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A New Approach to the Solution of some Problems of Rigid Body Dynamics

A representation of the turn-tensor of an axisymmetrical rigid body by using the moment of momentum vector is proposed. It is proved that for certain external moments the motion of an axisymmetrical rigid body differs from the motion of the spherical rigid body only by the additional rotation around its axis of symmetry. Analogy between the problems of the rotation of an axisymmetrical rigid body under the action of the moment, directed along the axis of symmetry of the body and under the action of the constantly directed moment, is exposed. Exact solution of the problem of free rotation of an axisymmetrical rigid body, taking account of a linear viscous friction, is constructed.

MSC (2000): 70B10, 70E15, 70E17, 70E18

0. Introduction

The part of rigid body dynamics, which is concerned with the solution of the problems of three-dimensional rotation of rigid bodies and their systems, began in the papers by Leonard Euler [2]. The methods of solution of the problems, worked out by Euler, are based on the use of vector technique. These methods are visual and the results obtained by using them are easy-to-interpret. That is why methods of vector mechanics are usually used for the solution of engineering problems. For the two hundred years of development of rigid body dynamics, these methods have not been modified, and now they exist in a form as formulated by Euler. This fact is reflected both on the results, obtained up to now in rigid body dynamics, and on tendencies of this science development.

Let us state briefly the results. 1. All problems of rigid body dynamics, exact analytical solutions of which are known, were solved in the last century. These are cases of integrability by Euler, Lagrange, and Kovalevskaya. 2. A number of problems, which have been solved incompletely, exist. In these problems analytical relations, defining the motion of the angular velocity vector with respect to the rigid body, were found several tens of years ago [3], [10], but complete solutions of these problems have not been constructed up to now. 3. The problems, which look very easy (for example, the problem of rigid body rotation under the action of a constant moment) exist. But the opportunity to solve these problems analytically even has not been discussed.

While analytical solutions of many "elementary" problems remain unknown, the main tendency of development of rigid body dynamics consists in complication of the rigid body systems under consideration. Sometimes, the rigid bodies involved in the system reach several tens in number. Solving such problems computers are used both for derivation of the differential equations and for their numerical solution. There are many publications on this subject. We refer to [13] and some papers published for the last time [5], [11] only.

Not underestimating the importance of investigations, carried out in the field of the multibody systems, let us pay attention to the fact, that absence of analytical solutions of the "simple" problems of rigid body motion under the action of elementary moments is grate gap in rigid body dynamics. What is a barrier to the solution of such "elementary" problems and why is the interest in these problems lost? Evidently, this is caused by the fact, that the used mathematical technique is not adequate to the problems which are solved. Vector technique, which can be used for the solution of any rigid body dynamics problems, has some limitation. It is not convenient for description of finite turns of a rigid body in three-dimensional space. Firstly, vectors of turn, defined as $\boldsymbol{\theta} = \theta \boldsymbol{m}$ (θ is the angle of turn, $\boldsymbol{m}$ is the unit vector, defining the direction of the axis of turn) do not obey the rule of vector composition, i.e., the vector of turn $\boldsymbol{\theta}$, which is the composition of two vectors of turn $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, cannot be represented as their sum: $\boldsymbol{\theta} \neq \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2$. Secondly, using the concept of vector of turn, it is difficult to introduce the simple and correct definition of the angular velocity, as $\boldsymbol{\omega} \neq \dot{\boldsymbol{\theta}}$. These and other difficulties are overcome, if the concept of the turn-tensor is used for description of turns of a rigid body [14], [15]. At times, when Euler lived, tensor technique was not known and characteristics of turns of rigid bodies in three-dimensional space were introduced by him descriptively. In the following years, when the tensor technique was developed and introduced in many parts of mechanics, rigid body dynamics continued to use descriptive manner of specifying rigid body turns, and this retarded the advance of rigid body dynamics. It is important to note, that all known manners of specifying rigid body turns can be easily translated into turn-tensor language. At the same time, the methods based on use of the turn-tensor give wide facilities for the solution of rigid body dynamics problems compared with any other manner of turns description. An advantage of this approach is stipulated as follows:

1. The turn-tensor admits an infinite number of different representations. This is important for solving many problems that cannot be solved analytically by using Eulerian angles or other known angles. Some of the problems look

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very simple, that allows to hope they will be solved analytically if the turn-tensor is represented in a suitable manner.

2. Representation of the turn-tensor is not sure to be chosen beforehand. It can be chosen partially or completely while solving the problem. Such approach is used in the paper by P. A. Zhilin [15] to study the problem of free rotation of a rigid body. The new approach to the solution of a known problem allows to expose some important facts concerned with the existence of three types of the rotation of a rigid body. These facts were not noted earlier.

One of the feasible representations of the turn-tensor will be proposed in this paper. This representation differs from known representations of the turn-tensor by its dependence on the dynamic characteristics of a rigid body. The methods of solving rigid body dynamics problems, based on the choice of dynamic characteristics of body motion (kinetic moment vector, magnitude of the kinetic moment vector, angle between the kinetic moment vector and some characteristic direction, and others) as the base variables, are worked out both in the theoretical investigations (e.g. [6]) and in applied works (e.g. [1]). Development of these methods is the promising trend in rigid body dynamics, especially for nonconservative problems.

1. The turn-tensor and its use in the kinematics of a rigid body

In this section necessary information on the turn-tensor and the base formulae and definitions of rigid body kinematics stated by using the direct tensor calculus is expounded. Detailed information on this question can be found in P. A. Zhilin [14], [15].

Definition: A properly orthogonal tensor, i.e. a tensor which is a solution of the equations

$$\boldsymbol{P}\cdot\boldsymbol{P}^T=\boldsymbol{P}^T\cdot\boldsymbol{P}=\boldsymbol{E}\,,\qquad \det\boldsymbol{P}=+1\,,\tag{1.1}$$

where $\boldsymbol{E}$ is the unit tensor, is called a *turn-tensor*.

The turn-tensor usually term as the rotation tensor. The author follows P. A. Zhilin [14], [15], where the term "turn-tensor" is used. There the concepts "turn" and "rotation" have different sense. The term "turn" is used for defining quantities describing instantaneous passage of a rigid body from one position into another. The term "rotation" is used for defining quantities describing the motion of the body (process of passage from one position into another). The turn-tensor does not depend on process of motion; it depends on two positions of a rigid body only (initial and terminating). Therefore we use for it the term "turn-tensor" instead of "rotation tensor" as in the majority of the publications.

Definition: The tensors $\boldsymbol{S}(t)$ and $\boldsymbol{S}_r(t)$, defined by the formulae

$$\boldsymbol{S}(t)=\dot{\boldsymbol{P}}(t)\cdot\boldsymbol{P}^T(t)\,,\qquad \boldsymbol{S}_r(t)=\boldsymbol{P}^T(t)\cdot\dot{\boldsymbol{P}}(t)\tag{1.2}$$

are called respectively the *left* and the *right spin-tensor*.

Definition: The accompanying vector of the left spin-tensor $\boldsymbol{\omega}(t)$ and of the right spin-tensor $\boldsymbol{\Omega}(t)$,

$$\boldsymbol{S}(t)=\boldsymbol{\omega}(t)\times\boldsymbol{E}\,,\qquad \boldsymbol{S}_r(t)=\boldsymbol{\Omega}(t)\times\boldsymbol{E}\,,\tag{1.3}$$

are called respectively the *left* and the *right angular velocity vector*.

The definition of the vector product as operator between vector and tensor can be found in the books on the direct tensor calculus (see Lagally [7] or the appendix to Lurie [9]).

In rigid body dynamics the vector $\boldsymbol{\omega}(t)$ is called the angular velocity in the space and the vector $\boldsymbol{\Omega}(t)$ is called the angular velocity in the body. In continuum mechanics the left and the right angular velocity vectors are also known as spatial and material angular velocities.

Eq. (1.3) can be rewritten in equivalent form known as the equations by Poisson:

$$\dot{\boldsymbol{P}}(t)=\boldsymbol{\omega}(t)\times\boldsymbol{P}(t)\,,\qquad \dot{\boldsymbol{P}}(t)=\boldsymbol{P}(t)\times\boldsymbol{\Omega}(t)\,.\tag{1.4}$$

The left and the right angular velocities are related by the formulae

$$\boldsymbol{\omega}(t)=\boldsymbol{P}(t)\cdot\boldsymbol{\Omega}(t)\,,\qquad \boldsymbol{\Omega}(t)=\boldsymbol{P}^T(t)\cdot\boldsymbol{\omega}(t)\,.\tag{1.5}$$

If the turn-tensor is known, the left and the right angular velocities can be rapidly calculated:

$$\boldsymbol{\omega}(t)=-\tfrac{1}{2}(\dot{\boldsymbol{P}}(t)\cdot\boldsymbol{P}^T(t))_\times\,,\qquad \boldsymbol{\Omega}(t)=-\tfrac{1}{2}(\boldsymbol{P}^T(t)\cdot\dot{\boldsymbol{P}}(t))_\times\,,\tag{1.6}$$

where operator $(\,)_\times$ called vector invariant has the following sense (see [7], [9]): If the tensor $\boldsymbol{A}$ is represented as a sum of diads, for example $\boldsymbol{A}=\boldsymbol{ab}+\boldsymbol{cd}$, then $\boldsymbol{A}_\times=\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{c}\times\boldsymbol{d}$.

The inverse problem (the problem of determination of the turn-tensor to the known angular velocity) is called the problem by Darboux. There are two formulations of the Darboux problem:

- The left Darboux problem is

$$\dot{\boldsymbol{P}}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{P}(t)\,, \qquad \boldsymbol{P}(0) = \boldsymbol{P}_0\,, \qquad \boldsymbol{P}_0 \cdot \boldsymbol{P}_0^T = \boldsymbol{E}\,, \qquad \det \boldsymbol{P}_0 = 1\,. \tag{1.7}$$

- The right Darboux problem is

$$\dot{\boldsymbol{P}}(t) = \boldsymbol{P}(t) \times \boldsymbol{\Omega}(t)\,, \qquad \boldsymbol{P}(0) = \boldsymbol{P}_0\,, \qquad \boldsymbol{P}_0 \cdot \boldsymbol{P}_0^T = \boldsymbol{E}\,, \qquad \det \boldsymbol{P}_0 = 1\,. \tag{1.8}$$

Theorem: *Every turn-tensor can be represented as the composition of any number n of turn-tensors:*

$$\boldsymbol{P}(t) = \boldsymbol{P}_n(t) \cdot \boldsymbol{P}_{n-1}(t) \cdot \ldots \cdot \boldsymbol{P}_3(t) \cdot \boldsymbol{P}_2(t) \cdot \boldsymbol{P}_1(t)\,. \tag{1.9}$$

Theorem: *If the turn-tensor $\boldsymbol{P}(t)$ is represented as the composition of two turn-tensors $\boldsymbol{P}_1(t)$ and $\boldsymbol{P}_2(t)$, then the angular velocity vector $\boldsymbol{\omega}(t)$, corresponding to the turn-tensor $\boldsymbol{P}(t)$, is expressed in terms of the angular velocity vectors $\boldsymbol{\omega}_1(t)$ and $\boldsymbol{\omega}_2(t)$, corresponding to the turn-tensors $\boldsymbol{P}_1(t)$ and $\boldsymbol{P}_2(t)$, as follows:*

$$\boldsymbol{P}(t) = \boldsymbol{P}_2(t) \cdot \boldsymbol{P}_1(t) \quad \Rightarrow \quad \boldsymbol{\omega}(t) = \boldsymbol{\omega}_2(t) + \boldsymbol{P}_2(t) \cdot \boldsymbol{\omega}_1(t)\,. \tag{1.10}$$

Euler theorem: *Every turn-tensor $\boldsymbol{P}(t) \neq \boldsymbol{E}$ can be represented uniquely in the form*

$$\boldsymbol{P}(\theta\boldsymbol{m}) = (1 - \cos\theta(t))\, \boldsymbol{m}(t)\, \boldsymbol{m}(t) + \cos\theta(t)\, \boldsymbol{E} + \sin\theta(t)\, \boldsymbol{m}(t) \times \boldsymbol{E}\,, \qquad |\boldsymbol{m}(t)| = 1\,, \tag{1.11}$$

where $\boldsymbol{m}(t)$ is the fixed vector of the turn-tensor $\boldsymbol{P}(t)$:

$$\boldsymbol{P}(t) \cdot \boldsymbol{m}(t) = \boldsymbol{m}(t) \cdot \boldsymbol{P}(t) = \boldsymbol{m}(t)\,. \tag{1.12}$$

Definition: The straight line, spanned on the fixed vector of the turn-tensor $\boldsymbol{m}(t)$, is called the *axis of turn* of a rigid body.

Using Euler representation of the turn-tensor, it is easy to derive the following expressions for the angular velocities:

$$\boldsymbol{\omega}(t) = \dot{\theta}(t)\, \boldsymbol{m}(t) + \sin\theta(t)\, \dot{\boldsymbol{m}}(t) + (1 - \cos\theta(t))\, \boldsymbol{m}(t) \times \dot{\boldsymbol{m}}(t)\,, \tag{1.13}$$

$$\boldsymbol{\Omega}(t) = \dot{\theta}(t)\, \boldsymbol{m}(t) + \sin\theta(t)\, \dot{\boldsymbol{m}}(t) - (1 - \cos\theta(t))\, \boldsymbol{m}(t) \times \dot{\boldsymbol{m}}(t)\,. \tag{1.14}$$

Definition: The straight line, spanned on the left angular velocity vector $\boldsymbol{\omega}(t)$, is called the *axis of rotation* of a rigid body.

Theorem: *If the fixed vector of the turn-tensor $\boldsymbol{P}(\theta\boldsymbol{m})$ does not depend on time* ($\boldsymbol{m} = \text{const}$), *then the axis of rotation of a rigid body coincides with the axis of turn of the rigid body, the right and the left angular velocities are equal and they are calculated by the formula $\boldsymbol{\omega}(t) = \boldsymbol{\Omega}(t) = \dot{\theta}(t)\, \boldsymbol{m}$. If the left angular velocity $\boldsymbol{\omega}(t)$ or the right angular velocity $\boldsymbol{\Omega}(t)$ has a constant direction and this angular velocity is the fixed vector of the turn-tensor $\boldsymbol{P}_0 = \boldsymbol{P}(0)$, then the axis of turn of a rigid body coincides with the axis of rotation of the rigid body, the right and the left angular velocities are equal, and they are expressed in terms of the angle of turn θ and the fixed vector of the turn-tensor $\boldsymbol{m}$ by the formula $\boldsymbol{\omega}(t) = \boldsymbol{\Omega}(t) = \dot{\theta}(t)\, \boldsymbol{m}$.*

The base equation of the kinematics of a rigid body has the form

$$\boldsymbol{R}_A(t) = \boldsymbol{R}_B(t) + \boldsymbol{P}(t) \cdot (\boldsymbol{r}_A - \boldsymbol{r}_B)\,, \qquad \boldsymbol{r} = \boldsymbol{R}(0)\,, \qquad \boldsymbol{P}(0) = \boldsymbol{E}\,. \tag{1.15}$$

Here $\boldsymbol{R}_A(t)$ and $\boldsymbol{R}_B(t)$ are the position vectors of the points A and B of the rigid body, $\boldsymbol{P}(t)$ is the turn-tensor of the rigid body.

2. Representation of the turn-tensor of an axisymmetrical rigid body by using of the kinetic moment vector

Theorem: *Let the inertia tensor of a rigid body in the reference position be*

$$\boldsymbol{\theta} = \lambda \boldsymbol{k}\boldsymbol{k} + \mu(\boldsymbol{E} - \boldsymbol{k}\boldsymbol{k})\,. \tag{2.1}$$

Let the actual position of the rigid body be determined by the turn-tensor $\boldsymbol{P}(t)$ and the rigid body have the angular velocity $\boldsymbol{\omega}(t)$, and the kinetic moment vector (the moment of momentum vector) of the rigid body, calculated with respect to some point of the body, be $\boldsymbol{L}(t)$:

$$\boldsymbol{L}(t) = \boldsymbol{P}(t) \cdot \boldsymbol{\theta} \cdot \boldsymbol{P}^T(t) \cdot \boldsymbol{\omega}(t) \tag{2.2}$$

In this case (see Fig. 1) the turn-tensor of the rigid body $\boldsymbol{P}(t)$ can be represented as the composition of two turn-tensors:

$$\boldsymbol{P}(t) = \boldsymbol{P}_L(t) \cdot \boldsymbol{P}_*(t)\,. \tag{2.3}$$

The turn-tensor $\boldsymbol{P}_L(t)$ is determined by the kinetic moment vector of the rigid body $\boldsymbol{L}(t)$ as solution of the left Darboux problem:

$$\dot{\boldsymbol{P}}_L(t) = \boldsymbol{\omega}_L(t) \times \boldsymbol{P}_L(t)\,, \qquad \boldsymbol{\omega}_L(t) = \mu^{-1}\boldsymbol{L}(t)\,, \qquad \boldsymbol{P}_L(0) = \boldsymbol{P}(0)\,. \tag{2.4}$$

The turn-tensor $\boldsymbol{P}_*(t)$ has the form

$$\boldsymbol{P}_*(t) = (1 - \cos\beta(t))\ \boldsymbol{kk} + \cos\beta(t)\ \boldsymbol{E} + \sin\beta(t)\ \boldsymbol{k}\times\boldsymbol{E} \tag{2.5}$$

where angle $\beta(t)$ is expressed in terms of the right angular velocity corresponding to turn-tensor $\boldsymbol{P}_L(t)$:

$$\beta(t) = \int (\mu - \lambda)\,\lambda^{-1}\boldsymbol{k}\cdot\boldsymbol{\Omega}_L(t)\ dt\,, \qquad \boldsymbol{\Omega}_L(t) = \boldsymbol{P}_L^T(t)\cdot\boldsymbol{\omega}_L(t)\,, \qquad \beta(0) = 0\,. \tag{2.6}$$

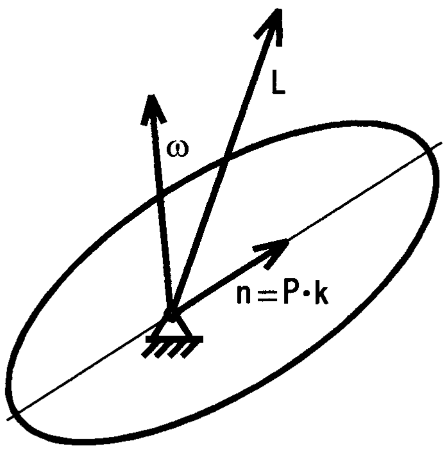

Fig. 1. The theorem of representation of the turn-tensor

P r o o f: Let us represent the turn-tensor of the rigid body $\boldsymbol{P}(t)$ as the composition of two turn-tensors:

$$\boldsymbol{P}(t) = \boldsymbol{P}_L(t)\cdot\boldsymbol{P}_*(t)\,. \tag{2.7}$$

Here $\boldsymbol{P}_\mathrm{L}(t)$ is the turn-tensor, that is defined by formulae (2.4), and $\boldsymbol{P}_*(t)$ is an unknown turn-tensor. Let us prove that $\boldsymbol{P}_*(t)$ has the form (2.5), (2.6). According to (1.10), the angular velocity, corresponding to the turn-tensor $\boldsymbol{P}(t)$ represented by (2.7), is calculated by the formula

$$\boldsymbol{\omega}(t) = \boldsymbol{\omega}_L(t) + \boldsymbol{P}_L(t)\cdot\boldsymbol{\omega}_*(t) \tag{2.8}$$

where $\boldsymbol{\omega}_*(t)$ is the angular velocity, corresponding to turn-tensor $\boldsymbol{P}_*(t)$. On the other hand, according to formulae (2.1), (2.2), the angular velocity $\boldsymbol{\omega}_L(t)$ can be represented as

$$\boldsymbol{\omega}_L(t) = \boldsymbol{P}(t)\cdot[(\lambda-\mu)\,\mu^{-1}\boldsymbol{kk} + \boldsymbol{E}]\cdot\boldsymbol{P}^T(t)\cdot\boldsymbol{\omega}(t)\,. \tag{2.9}$$

Multiplying eq. (2.9) from the left through by $\boldsymbol{P}^\mathrm{T}(t)$ scalarly and substituting into it expressions (2.7), (2.8), we obtain

$$(\lambda-\mu)\,\mu^{-1}\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{P}_*^T(t)\cdot\boldsymbol{\Omega}_L(t) + \boldsymbol{k}\cdot\boldsymbol{\Omega}_*(t)) + \boldsymbol{\Omega}_*(t) = 0\,. \tag{2.10}$$

Direction of vector $\boldsymbol{\Omega}_*(t)$ is seen from (2.10) to be constant. Hence, vector $\boldsymbol{\Omega}_*(t)$ can be represented as

$$\boldsymbol{\Omega}_*(t) = \dot{\beta}(t)\ \boldsymbol{k}\,, \qquad \dot{\beta}(t) = (\mu-\lambda)\,\lambda^{-1}\boldsymbol{k}\cdot\boldsymbol{P}_*^T(t)\cdot\boldsymbol{\Omega}_L(t)\,. \tag{2.11}$$

According to formula (2.11) and the theorems above mentioned turn-tensor $\boldsymbol{P}_*(t)$ has the form (2.5). Hence $\boldsymbol{k}\cdot\boldsymbol{P}_*^T(t) = \boldsymbol{k}$ and the angle $\beta(t)$ is determined by formula (2.6). □

Note: The theorem holds true if tensor $\boldsymbol{\theta}$ is not the tensor of inertia of a rigid body and even if quantities λ, μ are not constant.

3. The use of representation of the turn-tensor by the kinetic moment vector in rigid body dynamics problems

The second law of dynamics by Euler for a rigid body, having one fixed point, is formulated as follows:

$$\dot{\boldsymbol{L}} = \boldsymbol{M}(\boldsymbol{P},\boldsymbol{\omega},t)\,. \tag{3.1}$$

If the inertia tensor of a rigid body is axisymmetrical then ψ, according to formulae (2.1)–(2.6)

$$\boldsymbol{L} = \mu \boldsymbol{\omega}_L\,, \qquad \boldsymbol{P} = \boldsymbol{P}(\boldsymbol{P}_L,\, \boldsymbol{\omega}_L)\,, \qquad \boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{P}_L,\, \boldsymbol{\omega}_L)\,.$$

Hence the second law of dynamics by Euler can be rewritten as

$$(\mu \boldsymbol{\omega}_L)^{\cdot} = \boldsymbol{M}(\boldsymbol{P}_L, \boldsymbol{\omega}_L, t)\,. \tag{3.2}$$

Using the formulation of the second law of dynamics by Euler in the form (3.2) is the most efficient way for solving problems of rigid body rotation under the action of the moments independent of the turns of the body around its axis of symmetry. In this case, eq. (3.2) becomes more simple as it does not include trigonometric functions of angle β:

$$(\mu \boldsymbol{\omega}_L)^{\cdot} = \boldsymbol{M}(\boldsymbol{n}, \dot{\boldsymbol{n}}, t)\,, \qquad \boldsymbol{n} = \boldsymbol{P}_L \cdot \boldsymbol{k}\,. \tag{3.3}$$

Theorem: *The motion of an axisymmetrical rigid body, having one fixed point, under the action of an external moment, independent of the turn of the body around its axis of symmetry and the angular velocity corresponding to this turn, differs from the motion of the spherical rigid body, having the same transverse moment of inertia, only by the additional rotation around its axis of symmetry.*

The Proof of this theorem follows from the equation of the motion of a rigid body in the form (3.3) and the theorem of representation of the turn-tensor of an axisymmetrical rigid body (2.1)–(2.6). □

As applied to some particular problems this theorem is known for a long time. For example, it has been stated in the book by E. T. WHITTAKER [12] as applied to the case of integrability by Lagrange. In the general form this theorem is stated for the first time.

4. The rigid body dynamics problems, that can be reduced to the Darboux problem

The objective of this section is to show the use of the proposed representation of the turn-tensor for the solution of the concrete dynamics problems. Four problems will be considered below. Two of them (the problem of free rotation of a rigid body and the problem of the rigid body rotation taking account of the resistance proportional to the kinetic moment vector) have exact analytical solutions, that can be expressed in terms of elementary functions and visualized. These problems are considered to show as the known results are obtained by using the proposed representation of the turn-tensor. Two other problems (the problem of the rigid body rotation under the action of the moment directed along the axis of symmetry of the body and the problem of the rigid body rotation under the action of the constantly directed moment) are interesting because the method of their solving, based on using proposed representation of the turn-tensor, allows to expose analogy between these problems. The analogy consists in the fact that both the problems can be reduced to the Darboux problem (the former one can be reduced to the right Darboux problem, the latter one can be reduced to the left Darboux problem), and moreover the right angular velocity in the former problem and the left angular velocity in the latter problem are the same. In the case of constant magnitudes of the external moments the analogy has practical importance, as the latter problem has not been solved up to now and a solution of the former problem is known [8].

4.1 The rotation of a free axisymmetrical rigid body (see Fig. 2)

The problem is formulated as follows:

$$\dot{\boldsymbol{L}} = 0\,, \qquad \boldsymbol{L}(0) = \boldsymbol{L}_0\,, \qquad \boldsymbol{P}(0) = \boldsymbol{E}\,. \tag{4.1}$$

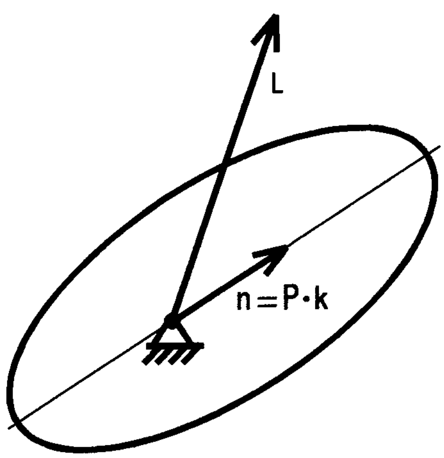

Fig. 2. Free rigid body

Using (3.2), the problem can be reformulated as

$$\dot{\boldsymbol{\omega}}_L = 0\,, \qquad \boldsymbol{\omega}_L(0) = \mu^{-1}\boldsymbol{L}_0\,. \tag{4.2}$$

After integrating eq. (4.2), the problem is reduced to the left Darboux problem

$$\dot{\boldsymbol{P}}_L = \boldsymbol{\omega}_L \times \boldsymbol{P}_L\,, \qquad \boldsymbol{\omega}_L = \mu^{-1}\boldsymbol{L}_0\,, \qquad \boldsymbol{P}_L(0) = \boldsymbol{E}\,. \tag{4.3}$$

The solution of the Darboux problem (4.3) is

$$\boldsymbol{P}_L = \boldsymbol{P}_L(\psi\boldsymbol{l})\,, \qquad \psi = \mu^{-1}L_0 t\,, \qquad \boldsymbol{l} = \boldsymbol{L}_0/L_0\,, \qquad L_0 = |\boldsymbol{L}_0|\,. \tag{4.4}$$

According to (2.3)–(2.6), the turn-tensor of the body can be represented as

$$\boldsymbol{P} = \boldsymbol{P}_L(\psi\boldsymbol{l})\cdot\boldsymbol{P}_*(\beta\boldsymbol{k})\,, \qquad \beta = (\lambda^{-1} - \mu^{-1})\,\boldsymbol{k}\cdot\boldsymbol{L}_0 t\,. \tag{4.5}$$

The turn-tensor $\boldsymbol{P}_L$ describes the precession of the rigid body around the kinetic moment vector. The turn-tensor $\boldsymbol{P}_*(\beta\boldsymbol{k})$ describes the rotation of the rigid body around its axis of symmetry.

4.2 Rotation of an axisymmetrical rigid body taking account of the resistance proportional to the kinetic moment vector (see Fig. 3)

The problem is formulated as follows:

$$\dot{\boldsymbol{L}} = -k\boldsymbol{L}\,, \qquad \boldsymbol{L}(0) = \boldsymbol{L}_0\,, \qquad \boldsymbol{P}(0) = \boldsymbol{E}\,. \tag{4.6}$$

Using (3.2), the problem can be reformulated as

$$\dot{\boldsymbol{\omega}}_L = -k\boldsymbol{\omega}_L\,, \qquad \boldsymbol{\omega}_L(0) = \mu^{-1}\boldsymbol{L}_0\,. \tag{4.7}$$

Having solved the differential equation (4.7), the problem is reduced to the left Darboux problem

$$\dot{\boldsymbol{P}}_L = \boldsymbol{\omega}_L \times \boldsymbol{P}_L\,, \qquad \boldsymbol{\omega}_L = \mu^{-1}\boldsymbol{L}_0\,\mathrm{e}^{-kt}\,, \qquad \boldsymbol{P}_L(0) = \boldsymbol{E}\,. \tag{4.8}$$

The solution of the Darboux problem (4.8) is

$$\boldsymbol{P}_L = \boldsymbol{P}_L(\psi\boldsymbol{l})\,, \qquad \psi = (k\mu)^{-1}\,L_0(1 - \mathrm{e}^{-kt})\,, \qquad \boldsymbol{l} = \boldsymbol{L}_0/L_0\,, \qquad L_0 = |\boldsymbol{L}_0|\,. \tag{4.9}$$

According to (2.3)–(2.6), the turn-tensor of the body can be represented as

$$\boldsymbol{P} = \boldsymbol{P}_L(\psi\boldsymbol{l})\cdot\boldsymbol{P}_*(\beta\boldsymbol{k})\,, \qquad \beta = (\mu - \lambda)\,\lambda^{-1}\boldsymbol{k}\cdot\boldsymbol{l}\psi\,. \tag{4.10}$$

The turn-tensor $\boldsymbol{P}_L(\psi\boldsymbol{l})$ describes the rotation of the axis of symmetry of the body around the constantly directed kinetic moment vector. The turn-tensor $\boldsymbol{P}_*(\beta\boldsymbol{k})$ characterizes the rotation of the body around its axis of symmetry.

4.3 Rotation of an axisymmetrical rigid body under the action of the moment directed along the axis of symmetry of the body (see Fig. 4)

The problem is formulated as follows:

$$\dot{\boldsymbol{L}} = M(t)\,\boldsymbol{n}\,, \qquad \boldsymbol{n} = \boldsymbol{P}\cdot\boldsymbol{k}\,. \tag{4.11}$$

In the form (3.2) the problem is formulated as

$$\mu\dot{\boldsymbol{\omega}}_L = M(t)\,\boldsymbol{n}\,, \qquad \boldsymbol{n} = \boldsymbol{P}_L\cdot\boldsymbol{k}\,. \tag{4.12}$$

Multiplying eq. (4.12) from the left through by $\boldsymbol{P}_L^T$ and taking into account relations (1.4), (1.5), we obtain

$$\mu\dot{\boldsymbol{\Omega}}_L = M(t)\,\boldsymbol{k}\,. \tag{4.13}$$

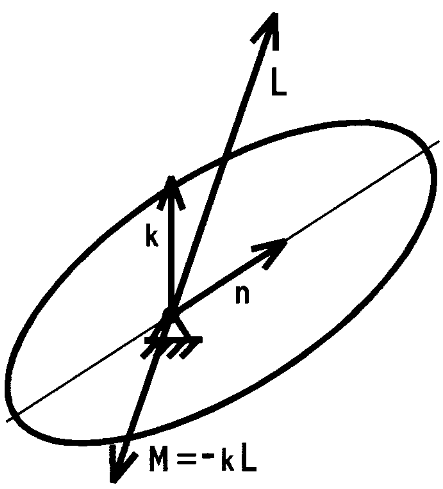

Fig. 3. Resistance: $\boldsymbol{M} = -k\boldsymbol{L}$

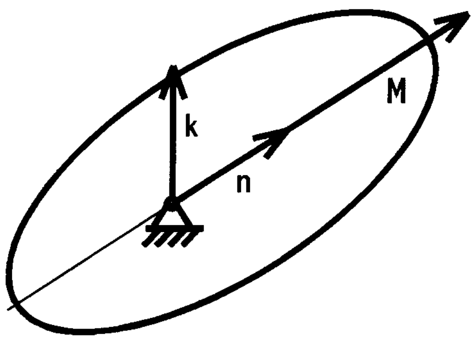

Fig. 4. The external moment: $\boldsymbol{M} = M\boldsymbol{n}$

The following manipulations have been done:

$$\boldsymbol{P}_L^T \cdot \dot{\boldsymbol{\omega}}_L = \dot{\boldsymbol{\omega}}_L \cdot \boldsymbol{P}_L = (\boldsymbol{\omega}_L \cdot \boldsymbol{P}_L)^{\cdot} - \boldsymbol{\omega}_L \cdot \dot{\boldsymbol{P}}_L = (\boldsymbol{P}_L^T \cdot \boldsymbol{\omega}_L)^{\cdot} - \boldsymbol{\omega}_L \cdot (\boldsymbol{\omega}_L \times \boldsymbol{P}_L) = \dot{\boldsymbol{\Omega}}_L \,.$$

After integrating (4.13), we have

$$\boldsymbol{\Omega}_L = \mu^{-1} \int_0^t M(t)\, dt\, \boldsymbol{k} + \boldsymbol{\Omega}_{L0} \,. \tag{4.14}$$

Thus, the problem is reduced to the right Darboux problem

$$\dot{\boldsymbol{P}}_L = \boldsymbol{P}_L \times \boldsymbol{\Omega}_L \,, \qquad \boldsymbol{P}_L(0) = \boldsymbol{E} \,. \tag{4.15}$$

When the turn-tensor $\boldsymbol{P}_L$ has been found, the turn-tensor of the body $\boldsymbol{P}$ can be determined by formulae (2.3)–(2.6); that gives a complete solution of the problem. Let us discuss the solution of the Darboux problem (4.14), (4.15) in two particular cases.

1. At the instant $t = 0$ the angular velocity of the rigid body is directed along its axis of symmetry. Then

$$\boldsymbol{\Omega}_L = \mu^{-1} \left(\int_0^t M(t)\, dt + L_0 \right) \boldsymbol{k} \,. \tag{4.16}$$

In this case the turn-tensor of the body (2.3) takes the form $\boldsymbol{P} = \boldsymbol{P}_L(\psi \boldsymbol{k}) \cdot \boldsymbol{P}_*(\beta \boldsymbol{k})$, where β and ψ are related by $\beta = (\mu - \lambda)\, \lambda^{-1} \psi$ and $\psi = \int_0^t \boldsymbol{k} \cdot \boldsymbol{\Omega}_L\, dt$. Hence

$$\boldsymbol{P} = \boldsymbol{P}(\theta \boldsymbol{k}) \,, \qquad \theta = \lambda^{-1} \left(\int_0^t \int_0^t M(t)\, dt\, dt + L_0 t \right) . \tag{4.17}$$

2. The magnitude of the external moment is constant. Then the right angular velocity (4.14) becomes a linear function of time:

$$\boldsymbol{\Omega}_L = \mu^{-1} M t \boldsymbol{k} + \boldsymbol{\Omega}_{L0} \tag{4.18}$$

This case has been studied in the book by A. I. LURIE [8], where the discussed dynamics problem is reduced to Darboux problem and the obtained Darboux problem is reduced to the equation by Weber.

4.4 Rotation of an axisymmetrical rigid body under the action of a constantly directed moment (see Fig. 5)

The problem is formulated as follows:

$$\dot{\boldsymbol{L}} = M(t)\, \boldsymbol{m} \,, \qquad \boldsymbol{m} = \text{const} \,, \qquad |\boldsymbol{m}| = 1 \,, \qquad \boldsymbol{P}(0) = \boldsymbol{E} \,. \tag{4.19}$$

In general case the direction of the external moment does not coincide with the direction of the axis of symmetry of the body at the instant $t = 0$. Using (3.2), the problem can be rewritten in the form

$$\mu \dot{\boldsymbol{\omega}}_L = M(t)\, \boldsymbol{m} \,, \qquad \boldsymbol{m} = \text{const} \,, \qquad |\boldsymbol{m}| = 1 \,. \tag{4.20}$$

After integrating (4.20), we have

$$\boldsymbol{\omega}_L = \mu^{-1} \left(\int_0^t M(t)\, dt\, \boldsymbol{m} + \boldsymbol{L}_0 \right) . \tag{4.21}$$

As a result, the problem is reduced to the left Darboux problem

$$\dot{\boldsymbol{P}}_L = \boldsymbol{\omega}_L \times \boldsymbol{P}_L \,, \qquad \boldsymbol{P}_L(0) = \boldsymbol{E} \,. \tag{4.22}$$

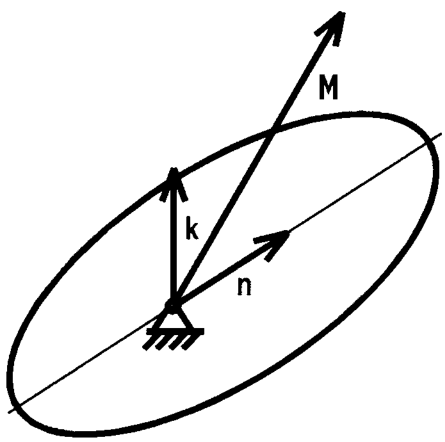

Fig. 5. The constantly directed external moment

When the turn-tensor $\boldsymbol{P}_L$ has been found, the turn-tensor of the rigid body $\boldsymbol{P}$ can be expressed in terms of $\boldsymbol{P}_L$ by formulae (2.3)–(2.6). Let us consider two particular cases of the Darboux problem (4.21)–(4.22).

1. At the instant $t = 0$ the direction of the kinetic moment vector coincides with the direction of the external moment. Then

$$\boldsymbol{\omega}_L = \mu^{-1} \left(\int_0^t M(t)\, dt + L_0 \right) \boldsymbol{m}\,. \tag{4.23}$$

The solution of the Darboux problem (4.22), (4.23) takes the form

$$\boldsymbol{P}_L = \boldsymbol{P}_L(\psi \boldsymbol{m})\,, \qquad \psi = \mu^{-1} \left(\int_0^t \int_0^t M(t)\, dt\, dt + L_0 t \right). \tag{4.24}$$

According to (2.3)–(2.6), the turn-tensor of the body can be represented in the form

$$\boldsymbol{P} = \boldsymbol{P}_L(\psi \boldsymbol{m}) \cdot \boldsymbol{P}_*(\beta \boldsymbol{k})\,, \qquad \beta = (\mu - \lambda)\, \lambda^{-1} \boldsymbol{k} \cdot \boldsymbol{m} \psi\,. \tag{4.25}$$

The rigid body moves as follows. The axis of symmetry of the body rotates around the external moment. The angle between the axis of symmetry of the body and the external moment is constant. Moreover, the body rotates around its axis of symmetry. The angle of spin of the body differs from the angle of precession to a constant multiplier.

2. The magnitude of the external moment is constant. In this case the left angular velocity (4.21) becomes a linear function of time:

$$\boldsymbol{\omega}_L = \mu^{-1} (\boldsymbol{M} t + \boldsymbol{L}_0)\,. \tag{4.26}$$

The solution of the left Darboux problem (4.22), (4.26) can be expressed in terms of the solution of the right Darboux problem (4.15), (4.18), that takes place in the problem of the rigid body rotation under the action of the moment, directed along its axis of symmetry, and the solution of that is known.

Theorem: *Let the turn-tensors $\boldsymbol{P}$ and $\boldsymbol{P}_*$ be respectively solutions of the right and the left Darboux problems*

$$\dot{\boldsymbol{P}} = \boldsymbol{P} \times \boldsymbol{\Omega}\,, \qquad \boldsymbol{P}(0) = \boldsymbol{E}\,; \qquad \dot{\boldsymbol{P}}_* = \boldsymbol{\omega}_* \times \boldsymbol{P}_*\,, \qquad \boldsymbol{P}_*(0) = \boldsymbol{E}\,. \tag{4.27}$$

If the angular velocities $\boldsymbol{\omega}_*$ and $\boldsymbol{\Omega}$ are related by $\boldsymbol{\omega}_* = -\boldsymbol{\Omega}$, then the turn-tensor $\boldsymbol{P}_*$ is reversed to the turn-tensor $\boldsymbol{P}$: $\boldsymbol{P}_* = \boldsymbol{P}^T$.

To prove the theorem it is sufficient to use the Euler representation of the turn-tensor (1.11), (1.12) and the expressions for the angular velocities (1.13), (1.14). □

5. The rotation of a free axisymmetrical rigid body in the resisting medium

In the previous section we discussed some problems that were solved by the method based on reduction of the dynamics problem to the Darboux problem. This method is efficient for some particular cases only. In case of an arbitrary angular velocity vector the solution of the Darboux problem is very difficult and a reduction of the dynamics problem to a corresponding Darboux problem does not simplify the solution. The majority of rigid body dynamics problems cannot be reduced to the Darboux problem. To solve these problems, we must integrate the dynamics and the kinematics equations simultaneously. However, there exist problems that can be reduced to the Darboux problem, but where nevertheless a simultaneous integration of the system of dynamics and kinematics equations is the more efficient method of their solving. One such problem is discussed in this section.

An axisymmetrical rigid body is considered. The mass center of the body is fixed. Free rotation of the rigid body, taking account of the resistance of surrounding medium, is studied. Interaction between the rigid body and the surrounding medium is simulated by the moment of the linear viscous friction $\boldsymbol{M}_{vf} = -\boldsymbol{K}_{vf} \cdot \boldsymbol{\omega}$ (see Fig. 6). The tensor of viscous friction $\boldsymbol{K}_{vf}$ is supposed to be axisymmetrical and coaxial with the inertia tensor of the rigid body. The accepted assumptions give the following formulation of the problem:

$$\dot{\boldsymbol{L}} = -\boldsymbol{K}_{vf} \cdot \boldsymbol{\omega}\,, \qquad \boldsymbol{K}_{vf} = k_3 \boldsymbol{n}\boldsymbol{n} + k_{12}(\boldsymbol{E} - \boldsymbol{n}\boldsymbol{n})\,, \qquad \boldsymbol{n} = \boldsymbol{P} \cdot \boldsymbol{k}\,, \tag{5.1}$$

The discussed problem in the same formulation was studied for the first time in the book by K. Magnus [10]. A particular case of this problem, when $\boldsymbol{K}_{vf} = k\boldsymbol{E}$, was studied earlier in the book by R. Grammel [3]. (From the point of view of mathematical complexity of a problem, there is no principal difference between the spherical and the axisymmetrical tensors of friction.) In the books by Grammel and Magnus the discussed problem is reduced to the right Darboux problem. However, the obtained Darboux problem is not solved in these books.

Let us consider an alternative approach to the solution of the discussed problem. Let us reformulate the problem in the form (3.2):

$$\dot{\boldsymbol{\omega}}_L = -k_{12}\mu^{-1} \boldsymbol{\omega}_L + (k_{12}\mu^{-1} - k_3\lambda^{-1})\, (\boldsymbol{\omega}_L \cdot \boldsymbol{n})\, \boldsymbol{n}\,, \qquad \dot{\boldsymbol{n}} = \boldsymbol{\omega}_L \times \boldsymbol{n}\,. \tag{5.2}$$

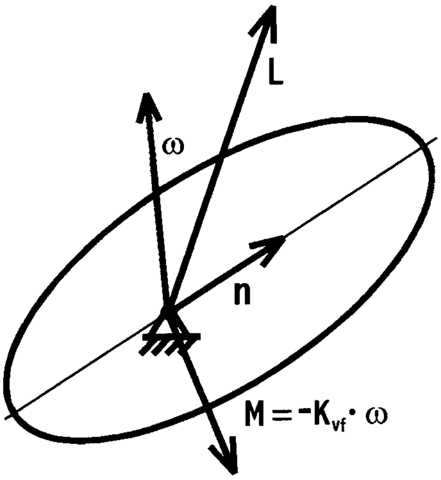

Fig. 6. Resistance: $\boldsymbol{M} = -\boldsymbol{K}_{vf} \cdot \boldsymbol{\omega}$

It is easy to show that the scalar equation

$$(\boldsymbol{\omega}_L \cdot \boldsymbol{n})^{\cdot} = -k_3 \lambda^{-1} (\boldsymbol{\omega}_L \cdot \boldsymbol{n}) \tag{5.3}$$

characterizing the rotation of the body around its axis of symmetry follows from the eqs. (5.2). Eq. (5.3) can be integrated. This allows to express vector $\boldsymbol{n}$, defining the direction of the axis of symmetry of the body, in terms of vector $\boldsymbol{\omega}_L$:

$$\boldsymbol{n} = \frac{\dot{\boldsymbol{\omega}}_L + k_{12}\mu^{-1}\boldsymbol{\omega}_L}{(k_{12}\mu^{-1} - k_3\lambda^{-1})\,(\boldsymbol{\omega}_L \cdot \boldsymbol{n})}\,, \qquad \boldsymbol{\omega}_L \cdot \boldsymbol{n} = \mu^{-1} \boldsymbol{k} \cdot \boldsymbol{L}_0 \,\mathrm{e}^{-(k_3/\lambda)\,t}\,. \tag{5.4}$$

Thus, if vector $\boldsymbol{\omega}_L$ is known, eqs. (5.4) give a complete solution of the problem: the first equation defines the motion of the axis of symmetry of the body, the second one characterizes the rotation of the body around its axis of symmetry. Vector $\boldsymbol{\omega}_L$ can be found as solution of the nonlinear differential equation of the second order, that follows from system (5.2):

$$\ddot{\boldsymbol{\omega}}_L + \left(\frac{k_3}{\lambda} + \frac{k_{12}}{\mu}\right) \dot{\boldsymbol{\omega}}_L + \frac{k_3 k_{12}}{\lambda\mu}\, \boldsymbol{\omega}_L = \boldsymbol{\omega}_L \times \dot{\boldsymbol{\omega}}_L\,. \tag{5.5}$$

It is easy to show that the differential equation of the third order and the first integral

$$\begin{gathered}\dddot{\boldsymbol{\omega}}_L + 2\left(\frac{k_3}{\lambda} + \frac{k_{12}}{\mu}\right) \ddot{\boldsymbol{\omega}}_L + \left[\left(\frac{k_3}{\lambda} + \frac{k_{12}}{\mu}\right)^2 + \frac{k_3 k_{12}}{\lambda\mu} + \omega_L^2\right] \dot{\boldsymbol{\omega}}_L + \left[\frac{k_3 k_{12}}{\lambda\mu}\left(\frac{k_3}{\lambda} + \frac{k_{12}}{\mu}\right) - \frac{(\omega_L^2)^{\cdot}}{2}\right] \boldsymbol{\omega}_L = 0\,, \\ \omega_L^2 = \mu^{-2}[(\boldsymbol{k} \cdot \boldsymbol{L}_0)^2\, \mathrm{e}^{-(2k_3/\lambda)\,t} + (L_0^2 - (\boldsymbol{k} \cdot \boldsymbol{L}_0)^2)\, \mathrm{e}^{-(2k_{12}/\mu)\,t}]\end{gathered} \tag{5.6}$$

follows from eq. (5.5). Taking into account the expression for ω_L^2, the differential equation of the third order can be considered as a linear equation with variable coefficients in $\boldsymbol{\omega}_L$. Let us look for the solution of the differential equation (5.6) in the form of series

$$\boldsymbol{\omega}_L = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \boldsymbol{C}_{nm}\, \mathrm{e}^{-(nk_3/\lambda + mk_{12}/\mu)\,t}\,. \tag{5.7}$$

Substituting series (5.7) into eq. (5.6) and equating to zero coefficients at the exponentials to different powers, we obtain the following results:

$$\begin{gathered}\boldsymbol{\omega}_L(t) = \boldsymbol{A} R_{10}(t) + \boldsymbol{B} R_{01}(t) + \boldsymbol{D} R_{11}(t)\,, \\ R_{pq}(t) = \sum_{n=p}^{\infty} \sum_{m=q}^{\infty} C_{nm}\, \mathrm{e}^{-(nk_3/\lambda + mk_{12}/\mu)\,t}\,, \\ p=1,\ q=0 \quad \Rightarrow \quad n \text{ odd},\ m \text{ even}\,, \\ p=0,\ q=1 \quad \Rightarrow \quad n \text{ even},\ m \text{ odd}\,, \\ p=1,\ q=1 \quad \Rightarrow \quad n, m \text{ odd}\,; \\ C_{10} = 1\,, \qquad C_{01} = 1\,, \qquad C_{11} = 1\,, \\ C_{nm} = -\frac{L_3 K(n-3, m)\, C_{n-2,m} + L_{12} K(n, m-3)\, C_{n,m-2}}{\mu^2 K(n-1, m-1)\, [K(n,m)\, K(n-1, m-1) + K]} \\ K(i,j) = \frac{ik_3}{\lambda} + \frac{jk_{12}}{\mu}\,, \qquad K = \frac{k_3 k_{12}}{\lambda\mu}\,, \qquad L_3 = (\boldsymbol{k} \cdot \boldsymbol{L}_0)^2\,, \qquad L_{12} = L_0^2 - (\boldsymbol{k} \cdot \boldsymbol{L}_0)^2\,.\end{gathered} \tag{5.8}$$

It has been proved that the series' $R_{pq}(t)$ uniformly converge at $t \in [0, \infty[$. Solution (5.8) includes three arbitrary constants $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{D}$. They can be determined by the initial conditions that are: values $\boldsymbol{\omega}_L$ at $t = 0$, the first equation of (5.2) at $t = 0$, and eq. (5.5) at $t = 0$. The solution of the problem is considered in [4] in more detail.

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