.3) and the scalar relations (3.1), (3.3), and (3.5) between the angular velocities; we have

$$a(\Omega) = -\frac{d}{dt}(\ln|\Omega \times \dot{\Omega}|), \quad b(\Omega) = \frac{(\Omega \times \dot{\Omega}) \cdot (\dot{\Omega} \times \ddot{\Omega})}{(\Omega \times \dot{\Omega})^2}, \quad c(\Omega) = \frac{(\ddot{\Omega} \cdot (\Omega \times \dot{\Omega})}{(\Omega \times \dot{\Omega})^2} + 1. \tag{5.7}$$

Statement 2 follows from Statement 1. The differential equation has the form

$$\dddot{\omega} + A(\Omega)\ddot{\omega} + B(\Omega)\omega + C(\Omega)\dot{\omega} = \mathbf{0}. \tag{5.8}$$

The initial conditions are formed by the conditions of (5.6) and Eq. (5.5) at $t = 0$. The coefficients $A(\Omega)$, $B(\Omega)$, and $C(\Omega)$ are expressed in terms of $a(\Omega)$, $b(\Omega)$, and $c(\Omega)$ by relations (4.5).

6. COMPARISON OF THE SUGGESTED AND CLASSICAL STATEMENTS OF THE DARBOUX PROBLEM

In the classical formulation [9], the Darboux problem is reduced to a Riccati-type equation for a complex unknown. This equation has the form

$$\dot{z} = \frac{\Omega_2 - i\Omega_1}{2} - i\Omega_3 z + \frac{\Omega_2 + i\Omega_1}{2} z^2. \tag{6.1}$$

The change of variables $u = \exp\left[-\frac{1}{2}\int(\Omega_2 + i\Omega)z\,dt\right]$ reduces Eq. (6.1) to the second-order linear equation

$$\begin{gathered} \ddot{u} + (a + ib)\dot{u} + cu = 0, \\ a = -\frac{d}{dt}\ln\sqrt{\Omega_1^2 + \Omega_2^2}, \quad b = \Omega_3 + \frac{\Omega_1\dot{\Omega}_2 - \dot{\Omega}_1\Omega_2}{\Omega_1^2 + \Omega_2^2}, \quad c = \frac{\Omega_1^2 + \Omega_2^2}{4}. \end{gathered} \tag{6.2}$$

The second-order linear equation (6.2) for a complex unknown is equivalent to the following fourth-order system of equations for real unknowns x and y:

$$\ddot{x} + a\dot{x} - b\dot{y} + cx = 0, \quad \ddot{y} + a\dot{y} + b\dot{x} + cy = 0. \tag{6.3}$$

A statement of the Darboux problem suggested in the present paper leads to a third-order linear equation for an unknown vector function. In the coordinate notation this vector equation is represented by three independent scalar equations of the form

$$\begin{gathered} \dddot{\omega}_x + A(\Omega)\ddot{\omega}_x + B(\Omega)\dot{\omega}_x + C(\Omega)\omega_x = 0, \\ \dddot{\omega}_y + A(\Omega)\ddot{\omega}_y + B(\Omega)\dot{\omega}_y + C(\Omega)\omega_y = 0, \\ \dddot{\omega}_z + A(\Omega)\ddot{\omega}_z + B(\Omega)\dot{\omega}_z + C(\Omega)\omega_z = 0. \end{gathered} \tag{6.4}$$

The equations of system (6.4) differ only in the notation of the unknown variables. Hence, having known the solution of one of these equation, we thereby know the solution of the entire system.

Conclusion. The approach suggested in the present paper permits one to reduce the Darboux problem to a third-order linear differential equation, whereas the approach based on the Riccati equation leads to a fourth-order system of linear differential equations.

7. ROTATION OF A BALL SUBJECTED TO VISCOUS FRICTION

Consider a rigid body fixed at its center of mass by means of a spherical joint. The tensor of inertia $\boldsymbol{\theta}$ of the body at the center of mass is spherical, i.e., $\boldsymbol{\theta} = \theta \mathbf{E}$, where θ is a positive number and $\mathbf{E}$ is the identity matrix. The interaction of this body with the ambient medium is modeled by the torque due to linear viscous friction acting on the body,

$$\mathbf{M}_{\text{vf}} = -\mathbf{K}_{\text{vf}} \cdot \boldsymbol{\omega}, \qquad \mathbf{K}_{\text{vf}} = k_3 \mathbf{nn} + k_{12}(\mathbf{E} - \mathbf{nn}), \quad \mathbf{n} = \mathbf{P} \cdot \mathbf{n}_0. \tag{7.1}$$

The viscous friction tensor $\mathbf{K}_{\text{vf}}$ is characterized by constant coefficients, k_{12} and k_3, depending on the properties of the ambient medium and the surface of the body, and also by the direction of the anisotropy axis, $\mathbf{n}$, fixed in the body. In Eq. (7.1), $\mathbf{P}$ is the tensor of rotation of the body and $\mathbf{n}_0$ is the initial value of the vector $\mathbf{n}$. In the case in question, Euler's equations of motion of the body have the form

$$\theta \dot{\boldsymbol{\omega}} = -[k_3 \mathbf{nn} + k_{12}(\mathbf{E} - \mathbf{nn})] \cdot \boldsymbol{\omega}, \quad \mathbf{n} = \mathbf{P} \cdot \mathbf{n}_0. \tag{7.2}$$

By premultiplying Eq. (7.2) by $\mathbf{P}^{\mathrm{T}}$, we obtain (after some transformations) the following equation for the right angular velocity:

$$\theta \dot{\boldsymbol{\Omega}} = -[k_3 \mathbf{n}_0 \mathbf{n}_0 + k_{12}(\mathbf{E} - \mathbf{n}_0 \mathbf{n}_0)] \cdot \boldsymbol{\Omega}. \tag{7.3}$$

The solution of this equation has the form

$$\boldsymbol{\Omega} = (\mathbf{n}_0 \cdot \boldsymbol{\Omega}_0)\mathbf{n}_0 \exp\left(-\frac{k_3}{\theta} t\right) + [\boldsymbol{\Omega}_0 - (\mathbf{n}_0 \cdot \boldsymbol{\Omega}_0)\mathbf{n}_0] \exp\left(-\frac{k_{12}}{\theta} t\right). \tag{7.4}$$

After determining the right angular velocity, the dynamic problem is reduced to the Darboux problem. By making use of Statement 2 for the right Darboux problem and relation (7.4) for the right angular velocity, we obtain the equation for the left angular velocity in the form

$$\begin{gathered} \dddot{\boldsymbol{\omega}} + 2\frac{k_1 + k_{12}}{\theta} \ddot{\boldsymbol{\omega}} + \left[\frac{k_3^2 + 3k_3 k_{12} + k_{12}^2}{\theta^2} + \Omega^2\right] \dot{\boldsymbol{\omega}} + \left[\frac{k_3 k_{12}(k_3 + k_{12})}{\theta^3} - \frac{1}{2}\frac{d}{dt}(\Omega^2)\right] \boldsymbol{\omega} = \mathbf{0}, \\ \Omega^2 = (\mathbf{n}_0 \cdot \boldsymbol{\Omega}_0)^2 \exp\left(-\frac{2k_3}{\theta} t\right) + [\boldsymbol{\Omega}_0 - (\mathbf{n}_0 \cdot \boldsymbol{\Omega}_0)\mathbf{n}_0]^2 \exp\left(-\frac{2k_{12}}{\theta} t\right). \end{gathered} \tag{7.5}$$

Equation (7.5) has an analytical solution which can be represented by a uniformly convergent series the coefficients of which are defined by recurrence relations. For the solution of Eq. (7.5), see, for example, [10], where the problem in question is considered in a more general formulation for a body with a transversely isotropic tensor of inertia.

8. ROTATION OF A BALL ACTED UPON BY A CONSTANT TORQUE

Consider a ball with the tensor of inertia $\boldsymbol{\theta} = \theta \mathbf{E}$. The center of mass of the body is fixed. The ball is acted upon by a constant torque $\mathbf{M}$. The equation of motion of the ball has the form

$$\theta \dot{\boldsymbol{\omega}} = \mathbf{M}. \tag{8.1}$$

This equation is easy to integrate to obtain the following expression for the left angular velocity:

$$\boldsymbol{\omega} = \theta^{-1} \mathbf{M} t + \boldsymbol{\omega}_0. \tag{8.2}$$

After that the dynamic problem is reduced to the Darboux problem. Using Statement 2 of the left Darboux problem, we obtain, with reference to relation (8.2), the following equation for the right angular velocity:

$$\dddot{\boldsymbol{\Omega}} + (\theta^{-2} M^2 t^2 + 2\theta^{-1} \mathbf{M} \cdot \boldsymbol{\omega}_0 t + \omega_0^2)\dot{\boldsymbol{\Omega}} - (\theta^{-2} M^2 t + \theta^{-1} \mathbf{M} \cdot \boldsymbol{\omega}_0)\boldsymbol{\Omega} = \mathbf{0}. \tag{8.3}$$

This equation can be solved in closed form in terms of hypergeometric functions. A detailed discussion of this solution is beyond the scope of the present paper.

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