4. Rigid body oscillator: a general model and some results

4.1 Introduction

The presented discourse develops a new model named rigid body oscillator. This model plays the same role in Eulerian mechanics as the model of nonlinear oscillator in Newtonian mechanics. The importance of introducing a rigid body oscillator, or in other words a rigid body on an elastic foundation of general kind, was pointed out by many scientists. However, the problem is not formalized up to now. In the paper, all concepts necessary for a mathematical description are introduced. Some of them are new. The equations of motion are represented in an unusual form for rigid body dynamics, have simple structure, but contain a nonlinearity of complicated kind. These equations may be an interesting object for the theory of nonlinear differential equations. The solutions of some problems are presented. For the simplest case, the exact solution is found by an explicit integration of the basic equations.

The nonlinear (linear) oscillator is the most important model of classical physics. An investigation of many physical phenomena and a development of many methods of nonlinear mechanics had arisen due to this model. At the same time, the necessity of construction of models with new properties was recognized. Especially, it was important in quantum mechanics where many authors pointed out that a new model must be something like a rigid body on an elastic foundation. However, such a model was not created up to now. At the present time, two huge branches of mechanics, i.e. continuum mechanics and rigid body dynamics, are existing without close contacts. While, maybe, rigid body dynamics does not need in methods of continuum mechanics, the same can not be said with respect to continuum mechanics. This is clear from the end of the last century. The theories of rods and shells, the theory of Cosserat continuum, the theory of liquid crystals, the theory of ferromagnetic media, and other theories involve ideas from rigid body dynamics. In the theory of liquid crystals, each point of the medium is a rigid body. In the theory of multi-polar continuum, each point is a gyrostat with many rotors inside. Thus, it is clear that the theory of multi-polar continuum can not be constructed without basic ideas of rigid body dynamics. In linear theories, there is no problem. In this case, continuum mechanics and rigid body dynamics use the same language. However, rotations of particles of media are not small in many cases. Therefore, we have to use nonlinear dynamics. In nonlinear theories, the difference between methods of rigid body dynamics and continuum mechanics is essential. Rigid body dynamics uses matrix methods or quaternion methods [13] which are not suited for aims of continuum mechanics. As a matter of fact, the only language which can be used in continuum mechanics is the tensor calculus. Thus, if we are going to apply the methods of rigid body dynamics to continuum mechanics, it is necessary to describe rigid body dynamics in terms of tensors. There are different versions of the tensor calculus. In this paper, the direct tensor calculus is used [3], [6], [9]. In Appendix A, it is shown how to transform the tensor notation into the matrix notation.

A rigid body on an elastic foundation will be called the rigid body oscillator in the following. A general model of such an object can be used in many cases, e.g. in mechanics of multi-polar continuum. For the construction of the model, three new elements are required: the turn-vector (see Appendix A for the terminology explanation), the integrating tensor, and the potential moment. Let us briefly discuss these concepts.

An unusual situation takes place with the turn-vector. On the one side, the well-known Euler theorem proves that any turn of the body can be realized as the turn around an unit vector $\mathbf{n}$ by an angle θ. Thus, the turn can be described by a vector $\boldsymbol{\theta} = \theta\mathbf{n}$. This fact can be found in many works on mechanics. On the other side, the same works [1] claim that the vector $\theta\mathbf{n}$ is not a vector, and a description of a turn in terms of a vector is impossible. Maybe by this reason, the turn-vector has not found great acceptance in conventional rigid body dynamics. However, namely the turn-vector plays a major role in dynamics of a rigid body on an elastic foundation. In classical mechanics, the linear differential form $\mathbf{v}dt$ is the total differential of the vector of position, i.e. $\mathbf{v}dt = d\mathbf{R}$. This is not true for rotations. If the vector $\boldsymbol{\omega}$ is a vector of angular velocity, then the linear differential form $\boldsymbol{\omega}dt$ is not a total differential of the turn-vector. However, it can be proved that there exists an integrating tensor $\mathbf{Z}$ that transforms the linear differential form $\boldsymbol{\omega}dt$ into the total differential $d\boldsymbol{\theta}$ of the turn-vector $\boldsymbol{\theta}$ [11]. The integrating tensor $\mathbf{Z}$ plays the decisive role for an introduction of a potential moment which expresses an action of the elastic foundation on the rigid body. Thus, it is an essential element of a general model of a rigid body oscillator.

The basic equations of dynamics of a rigid body oscillator contain a strong non-linearity, but their form is rather simple. These equations give a very interesting object for applying methods of nonlinear mechanics. In the paper, some simple examples are considered. In particular, the basic equations are integrated explicitly in the case of the simplest model.

4.2 Mathematical preliminaries

In this section, certain aspects of the turn-tensor and the turn-vector will be briefly presented. Some initial definitions can be found in [10].

4.2.1 Vector of turn

The turn-vector is a very old concept. It is difficult to find another concept, for which there exist so many inconsistent propositions as for the turn-vector. Because of this, it seems to be necessary to give the strict introduction of the turn-vector and to describe its basic properties. The introduction of the turn-vector is determined by the well-known Euler statement: any turn can be represented as the turn around some axis $\mathbf{n}$ by a certain angle θ. The vector $\boldsymbol{\theta} = \theta\mathbf{n}$, $|\mathbf{n}| = 1$, is called a turn-vector. Note that two different mathematical concepts correspond to one physical (or geometrical) idea of turn. One of them is described by a turn-tensor, the other is described by a turn-vector. Of course, both of them are connected uniquely. For the turn-tensor we shall use the notation [10] — see Appendix A for additional explanations of notations and terminology

$$\mathbf{Q}\,(\theta\mathbf{n}) = (1 - \cos\theta)\,\mathbf{n}\otimes\mathbf{n} + \cos\theta\mathbf{E} + \sin\theta\mathbf{n}\times\mathbf{E}. \tag{4.1}$$

An action of the tensor $\mathbf{Q}\,(\theta\mathbf{n})$ on the vector $\mathbf{a}$ can be expressed in the form

$$\mathbf{a}' = \mathbf{Q}\,(\theta\mathbf{n})\cdot\mathbf{a} = (\mathbf{a}\cdot\mathbf{n})\,\mathbf{n} + \cos\theta\,(\mathbf{a} - (\mathbf{a}\cdot\mathbf{n})\,\mathbf{n}) + \sin\theta\mathbf{n}\times\mathbf{a}. \tag{4.2}$$

If $\mathbf{n}\times\mathbf{a} = \mathbf{0}$, then $\mathbf{a}' = \mathbf{a}$. If $\mathbf{a}\cdot\mathbf{n} = 0$, then we have

$$\mathbf{a}' = \cos\theta\,\mathbf{a} + \sin\theta\mathbf{n}\times\mathbf{a}.$$

This means that the vector $\mathbf{a}'$ represents the vector $\mathbf{a}$ turned around the vector $\mathbf{n}$ by an angle θ. Representation (4.1) can be rewritten also as

$$\mathbf{Q}\,(\boldsymbol{\theta}) = \mathbf{E} + \frac{\sin\theta}{\theta}\mathbf{R} + \frac{1-\cos\theta}{\theta^2}\mathbf{R}^2 = \exp\mathbf{R}, \tag{4.3}$$

where

$$\mathbf{R} = \boldsymbol{\theta}\times\mathbf{E}, \quad \mathbf{R}^2 = \boldsymbol{\theta}\otimes\boldsymbol{\theta} - \theta^2\mathbf{E}, \quad \theta = |\boldsymbol{\theta}|\,. \tag{4.4}$$

Let us show the derivation of the second equality in (4.3). By definition we have

$$\exp\mathbf{R} = \sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{R}^k, \quad \mathbf{R}^{2k+1} = \left(-\theta^2\right)^k\mathbf{R}, \quad \mathbf{R}^{2k} = \left(-\theta^2\right)^{k-1}\mathbf{R}^2, \quad k \geq 1.$$

From this follows

$$\exp \mathbf{R} = \mathbf{E} + \left(\sum_{k=0}^{\infty} \frac{\left(-\theta^2\right)^k}{(2k+1)\,!} \right) \mathbf{R} + \left(\sum_{k=1}^{\infty} \frac{\left(-\theta^2\right)^{k-1}}{(2k)\,!} \right) \mathbf{R}^2.$$

If we take into account the power series for $\cos\theta$ and $\sin\theta$, we obtain (4.3), which in matrix notations can be found in [1]. Note that there exists a certain difference between representations (4.1) and (4.3). In (4.1) the quantity θ is the angle of turn and can be both positive and negative. In (4.3) the quantity θ is the modulus of the turn-vector, i.e. the modulus of the angle of turn. Such an interpretation is possible since, e.g. $\sin\theta/\theta = \sin|\theta| \,/\, |\theta|$. As a rule, representation (4.3) is more convenient for applications then expression (4.1).

Let us consider a superposition of two turns

$$\mathbf{Q}(\boldsymbol{\theta}) = \mathbf{Q}(\boldsymbol{\varphi}) \cdot \mathbf{Q}(\boldsymbol{\psi}). \tag{4.5}$$

The vector of total turn $\boldsymbol{\theta}$ is connected with the turn-vectors $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ by the formulas

$$\begin{aligned} \mathrm{tr}\mathbf{Q}(\boldsymbol{\theta}) = 1 + 2\cos\theta = \cos\varphi + \cos\psi + \cos\varphi\cos\psi - \\ -2\frac{\sin\varphi}{\varphi}\frac{\sin\psi}{\psi}\boldsymbol{\varphi}\cdot\boldsymbol{\psi} + \frac{(1-\cos\varphi)}{\varphi^2}\frac{(1-\cos\psi)}{\psi^2}(\boldsymbol{\varphi}\cdot\boldsymbol{\psi})^2, \end{aligned} \tag{4.6}$$

$$\begin{aligned} -\left[\mathbf{Q}(\boldsymbol{\theta})\right]_{\times} = 2\frac{\sin\theta}{\theta}\boldsymbol{\theta} = \\ = \left(\frac{\sin\varphi}{\varphi}(1+\cos\psi) - \frac{(1-\cos\varphi)}{\varphi^2}\frac{\sin\psi}{\psi}\boldsymbol{\varphi}\cdot\boldsymbol{\psi} \right)\boldsymbol{\varphi} + \\ + \left(\frac{\sin\psi}{\psi}(1+\cos\varphi) - \frac{(1-\cos\psi)}{\psi^2}\frac{\sin\varphi}{\varphi}\boldsymbol{\varphi}\cdot\boldsymbol{\psi} \right)\boldsymbol{\psi} + \\ + \left(\frac{\sin\varphi\sin\psi}{\varphi\psi} - \frac{(1-\cos\varphi)}{\varphi^2}\frac{(1-\cos\psi)}{\psi^2}\boldsymbol{\varphi}\cdot\boldsymbol{\psi} \right)\boldsymbol{\varphi}\times\boldsymbol{\psi}. \end{aligned} \tag{4.7}$$

Note that from expressions (4.3), (4.4) follows

$$\mathbf{R}\cdot\boldsymbol{\theta} = 0, \quad \mathbf{Q}(\boldsymbol{\theta})\cdot\boldsymbol{\theta} = \boldsymbol{\theta}. \tag{4.8}$$

4.2.2 Integrating tensor

For rotations the turn-vector $\boldsymbol{\theta}(t)$ plays the same role as the vector $\mathbf{R}(t)$ for translations. In the latter case, the translation velocity $\boldsymbol{v}$ can be found by means of a simple formula, i.e. $\boldsymbol{v} = \dot{\mathbf{R}}(t)$. This means that the linear form $\mathbf{v}dt$ is the total differential of the position vector. For the rotations the situation is more complicated, since the linear form ωdt is not the total differential of the turn-vector θ, where ω is the vector of angular velocity. Thus, it is necessary to find an integrating factor that transforms the linear form ωdt to the total differential $d\theta$ of the turn-vector. For this end, let us consider the left Poisson equation [10]

$$\frac{d}{dt}\mathbf{Q} = \dot{\mathbf{Q}} = \omega \times \mathbf{Q}\left(\theta\right). \tag{4.9}$$

This equation for the turn-tensor $\mathbf{Q}\left(\theta\right)$ is equivalent to a system of nine scalar equations but only three of them are independent. In order to find these independent equations, it is possible to substitute $\mathbf{Q}\left(\theta\right)$ in Eq. (4.9) by equality (4.3). After some transformations given in Appendix B we find

$$\dot{\theta}\left(t\right) = \mathbf{Z}\left(\theta\right) \cdot \omega\left(t\right), \tag{4.10}$$

where

$$\mathbf{Z}\left(\theta\right) = \mathbf{E} - \frac{1}{2}\mathbf{R} + \frac{1-g}{\theta^2}\mathbf{R}^2, \quad g = \frac{\theta\sin\theta}{2\left(1-\cos\theta\right)}. \tag{4.11}$$

The tensor $\mathbf{Z}\left(\theta\right)$ will be called the integrating tensor in the following. The tensor $\mathbf{Z}$ has the determinant

$$\det \mathbf{Z}\left(\theta\right) = \theta^2/2\left(1-\cos\theta\right) \neq 0.$$

Strictly speaking, we must exclude the singular points $\theta = 2\pi k$, $k \geq 1$, where k is a positive integer, from the consideration. However, it seems to be obvious that there exist only following alternatives: either $\omega = \dot{\theta}$ and θ may have an arbitrary value, or $\omega \neq \dot{\theta}$ and θ is forced to be less than π. In the first case, we have $\omega \times \theta = 0$ and expression (4.10) or, what is the same, (4.15) is valid. In the second case, the θ can not be a singular point. These alternatives are obvious from the geometrical point of view, but its strict proof is unknown. It can be shown [11] that if $\theta^2 >> 1$, then the equality $\theta \times \omega = 0$ must be valid. As a matter of fact, we do not know problems where the singular points lead to noticeable difficulties.

The integrating tensor has a number of useful properties. Let us describe some of them. First of all, the tensor $\mathbf{Z}\left(\theta\right)$ is an isotropic function of the turn-vector θ which means

$$\mathbf{Z}\left(\mathbf{S}\cdot\theta\right) = \mathbf{S}\cdot\mathbf{Z}\left(\theta\right)\cdot\mathbf{S}^T \quad \forall \mathbf{S} : \mathbf{S}\cdot\mathbf{S}^T = \mathbf{E}, \quad \det\mathbf{S} = 1. \tag{4.12}$$

If $\mathbf{S}$=$\mathbf{Q}\left(\theta\right)$, then it follows from (4.12) and (4.8)

$$\mathbf{Z}\left(\theta\right)\cdot\mathbf{Q}\left(\theta\right) = \mathbf{Q}\left(\theta\right)\cdot\mathbf{Z}\left(\theta\right).$$

Further, we can prove the identity

$$\mathbf{Z}^T\left(\theta\right) = \mathbf{Q}\left(\theta\right)\cdot\mathbf{Z}\left(\theta\right) = \mathbf{Z}\left(\theta\right)\cdot\mathbf{Q}\left(\theta\right). \tag{4.13}$$

For the right angular velocity $\boldsymbol{\Omega} = \mathbf{Q}^{\mathrm{T}}(\boldsymbol{\theta}) \cdot \boldsymbol{\omega}$, see [10], it follows from expressions (4.10) and (4.13)

$$\dot{\boldsymbol{\theta}}(t) = \mathbf{Z}^{\mathrm{T}}(\boldsymbol{\theta}) \cdot \boldsymbol{\Omega}(t). \tag{4.14}$$

This equation is equivalent to the right Poisson equation [10]. In an explicit form, Eqs. (4.10) and (4.14) can be rewritten as

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}), \quad \boldsymbol{\theta}(0) = \boldsymbol{\theta}_0; \tag{4.15}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\Omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\Omega}), \quad \boldsymbol{\theta}(0) = \boldsymbol{\theta}_0. \tag{4.16}$$

In [5] one can find expression (4.15) in terms of a vector of a finite rotation

$$\dot{\boldsymbol{\theta}}_* = \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta}_* \times \boldsymbol{\omega} + \frac{1}{4}\boldsymbol{\theta}_* (\boldsymbol{\theta}_* \cdot \boldsymbol{\omega}), \quad \boldsymbol{\theta}_* = \frac{2}{\theta}\left(\tan\frac{\theta}{2}\right)\boldsymbol{\theta}.$$

This expression coincides with (4.15). Thus, we see that generally speaking almost all our expressions are known. However, paper [1] deals with a turn-tensor without a turn-vector, book [5] deals with a turn-vector without a turn-tensor. Besides, the vector of finite rotation $\boldsymbol{\theta}_*$ is not convenient in some cases since it is discontinuous. Problem (4.15) is called the left Darboux problem[10]. If the left angular velocity is known, then the turn-vector (and therefore the turn-tensor) can be found as the solution of differential Eq. (4.15).

Expressions (4.15) and (4.16) can also be written in an equivalent form

$$\dot{\boldsymbol{\theta}} = g\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1-g}{\theta}\dot{\theta}\boldsymbol{\theta}, \tag{4.17}$$

$$\dot{\boldsymbol{\theta}} = g\boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\Omega} + \frac{1-g}{\theta}\dot{\theta}\boldsymbol{\theta} \tag{4.18}$$

by taking into account the identities

$$\boldsymbol{\theta} \cdot \boldsymbol{\omega} = \boldsymbol{\theta} \cdot \boldsymbol{\Omega} = \boldsymbol{\theta} \cdot \dot{\boldsymbol{\theta}} = \theta\dot{\theta}.$$

Sometimes it is more convenient to use an inverse form of Eqs. (4.10) – (4.14)

$$\boldsymbol{\omega}(t) = \mathbf{Z}^{-1}(\boldsymbol{\theta}) \cdot \dot{\boldsymbol{\theta}}(t), \quad \boldsymbol{\Omega}(t) = \mathbf{Z}^{-\mathrm{T}}(\boldsymbol{\theta}) \cdot \dot{\boldsymbol{\theta}}(t), \tag{4.19}$$

where

$$\mathbf{Z}^{-1}(\boldsymbol{\theta}) = \mathbf{E} + \frac{1-\cos\theta}{\theta^2}\mathbf{R} + \frac{\theta - \sin\theta}{\theta^3}\mathbf{R}^2 \tag{4.20}$$

according to Appendix B.

4.2.3 Potential moment

Let us introduce a concept of potential moment. This concept is necessary for a statement and an analysis of many problems. Nevertheless a general definition of potential moment is absent in the literature.

Definition: A moment $\mathbf{M}(t)$ is called potential, if there exists a scalar function $U(\boldsymbol{\theta})$ depending on a turn-vector such that

$$\mathbf{M} \cdot \boldsymbol{\omega} = -\dot{U}(\boldsymbol{\theta}) = -\frac{dU}{d\boldsymbol{\theta}} \cdot \dot{\boldsymbol{\theta}}, \tag{4.21}$$

see Appendix C. Making use of Eq. (4.10), this equality can be rewritten in the form

$$\left(\mathbf{M} + \frac{dU}{d\boldsymbol{\theta}} \cdot \mathbf{Z}\right) \cdot \boldsymbol{\omega} = 0.$$

This equality must be satisfied for any vector $\boldsymbol{\omega}$ which is possible if and only if

$$\mathbf{M} = -\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{dU}{d\boldsymbol{\theta}} + \mathbf{f}(\boldsymbol{\theta}, \boldsymbol{\omega}) \times \boldsymbol{\omega}, \tag{4.22}$$

where $\mathbf{f}(\boldsymbol{\theta}, \boldsymbol{\omega})$ is an arbitrary function of vectors $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$.

Definition: A moment $\mathbf{M}$ is called positional, if $\mathbf{M}$ depends on the turn-vector $\boldsymbol{\theta}$ only. For the positional moment $\mathbf{M}(\boldsymbol{\theta})$ we have

$$\mathbf{M}(\boldsymbol{\theta}) = -\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{dU(\boldsymbol{\theta})}{d\boldsymbol{\theta}}. \tag{4.23}$$

Let us show two simple examples. If the potential function has a form of an isotropic function of the turn-vector

$$U(\boldsymbol{\theta}) = F\left(\theta^2\right) = F(\boldsymbol{\theta} \cdot \boldsymbol{\theta}),$$

then we find from expressions (4.23) and (4.11)

$$\mathbf{M}(\boldsymbol{\theta}) = -2\frac{dF\left(\theta^2\right)}{d\left(\theta^2\right)}\boldsymbol{\theta}.$$

If a potential function has the simplest form $U(\boldsymbol{\theta}) = C\mathbf{k} \cdot \boldsymbol{\theta},\ C = const, \mathbf{k} = const, |\mathbf{k}| = 1$, then we have a rather complicated expression for the moment:

$$\mathbf{M} = -C\mathbf{Z}^T \cdot \mathbf{k} = -C\left(\mathbf{k} + \frac{1}{2}\boldsymbol{\theta} \times \mathbf{k} + \frac{1-g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \mathbf{k})\right).$$

Definition : The potential $U(\boldsymbol{\theta})$ is called transversally isotropic with an axis of symmetry $\mathbf{k}$, if the equality

$$U(\boldsymbol{\theta}) = U\left[\mathbf{Q}(\alpha\mathbf{k}) \cdot \boldsymbol{\theta}\right]$$

holds for any turn-tensor $\mathbf{Q}(\alpha \mathbf{k})$. It can be proved — see [11] and Appendix D — that a general form of a transversally isotropic potential can be expressed as a function of two arguments

$$U(\boldsymbol{\theta}) = F\left(\mathbf{k} \cdot \boldsymbol{\theta}, \theta^2\right). \tag{4.24}$$

For this potential one can derive the expression

$$\mathbf{M}(\boldsymbol{\theta}) = -2\frac{\partial F}{\partial\left(\theta^2\right)}\boldsymbol{\theta} - \frac{\partial F}{\partial\left(\mathbf{k}\cdot\boldsymbol{\theta}\right)}\mathbf{Z}^T \cdot \mathbf{k}. \tag{4.25}$$

There exists the obvious identity

$$\left(\mathbf{E} - \mathbf{Q}(\boldsymbol{\theta})\right) \cdot \boldsymbol{\theta} = \left(\mathbf{E} - \mathbf{Q}^T\right) \cdot \boldsymbol{\theta} = 0 \Longrightarrow \left(\mathbf{a} - \mathbf{a}'\right) \cdot \boldsymbol{\theta} = 0$$

for arbitrary $\mathbf{a}$, $\mathbf{a}' = \mathbf{Q} \cdot \mathbf{a}$. Taking into account this identity, we may obtain from expression (4.25)

$$\left(\mathbf{E} - \mathbf{Q}^T(\boldsymbol{\theta})\right) \cdot \mathbf{M} = \frac{\partial F}{\partial\left(\mathbf{k}\cdot\boldsymbol{\theta}\right)}\mathbf{k} \times \boldsymbol{\theta}.$$

Multiplying this equality by the vector $\mathbf{k}$ we obtain

$$\left(\mathbf{k} - \mathbf{k}'\right) \cdot \mathbf{M} = 0. \tag{4.26}$$

For the isotropic potential, equality (4.26) holds for any vector $\mathbf{a}$. Sometimes equality (4.26) is very important — see, for example, section 4.

4.2.4 The perturbation method on the set of orthogonal tensors

Turn-tensors are subjected to restrictions

$$\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{E}, \quad \det \mathbf{Q} = +1. \tag{4.27}$$

A perturbed turn-tensor $\mathbf{Q}_\varepsilon$ must be subjected to these conditions as well. In contrast, the turn-vector has no restrictions like (4.27). Thus the perturbed turn-vector can be simply defined as

$$\boldsymbol{\theta}_\varepsilon = \boldsymbol{\theta}_0 + \varepsilon\boldsymbol{\varphi}, \quad |\varepsilon| \ll 1, \tag{4.28}$$

where $\boldsymbol{\theta}_0$ is the unperturbed turn-vector and the vector $\boldsymbol{\varphi}$ is called the first variation of the turn-vector. The perturbed turn-tensor can be found from (4.3) and (4.4)

$$\mathbf{Q}_\varepsilon = \exp \mathbf{R}_\varepsilon = \exp\left(\boldsymbol{\theta}_\varepsilon \times \mathbf{E}\right). \tag{4.29}$$

Equations (4.27) are then satisfied by the tensor $\mathbf{Q}_\varepsilon$ for arbitrary vectors $\boldsymbol{\theta}_\varepsilon$. We shall consider the parameter ε as an independent variable. In such a case, it is possible to introduce the left and the right perturbation directions $\boldsymbol{\eta}_\varepsilon$ and $\boldsymbol{\zeta}_\varepsilon$, respectively, analogously to the angular velocities:

$$\frac{\partial}{\partial\varepsilon}\mathbf{Q}_\varepsilon = \boldsymbol{\eta}_\varepsilon \times \mathbf{Q}_\varepsilon = \mathbf{Q}_\varepsilon \times \boldsymbol{\zeta}_\varepsilon, \quad \boldsymbol{\eta}_\varepsilon = \mathbf{Q}_\varepsilon \cdot \boldsymbol{\zeta}_\varepsilon. \tag{4.30}$$

The perturbed angular velocities can be found from the Poisson equations (4.9)

$$\dot{\mathbf{Q}}_\varepsilon = \boldsymbol{\omega}_\varepsilon \times \mathbf{Q}_\varepsilon = \mathbf{Q}_\varepsilon \times \boldsymbol{\Omega}_\varepsilon, \quad \boldsymbol{\omega}_\varepsilon = \mathbf{Q}_\varepsilon \cdot \boldsymbol{\Omega}_\varepsilon. \tag{4.31}$$

The conditions of integrability for system (4.30), (4.31) can be written in the form

$$\frac{\partial}{\partial\varepsilon}\boldsymbol{\omega}_\varepsilon = \dot{\boldsymbol{\eta}}_\varepsilon + \boldsymbol{\eta}_\varepsilon \times \boldsymbol{\omega}_\varepsilon, \quad \frac{\partial}{\partial\varepsilon}\boldsymbol{\Omega}_\varepsilon = \dot{\boldsymbol{\zeta}}_\varepsilon - \boldsymbol{\zeta}_\varepsilon \times \boldsymbol{\Omega}_\varepsilon. \tag{4.32}$$

For the perturbation directions we have expressions analogous to Eqs. (4.19)

$$\boldsymbol{\eta}_\varepsilon = \mathbf{Z}^{-1}(\boldsymbol{\theta}_\varepsilon) \cdot \frac{\partial}{\partial\varepsilon}\boldsymbol{\theta}_\varepsilon = \mathbf{Z}_\varepsilon^{-1} \cdot \boldsymbol{\varphi}, \quad \boldsymbol{\zeta}_\varepsilon = \mathbf{Z}_\varepsilon^{-T} \cdot \boldsymbol{\varphi}. \tag{4.33}$$

According to Eqs. (4.19), the perturbed angular velocities can be found by

$$\boldsymbol{\omega}_\varepsilon = \mathbf{Z}_\varepsilon^{-1} \cdot \dot{\boldsymbol{\theta}}_\varepsilon, \quad \boldsymbol{\Omega}_\varepsilon = \mathbf{Z}_\varepsilon^{-T} \cdot \dot{\boldsymbol{\theta}}_\varepsilon.$$

If the unperturbed vector $\boldsymbol{\theta}_0$ does not depend on time (state of equilibrium), then

$$\boldsymbol{\omega}_\varepsilon = \varepsilon\mathbf{Z}_\varepsilon^{-1} \cdot \dot{\boldsymbol{\varphi}}, \quad \boldsymbol{\Omega}_\varepsilon = \varepsilon\mathbf{Z}_\varepsilon^{-T} \cdot \dot{\boldsymbol{\varphi}}. \tag{4.34}$$

Let there be given the function $f(\varepsilon, t)$. The quantity

$$f^*(t) = [\partial f(\varepsilon, t)/\partial\varepsilon]_{\varepsilon=0} \tag{4.35}$$

is called the first variation of the function $f(\varepsilon, t)$. For the first variation of the turn-tensor and of the perturbation directions we find from (4.30), (4.33), and (4.34)

$$\mathbf{Q}^* = \boldsymbol{\eta}_0 \times \mathbf{Q}_0, \quad \boldsymbol{\eta}_0 = \mathbf{Z}_0^{-1} \cdot \boldsymbol{\varphi}, \quad \boldsymbol{\omega}^* = \dot{\boldsymbol{\eta}}_0 + \boldsymbol{\eta}_0 \times \boldsymbol{\omega}_0, \tag{4.36}$$

where the subscripts 0 marks the unperturbed state, $\boldsymbol{\eta}_0 = \boldsymbol{\eta}_\varepsilon \,|_{\varepsilon=0}$. For the right quantities analogous relations are valid

$$\mathbf{Q}^* = \mathbf{Q}_0 \times \boldsymbol{\zeta}_0, \quad \boldsymbol{\zeta}_0 = \mathbf{Z}_0^{-T} \cdot \boldsymbol{\varphi}, \quad \boldsymbol{\Omega}^* = \dot{\boldsymbol{\zeta}}_0 - \boldsymbol{\zeta}_0 \times \boldsymbol{\Omega}_0. \tag{4.37}$$

Especially, if the perturbations are superposed on a state of equilibrium, these formulas may be simplified by $\boldsymbol{\omega}_0 = \boldsymbol{\Omega}_0 = \mathbf{0}$. Using Eqs. (4.28), (4.34)- (4.37) the first variation of the modulus of the turn-vector may be found from $\theta_\varepsilon^2 = \boldsymbol{\theta}_\varepsilon \cdot \boldsymbol{\theta}_\varepsilon$:

$$\theta^* = \frac{1}{\theta_0}\boldsymbol{\theta}_0 \cdot \boldsymbol{\varphi} = \frac{1}{\theta_0}\boldsymbol{\theta}_0 \cdot \boldsymbol{\eta}_0 = \frac{1}{\theta_0}\boldsymbol{\theta}_0 \cdot \boldsymbol{\zeta}_0. \tag{4.38}$$

4.3 The equations of motion of a rigid body oscillator

Let us consider a rigid body with a fixed point O. The body is supposed to be clamped on an elastic foundation, which is resisting to any turn of the body. The position of the body for an undeformed elastic foundation is chosen as reference position. The tensor of inertia of the body with respect to the fixed point O will be denoted as

$$\mathbf{A} = A_1 \mathbf{d}_1 \otimes \mathbf{d}_1 + A_2 \mathbf{d}_2 \otimes \mathbf{d}_2 + A_3 \mathbf{d}_3 \otimes \mathbf{d}_3, \tag{4.39}$$

where $A_i > 0$ are the principal moments of inertia and the vectors $\mathbf{d}_i$ are the principal axes of the inertia tensor. Of course, the tensor $\mathbf{A}$ can be represented in terms of an arbitrary basis $\mathbf{e}_i$

$$\mathbf{d}_i = \alpha_i^m \mathbf{e}_m, \quad \mathbf{A} = A^{mn} \mathbf{e}_m \otimes \mathbf{e}_n, \quad A^{mn} = \sum_{i=1}^{3} \alpha_i^m \alpha_i^n A_i.$$

If a body has the axis of symmetry $\mathbf{k}$, then the inertia tensor will be transversally isotropic

$$\mathbf{A} = A_1 (\mathbf{E} - \mathbf{k} \otimes \mathbf{k}) + A_3 \mathbf{k} \otimes \mathbf{k}, \quad \mathbf{d}_3 = \mathbf{k}, \quad A_1 = A_2. \tag{4.40}$$

The position of a body at the instant t is called the actual position of a body. The motion of the body can be defined either by the turn-tensor $\mathbf{P}(t)$ or by the turn-vector $\boldsymbol{\theta}(t)$

$$\mathbf{P}(t) = \mathbf{Q}(\boldsymbol{\theta}(t)).$$

The tensor of inertia $\mathbf{A}^{(t)}$ in the actual position is determined by

$$\mathbf{A}^{(t)} = \mathbf{P}(t) \cdot \mathbf{A} \cdot \mathbf{P}^T(t). \tag{4.41}$$

If the tensor $\mathbf{A}$ is transversally isotropic, this results in

$$\mathbf{A}^{(t)} = A_1 \left(\mathbf{E} - \mathbf{k}' \otimes \mathbf{k}'\right) + A_3 \mathbf{k}' \otimes \mathbf{k}', \quad \mathbf{k}' = \mathbf{P} \cdot \mathbf{k}. \tag{4.42}$$

The kinetic moment of the body can be expressed in two forms. In terms of the left angular velocity we obtain

$$\mathbf{L} = \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{P}^T \cdot \boldsymbol{\omega} = A_1 \boldsymbol{\omega} + (A_3 - A_1) \left(\mathbf{k}' \cdot \boldsymbol{\omega}\right) \mathbf{k}', \tag{4.43}$$

where the second term of Eq. (4.43) applies to the transversally isotropic tensor of inertia only. In terms of the right angular velocity the kinetic moment has the form

$$\mathbf{L} = \mathbf{P} \cdot \mathbf{A} \cdot \boldsymbol{\Omega} = \mathbf{P} \cdot [A_1 \boldsymbol{\Omega} + (A_3 - A_1)(\mathbf{k} \cdot \boldsymbol{\Omega}) \mathbf{k}]. \tag{4.44}$$

Let us note that

$$\mathbf{k}' \cdot \boldsymbol{\omega} = \mathbf{k} \cdot \mathbf{P}^{\mathsf{T}} \cdot \boldsymbol{\omega} = \mathbf{k} \cdot \boldsymbol{\Omega}. \tag{4.45}$$

An external moment $\mathbf{M}$ acting on the body can be represented in the form

$$\mathbf{M} = \mathbf{M}_{e} + \mathbf{M}_{ext},$$

where $\mathbf{M}_{e}$ is a reaction of the elastic foundation and $\mathbf{M}_{ext}$ is an additional external moment. The elastic moment $\mathbf{M}_{e}$ is supposed to be positional with a potential. In such a case, we may write according to (4.23)

$$\mathbf{M}_{e} = -\mathbf{Z}^{\mathsf{T}}(\boldsymbol{\theta}) \cdot \frac{dU(\boldsymbol{\theta})}{d\boldsymbol{\theta}}, \tag{4.46}$$

where the scalar function $U(\boldsymbol{\theta})$ is called the elastic energy. In the following, the elastic foundation is supposed to be transversally isotropic. Then the elastic moment can be represented in form (4.25), i.e.

$$\mathbf{M}_{e}(\boldsymbol{\theta}) = -C\left(\theta^{2}, \mathbf{k} \cdot \boldsymbol{\theta}\right)\boldsymbol{\theta} - D\left(\theta^{2}, \mathbf{k} \cdot \boldsymbol{\theta}\right)\mathbf{Z}^{\mathsf{T}}(\boldsymbol{\theta}) \cdot \mathbf{k}, \tag{4.47}$$

where the unit vector $\mathbf{k}$ is placed on the axis of isotropy of the body in the reference position, and

$$C = 2\frac{\partial}{\partial\left(\theta^{2}\right)}U\left(\theta^{2}, \mathbf{k} \cdot \boldsymbol{\theta}\right), \quad D = \frac{\partial}{\partial\left(\mathbf{k} \cdot \boldsymbol{\theta}\right)}U\left(\theta^{2}, \mathbf{k} \cdot \boldsymbol{\theta}\right). \tag{4.48}$$

Let us consider a possible expression of an elastic energy, e.g.

$$U = \frac{1}{2}\frac{\alpha^{2}c\theta^{2}}{\alpha^{2} - \theta^{2} + (\mathbf{k} \cdot \boldsymbol{\theta})^{2}} + \frac{1}{2}\frac{\beta^{2}(d - c)(\mathbf{k} \cdot \boldsymbol{\theta})^{2}}{\beta^{2} - (\mathbf{k} \cdot \boldsymbol{\theta})^{2}}, \tag{4.49}$$

where $\alpha^{2} > 0$, $\beta^{2} > 0$, $c > 0$ and $d > 0$ are constant parameters. The parameters c and d are the bending stiffness and torsional stiffness of the elastic foundation, respectively. If the parameters α^{2} and β^{2} go to infinity, we obtain as simplest form of an elastic potential

$$U = \frac{1}{2}c\left(\theta^{2} - (\mathbf{k} \cdot \boldsymbol{\theta})^{2}\right) + \frac{1}{2}d(\mathbf{k} \cdot \boldsymbol{\theta})^{2}. \tag{4.50}$$

In this case, the elastic moment (4.47) is

$$\mathbf{M}_{e}(\boldsymbol{\theta}) = -c\boldsymbol{\theta} - (d - c)(\mathbf{k} \cdot \boldsymbol{\theta})\mathbf{Z}^{\mathsf{T}}(\boldsymbol{\theta}) \cdot \mathbf{k}. \tag{4.51}$$

For an external moment $\mathbf{M}_{ext}$ let us accept the expression

$$\mathbf{M}_{ext} = -\mathbf{Z}^{\mathsf{T}}(\boldsymbol{\theta}) \cdot \frac{dV(\boldsymbol{\theta})}{d\boldsymbol{\theta}} + \mathbf{M}_{ex}, \tag{4.52}$$

where the first term describes the potential part of the external moment according to (4.23). The second law of dynamics by Euler can be represented in two equivalent forms. In terms of the left angular velocity we find from $\dot{\mathbf{L}} = \mathbf{M}$

$$\left[\mathbf{P}(\theta) \cdot \mathbf{A} \cdot \mathbf{P}^{\mathsf{T}}(\theta) \cdot \boldsymbol{\omega}\right]^{\cdot} + \mathbf{Z}^{\mathsf{T}}(\theta) \cdot \frac{d\,(U+V)}{d\theta} = \mathbf{M}_{ex}. \tag{4.53}$$

This equation has to be completed by the left Poisson Eq. (4.15)

$$\dot{\theta} = \boldsymbol{\omega} - \frac{1}{2}\theta \times \boldsymbol{\omega} + \frac{1-g}{\theta^2}\theta \times (\theta \times \boldsymbol{\omega}) \tag{4.54}$$

resulting in a general model of a rigid body oscillator. In terms of the right angular velocity, the model can be represented by

$$\mathbf{A} \cdot \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbf{A} \cdot \boldsymbol{\Omega} + \mathbf{Z}(\theta) \cdot \frac{d\,(U+V)}{d\theta} = \mathbf{P}^{\mathsf{T}}(\theta) \cdot \mathbf{M}_{ex}, \tag{4.55}$$

$$\dot{\theta} = \boldsymbol{\Omega} + \frac{1}{2}\theta \times \boldsymbol{\Omega} + \frac{1-g}{\theta^2}\theta \times (\theta \times \boldsymbol{\Omega}), \tag{4.56}$$

where Eqs. (4.13), (4.16), (4.44), (4.46), (4.52) and identities

$$(\mathbf{P} \cdot \mathbf{A} \cdot \boldsymbol{\Omega})^{\cdot} = \mathbf{P} \cdot \left(\mathbf{A} \cdot \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbf{A} \cdot \boldsymbol{\Omega}\right), \quad \mathbf{P} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{P} \cdot \mathbf{a}) \times (\mathbf{P} \cdot \mathbf{b})$$

have been used. It is important that the model of a rigid body oscillator is represented in terms of natural variables: the turn-vector and the vector of angular velocity. A significant merit of the equations stated above is that they contain the first derivatives of the unknown vectors only. Thus, it is possible to use standard methods for numerical analysis.
The remainder of the paper deals with applications of the derived equations.

4.4 Paradox by Nikolai

Let us consider a classical problem that was investigated by E.L.Nikolai [8]. Later it was studied by many other authors — see, for example [14], [2], where other references may be found.

The inertia tensor of the body is supposed to be transversally isotropic as defined by expression (4.40). An external follower moment is defined by

$$\mathbf{M}_{ext} = \mathbf{M}_{ex} = M\mathbf{k}' = M\mathbf{P}(\theta) \cdot \mathbf{k}, \quad M = \mathrm{const}, \tag{4.57}$$

where the unit vector $\mathbf{k}$ is placed on the symmetry axis of the body in the reference position. The Eqs. (4.55) – (4.56) then read as

$$A_1\dot{\boldsymbol{\Omega}} + (A_3 - A_1)\left(\mathbf{k}\cdot\dot{\boldsymbol{\Omega}}\right)\mathbf{k} - (A_3 - A_1)(\mathbf{k}\cdot\boldsymbol{\Omega})\,\mathbf{k}\times\boldsymbol{\Omega} + \\ + C\boldsymbol{\theta} + D\mathbf{Z}(\boldsymbol{\theta})\cdot\mathbf{k} = M\mathbf{k}, \quad (4.58)$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta}\times\boldsymbol{\Omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\Omega}), \quad g = \frac{\theta\sin\theta}{2(1-\cos\theta)}, \quad (4.59)$$

where the functions C and D are defined by expressions (4.48). It is easy to find the equilibrium solution of system of Eqs. (4.58) – (4.59)

$$\boldsymbol{\theta} = \theta\mathbf{k}, \quad \theta = const, \quad \boldsymbol{\Omega} = 0. \quad (4.60)$$

Substituting Eq. (4.60) into system (4.58) – (4.59), we obtain the equation

$$C\left(\theta^2,\theta\right)\theta + D\left(\theta^2,\theta\right) = M. \quad (4.61)$$

Especially, if the elastic energy has form (4.50), then Eq. (4.61) takes a linear form

$$C\left(\theta^2,\theta\right) = c, D\left(\theta^2,\theta\right) = (d-c)\,\mathbf{k}\cdot\boldsymbol{\theta} \Longrightarrow \boldsymbol{\theta} = M\mathbf{k}/d. \quad (4.62)$$

In order to investigate the stability of solution (4.62), we will use a method of superposition of small perturbations on the state of equilibrium. Let us consider the perturbed quantities

$$\boldsymbol{\theta}_\varepsilon = \theta_0\mathbf{k} + \varepsilon\boldsymbol{\varphi}(t), \quad \boldsymbol{\Omega}_\varepsilon = \varepsilon\boldsymbol{\eta}(t), \quad \theta_0 = M/d. \quad (4.63)$$

The perturbed Eqs. (4.58) – (4.59) take the form

$$A_1\dot{\boldsymbol{\Omega}}_\varepsilon + (A_3 - A_1)\left(\mathbf{k}\cdot\dot{\boldsymbol{\Omega}}_\varepsilon\right)\mathbf{k} - (A_3 - A_1)(\mathbf{k}\cdot\boldsymbol{\Omega}_\varepsilon)\,\mathbf{k}\times\boldsymbol{\Omega}_\varepsilon + \\ + c\boldsymbol{\theta}_\varepsilon + (d-c)(\mathbf{k}\cdot\boldsymbol{\theta}_\varepsilon)\,\mathbf{Z}(\boldsymbol{\theta}_\varepsilon)\cdot\mathbf{k} = M\mathbf{k}, \quad (4.64)$$

$$\dot{\boldsymbol{\theta}}_\varepsilon = \boldsymbol{\Omega}_\varepsilon + \frac{1}{2}\boldsymbol{\theta}_\varepsilon\times\boldsymbol{\Omega}_\varepsilon + \frac{1-g_\varepsilon}{\theta_\varepsilon^2}\boldsymbol{\theta}_\varepsilon\times(\boldsymbol{\theta}_\varepsilon\times\boldsymbol{\Omega}_\varepsilon), g_\varepsilon = \frac{\theta_\varepsilon\sin\theta_\varepsilon}{2(1-\cos\theta_\varepsilon)}. \quad (4.65)$$

Substituting expressions (4.63) and taking into account the unperturbed solution yields

$$A_1\dot{\boldsymbol{\eta}} + (A_3 - A_1)(\mathbf{k}\cdot\dot{\boldsymbol{\eta}})\,\mathbf{k} + c\boldsymbol{\varphi} + (d-c)(\mathbf{k}\cdot\boldsymbol{\varphi})\,\mathbf{k} + \\ + M\left(1-\frac{c}{d}\right)\left(\frac{1}{2}\mathbf{k}\times\boldsymbol{\varphi} + \frac{1-g}{\theta_0}(\boldsymbol{\varphi} - (\mathbf{k}\cdot\boldsymbol{\varphi})\,\mathbf{k})\right) = 0, \quad (4.66)$$

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{\eta} + \frac{1}{2}\frac{M}{d}\mathbf{k}\times\boldsymbol{\eta} - (1-g)(\boldsymbol{\eta} - (\mathbf{k}\cdot\boldsymbol{\eta})\,\mathbf{k}). \quad (4.67)$$

These equations may be decoupled in torsion and bending perturbations by

$$\boldsymbol{\eta} = \zeta \mathbf{k} + \mathbf{y}, \quad \mathbf{y} \cdot \mathbf{k} = 0; \quad \boldsymbol{\varphi} = \gamma \mathbf{k} + \boldsymbol{\psi}, \quad \boldsymbol{\psi} \cdot \mathbf{k} = 0. \tag{4.68}$$

Substituting (4.68) into (4.66) – (4.67) and projecting obtained equations on $\mathbf{k}$ and the plane orthogonal $\mathbf{k}$ we have

$$A_3 \dot{\zeta} + d\gamma = 0, \quad A_1 \dot{\mathbf{y}} + [gc + (1-g)\, d]\, \boldsymbol{\psi} + \frac{M}{2}\left(1 - \frac{c}{d}\right) \mathbf{k} \times \boldsymbol{\psi} = \mathbf{0},$$
$$\dot{\gamma} = \zeta, \quad \boldsymbol{\psi} = g\mathbf{y} + \frac{M}{2d} \mathbf{k} \times \mathbf{y}.$$

Excluding from these equations ζ and $\mathbf{y}$ we obtain

$$A_3 \ddot{\gamma} + d\gamma = 0,$$
$$A_1 \ddot{\boldsymbol{\psi}} + \left(c\left(g^2 + \frac{M^2}{4d^2}\right) - \frac{M^2}{4d} + (1-g)\, gd \right) \boldsymbol{\psi} + \frac{M}{2} \mathbf{k} \times \boldsymbol{\psi} = \mathbf{0}, \tag{4.69}$$

where

$$g = \frac{\theta_0 \sin \theta_0}{2\,(1 - \cos \theta_0)}, \quad \theta_0 = \frac{M}{d}.$$

If the quantity $|M|/d$ is small, i.e. $|M|/d \ll 1$, then the second Eq. (4.69) can be rewritten as

$$A_1 \ddot{\boldsymbol{\psi}} + \left(c + \frac{(c - 2d)\, M^2}{12 d^2} \right) \boldsymbol{\psi} + \frac{M}{2} \mathbf{k} \times \boldsymbol{\psi} = 0. \tag{4.70}$$

Taking into account that $\boldsymbol{\psi}$ is orthogonal $\mathbf{k}$ let us look for a particular solution of this equation in the form

$$\boldsymbol{\psi} = \mathbf{a} \exp(pt), \quad \mathbf{a} = const, \quad \mathbf{a} \cdot \mathbf{k} = 0.$$

For the vector $\mathbf{a}$ we have the system

$$\left(\left(A_1 p^2 + c + \frac{(c - 2d)\, M^2}{12 d^2} \right) \mathbf{E}_* + \frac{M}{2} \mathbf{k} \times \mathbf{E}_* \right) \cdot \mathbf{a} = \mathbf{0}, \quad \mathbf{E}_* = \mathbf{E} - \mathbf{k} \otimes \mathbf{k}. \tag{4.71}$$

This equation has the form

$$\mathbf{A} \cdot \boldsymbol{\psi} = \mathbf{0}, \quad \mathbf{A} = \lambda \mathbf{E}_* + \mu \mathbf{k} \times \mathbf{E}_*, \quad \mathbf{E}_* = \mathbf{E} - \mathbf{k} \otimes \mathbf{k}, \quad \boldsymbol{\psi} \cdot \mathbf{k} = 0,$$

where

$$\lambda = A_1 p^2 + c + \frac{(c - 2d)\, M^2}{12 d^2}, \quad \mu = \frac{M}{2}.$$

We see that

$$\mathbf{k} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{k} = \mathbf{0}, \quad \det \mathbf{A} = 0.$$

In such a case, only the determinant of the plane part of $\mathbf{A}$ is important. It can be defined by

$$\mathrm{Det}\mathbf{A} = \frac{1}{2}\left((\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}\mathbf{A}^2\right) = \frac{1}{2}\left(4\lambda^2 - 2\lambda^2 + \mu^2\right) = \lambda^2 + \mu^2.$$

Thus, nontrivial solutions of Eq. (4.71) are obtained for

$$\left(A_1 p^2 + c + \frac{(c-2d)\,M^2}{12d^2}\right)^2 + \frac{M^2}{4} = 0. \tag{4.72}$$

It is easy to see that at least one root of this equation has a positive real part resulting in an increasing solution of Eq. (4.70) since a total solution of (4.70) has a form

$$\boldsymbol{\psi} = \sum_{k=1}^{4} \mathbf{a}_k \exp\left(p_k t\right), \quad \mathbf{a}_k = \mathrm{const}, \quad \mathbf{a}_k \cdot \mathbf{k} = 0,$$

where p_k are the roots of (4.72), $\mathbf{a}_k$ are the solutions of (4.71). Thus, the state of equilibrium (4.62) is unstable for an arbitrarily small external twisting moment M. This phenomenon is well known under the name of paradox of Nikolai.

From the pure theoretical point of view it is no wonder that the state of equilibrium is unstable. However, from the practical point of view the situation is very disagreeable. Really, if an external moment is small, then it is supposed that the linear theory is valid. In this case, Eqs. (4.58) – (4.59) can be rewritten as

$$A_1\ddot{\boldsymbol{\theta}} + (A_3 - A_1)\,(\mathbf{k}\cdot\ddot{\boldsymbol{\theta}})\mathbf{k} + c\boldsymbol{\theta} + (d-c)\,(\mathbf{k}\cdot\boldsymbol{\theta})\,\mathbf{k} = M\mathbf{k}. \tag{4.73}$$

Using a decomposition $\boldsymbol{\theta} = \gamma\mathbf{k} + \boldsymbol{\psi}$, $\boldsymbol{\theta}\cdot\mathbf{k} = 0$ this equation may be written as

$$A_3\ddot{\gamma} + d\gamma = M, \quad A_1\ddot{\boldsymbol{\psi}} + c\boldsymbol{\psi} = 0. \tag{4.74}$$

The solution of this system has a small norm if the moment M and the norm of initial conditions are small. Namely this is done in most of investigations and there was no doubt that such an approach is quite accurate. However, as shown above, if we take into account second order quantities, then the solution will be unstable. Is it really so ? It is a well-known fact [7] that the equations in terms of variations may give a faulty result in some cases. In such doubtful cases the nonlinear analysis has to be used.

4.5 Rigorous justification of the Nikolai paradox

Let us consider an external moment of the kind

$$\mathbf{M}_{ex} = \gamma M\left(l_1\mathbf{k} + l_2\mathbf{P}\cdot\mathbf{k}\right), \quad \gamma = \left(l_1^2 + l_2^2 + 2l_1 l_2 \mathbf{k}\cdot\mathbf{P}\cdot\mathbf{k}\right)^{-\frac{1}{2}}. \tag{4.75}$$

If $l_1 = 1, l_2 = 0$, then $\mathbf{M}_{ex}$ is a dead moment; if $l_1 = 0, l_2 = 1$, then $\mathbf{M}_{ex}$ is a followed (tangential) moment; if $l_1 = l_2 = 1$, then $\mathbf{M}_{ex}$ is a semi-tangential moment. For the elastic moment let us accept expression (4.47) where $C\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right)$ and $D\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right)$ are functions of a general kind. The tensor of inertia is supposed to be transversally isotropic with $\mathbf{k}$ as axis of symmetry.

Using expressions (4.43) and (4.44) for the kinetic moment, the equations may be derived either in form (4.53) or (4.55):

$$\left[A_1\boldsymbol{\omega} + (A_3 - A_1)\left(\boldsymbol{\omega}\cdot\mathbf{k}'\right)\mathbf{k}'\right]^{\cdot} + C\boldsymbol{\theta} + D\mathbf{Z}^{T}\cdot\mathbf{k} = \gamma M\left(l_1\mathbf{k} + l_2\mathbf{k}'\right), \quad (4.76)$$

$$\begin{aligned}\left[A_1\boldsymbol{\Omega} + (A_3 - A_1)(\boldsymbol{\Omega}\cdot\mathbf{k})\mathbf{k}\right]^{\cdot} + (A_3 - A_1)(\mathbf{k}\cdot\boldsymbol{\Omega})\boldsymbol{\Omega}\times\mathbf{k} +\\ + C\boldsymbol{\theta} + D\mathbf{Z}\cdot\mathbf{k} = \gamma M\left(l_1\mathbf{P}^{T}\cdot\mathbf{k} + l_2\mathbf{k}\right),\end{aligned} \quad (4.77)$$

where $\mathbf{k}' = \mathbf{P}\cdot\mathbf{k}$. Although Eqs. (4.76) – (4.77) are equivalent, a nontrivial result can be found from a comparison. Subtracting Eq. (4.77) from Eq. (4.76) and taking into account $\boldsymbol{\omega}\cdot\mathbf{k}' = \boldsymbol{\Omega}\cdot\mathbf{k}$ yields

$$\begin{aligned}&\left[A_1(\boldsymbol{\omega} - \boldsymbol{\Omega}) + (A_3 - A_1)(\boldsymbol{\Omega}\cdot\mathbf{k})\left(\mathbf{k}' - \mathbf{k}\right)\right]^{\cdot} +\\ &+ (A_3 - A_1)(\mathbf{k}\cdot\boldsymbol{\Omega})\mathbf{k}\times\boldsymbol{\Omega} + D\boldsymbol{\theta}\times\mathbf{k} = \gamma M\left[(l_1 - l_2)\mathbf{k} + l_2\mathbf{k}' - l_1\mathbf{P}^{T}\cdot\mathbf{k}\right].\end{aligned}$$

By multiplying this equation by the vector $\mathbf{k}$, we obtain

$$\left[A_1(\boldsymbol{\omega} - \boldsymbol{\Omega})\cdot\mathbf{k} + (A_1 - A_3)\mathbf{k}\cdot\boldsymbol{\Omega}(1 - \cos\vartheta)\right]^{\cdot} = \gamma M(l_1 - l_2)(1 - \cos\vartheta), \quad (4.78)$$

where $\cos\vartheta = \mathbf{k}\cdot\mathbf{k}' = \mathbf{k}\cdot\mathbf{P}\cdot\mathbf{k}$. Let us note that Eq. (4.78) does not contain the characteristics of the elastic foundation.

Equation (4.78) can be rewritten in another form. From Eqs. (4.19) – (4.20) follows

$$\boldsymbol{\omega} - \boldsymbol{\Omega} = \left(\mathbf{Z}^{-1} - \mathbf{Z}^{-T}\right)\cdot\dot{\boldsymbol{\theta}} = 2\frac{1 - \cos\theta}{\theta^2}\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}. \quad (4.79)$$

The turn-vector $\boldsymbol{\theta}$ can be represented as a composition

$$\boldsymbol{\theta} = x\mathbf{k} + \mathbf{y}, \quad \mathbf{y}\cdot\mathbf{k} = 0, \quad \theta^2 = x^2 + y^2, \mathbf{y} = y(t)\,\mathbf{Q}(\psi(t)\,\mathbf{k})\cdot\mathbf{m}, \quad (4.80)$$

where

$$\mathbf{m}\cdot\mathbf{k} = 0, \quad |\mathbf{m}| = 1.$$

One can prove the formulas

$$\mathbf{k}\cdot\left(\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}\right) = \mathbf{k}\cdot(\mathbf{y}\times\dot{\mathbf{y}}) = \dot{\psi}y^2, \quad 1 - \cos\vartheta = \frac{y^2(1 - \cos\theta)}{\theta^2}. \quad (4.81)$$

Taking into account relations (4.79), (4.80) and (4.81), Eq. (4.78) can be rewritten as

$$[(1-\cos\vartheta)\,F]^{\cdot} = \gamma M\,(l_1 - l_2)\,(1-\cos\vartheta)\,, \quad F \equiv 2A_1\dot{\psi} + (A_1 - A_3)\,\mathbf{k}\cdot\boldsymbol{\Omega}. \tag{4.82}$$

Equality (4.82) was derived in another way and was shown to the author in a private talk by Dr. A. Krivtsov. In fact, equality (4.82) is due to the existence of property (4.26) for the elastic moment. Let us note that the right side of Eq. (4.82) has a constant sign which is defined by the sign of $M\,(l_1 - l_2)$. Let us suppose that $M\,(l_1 - l_2) > 0$. In such a case, let us choose the initial conditions in such a way that $F\,|_{t=0} > 0$. Then equality (4.82) shows to us that the function $F\,(t)$ tends to infinity as $t \to \infty$ which is equivalent to an infinitely big velocity of precession $\dot{\psi}$, i.e. the state of equilibrium (4.62) is unstable for an arbitrarily small value of twisting moment and for any transversally isotropic elastic foundation. Therefore, the analysis on the base of the equations in terms of variations gives the right result. The Nikolai paradox is due to an accumulation of energy in the system.

4.6 The simplest rigid body oscillator

Let us consider the simplest case of the rigid body oscillator given by

$$\mathbf{A} = A\mathbf{E}, \quad U = u\left(\theta^2\right), \quad \frac{d}{d\boldsymbol{\theta}}U = 2\frac{du\left(\theta^2\right)}{d\left(\theta^2\right)}\boldsymbol{\theta} = c\left(\theta^2\right)\boldsymbol{\theta}. \tag{4.83}$$

Further, let us introduce a friction moment

$$\mathbf{M}_{ex} = -b\boldsymbol{\omega}, \quad b = \mathrm{const} \geqslant 0. \tag{4.84}$$

In such a case, basic Eqs. (4.55) – (4.56) yield

$$A\dot{\boldsymbol{\Omega}} + b\boldsymbol{\Omega} + c\left(\theta^2\right)\boldsymbol{\theta} = 0, \tag{4.85}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta}\times\boldsymbol{\Omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\Omega})\,, \quad g = \frac{\theta\sin\theta}{2\,(1-\cos\theta)}. \tag{4.86}$$

It is seen that even in this simplest case the equations of motion are rather complicated. For the plane oscillations this system can be simplified

$$\boldsymbol{\omega} = \boldsymbol{\Omega} = \dot{\boldsymbol{\theta}}, \quad \boldsymbol{\theta}\times\boldsymbol{\Omega} = 0$$

satisfying Eq. (4.86). Equation (4.85) then simplifies to

$$A\boldsymbol{\theta}^{\cdot\cdot} + b\dot{\boldsymbol{\theta}} + c\left(\theta^2\right)\boldsymbol{\theta} = 0; \quad \boldsymbol{\theta}\,(0) = \boldsymbol{\theta}_0, \quad \boldsymbol{\Omega}\,(0) = \boldsymbol{\Omega}_0, \quad \boldsymbol{\theta}_0\times\boldsymbol{\Omega}_0 = 0. \tag{4.87}$$

This system can be investigated without any problems. Let us discuss system of Eqs. (4.85) – (4.86) in a more general case. In order to show the difference between conventional approaches and our method, let us consider both of them.

4.6.1 Conventional approach

Let us try to investigate system (4.85), (4.86) on the basis of the Euler angles. The turn-tensor can be represented [10] in the form

$$\mathbf{P}(\boldsymbol{\theta}) = \mathbf{Q}(\psi\mathbf{k})\cdot\mathbf{Q}(\vartheta\mathbf{p})\cdot\mathbf{Q}(\varphi\mathbf{k}) = \mathbf{Q}(\vartheta\mathbf{p}')\cdot\mathbf{Q}(\psi\mathbf{k})\cdot\mathbf{Q}(\varphi\mathbf{k}) = \\ = \mathbf{Q}(\vartheta\mathbf{p}')\cdot\mathbf{Q}(\beta\mathbf{k}), \quad (4.88)$$

where

$$\beta = \varphi + \psi, \quad \mathbf{p}' = \mathbf{Q}(\psi\mathbf{k})\cdot\mathbf{p}, \quad \mathbf{k}\cdot\mathbf{p} = \mathbf{k}\cdot\mathbf{p}' = 0. \quad (4.89)$$

The left angular velocity is determined by [10]

$$\boldsymbol{\omega} = \left(\dot\psi + \dot\varphi\cos\vartheta\right)\mathbf{k} + \dot\vartheta\mathbf{p}' + \dot\varphi\sin\vartheta\,\mathbf{p}'\times\mathbf{k}. \quad (4.90)$$

Making use of expressions (4.7), (4.88), (4.90) and substituting them into Eq. (4.93) yields

$$A\left(\dot\psi + \dot\varphi\cos\vartheta\right)^{\cdot} + b\left(\dot\psi + \dot\varphi\cos\vartheta\right) + \frac{c\left(\theta^2\right)\theta}{2\sin\theta}\sin\beta\,(1+\cos\vartheta) = 0,$$
$$A\left(\ddot\vartheta + \dot\psi\dot\varphi\sin\vartheta\right) + b\dot\vartheta + \frac{c\left(\theta^2\right)\theta}{2\sin\theta}\sin\vartheta\,(1+\cos\beta) = 0,$$
$$A\left((\dot\varphi\sin\vartheta)^{\cdot} - \dot\psi\dot\vartheta\right) + b\dot\varphi\sin\vartheta + \frac{c\left(\theta^2\right)\theta}{2\sin\theta}\sin\beta\sin\vartheta = 0. \quad (4.91)$$

In addition to this system we obtain from (4.6) and (4.89)

$$1 + 2\cos\theta = \cos\vartheta + \cos\beta + \cos\vartheta\cos\beta, \quad \beta = \varphi + \psi. \quad (4.92)$$

It is not so easy to find the total solution of Eqs. (4.91). Let us note that representation (4.88) is completely admissible. However, there are many other possibilities and the most of them will lead to complicated equations. If we want to find the best representation, then we have to look for this representation during solution process rather than to guess it a priori [10].

4.6.2 The total integrability of the basic equations

Multiplying Eq. (4.85) by the tensor $\mathbf{P}(\boldsymbol{\theta})$ from the left, one can obtain

$$A\dot{\boldsymbol{\omega}} + b\boldsymbol{\omega} + c\left(\theta^2\right)\boldsymbol{\theta} = 0, \quad (4.93)$$

where the identity

$$\mathbf{P}\cdot\dot{\boldsymbol{\Omega}} = (\mathbf{P}\cdot\boldsymbol{\Omega})^{\cdot} - \dot{\mathbf{P}}\cdot\boldsymbol{\Omega} = \dot{\boldsymbol{\omega}} - (\mathbf{P}\times\boldsymbol{\Omega})\cdot\boldsymbol{\Omega} = \dot{\boldsymbol{\omega}}$$

was taken into account. Equation (4.93) is equivalent to Eq. (4.85). However, from (4.85) and (4.93) a nontrivial result may be obtained

$$A\left(\boldsymbol{\omega}-\boldsymbol{\Omega}\right)^{\cdot}+b\left(\boldsymbol{\omega}-\boldsymbol{\Omega}\right)=0\Longrightarrow\boldsymbol{\omega}-\boldsymbol{\Omega}=\left(\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0\right)\exp\left(-\frac{bt}{A}\right),\quad(4.94)$$

where $\boldsymbol{\omega}_0$ and $\boldsymbol{\Omega}_0$ are the initial angular velocities. Expression (4.94) gives to us three integrals. Now it is necessary to consider two cases:

$$a)\quad\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0=0,\quad b)\quad\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0=\left|\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0\right|\boldsymbol{e}\neq 0.$$

In the first case we deal with plane vibrations of the oscillator. Really, in the first case from (4.94) it follows that

$$\boldsymbol{\omega}=\boldsymbol{\Omega}\Longrightarrow\boldsymbol{\Omega}\times\boldsymbol{\theta}=0.$$

The latter fact follows from (4.15) and (4.16). This is identical to the case of Eq. (4.87).

It is more interesting to investigate case b). From Eqs. (4.17) – (4.18) we find

$$g\left(\theta\right)\left(\boldsymbol{\omega}-\boldsymbol{\Omega}\right)=\frac{1}{2}\boldsymbol{\theta}\times\left(\boldsymbol{\omega}+\boldsymbol{\Omega}\right).$$

Taking into account integral (4.94) one can get

$$g\left(\theta\right)\exp\left(-\frac{bt}{A}\right)\left(\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0\right)=\frac{1}{2}\boldsymbol{\theta}\times\left(\boldsymbol{\omega}+\boldsymbol{\Omega}\right).$$

Further, we may derive from (4.19) and (4.20) the identity

$$\frac{1}{2}\boldsymbol{\theta}\times\left(\boldsymbol{\omega}+\boldsymbol{\Omega}\right)=\frac{\sin\theta}{\theta}\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}$$

which change the above relation to

$$\frac{2\left(1-\cos\theta\right)}{\theta^2}\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}=\left(\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0\right)\exp\left(-\frac{bt}{A}\right).\qquad(4.95)$$

From this equation, one more integral follows

$$\boldsymbol{\theta}\left(t\right)\cdot\left(\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0\right)=0\Longrightarrow\boldsymbol{\theta}\left(t\right)\cdot\boldsymbol{e}=0,\qquad(4.96)$$

where $\boldsymbol{e}=\left(\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0\right)/\left|\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0\right|$. This equation shows that the turn-vector $\boldsymbol{\theta}\left(t\right)$ can be represented as

$$\boldsymbol{\theta}\left(t\right)=\theta\left(t\right)\mathbf{Q}\left(\psi\boldsymbol{e}\right)\cdot\boldsymbol{m},\quad\boldsymbol{m}=\boldsymbol{\theta}_0/\theta_0,\quad\boldsymbol{m}\cdot\boldsymbol{e}=0,\quad\psi\left(0\right)=0.\qquad(4.97)$$

From this representation we can conclude

$$\boldsymbol{\theta} \times \dot{\boldsymbol{\theta}} = \dot{\psi}\theta^2 \boldsymbol{e}. \tag{4.98}$$

Substituting this expression in (4.95) and taking into account the resulting equation for $t = 0$ yields

$$\dot{\psi} = \frac{1 - \cos\theta_0}{1 - \cos\theta(t)} \dot{\psi}_0 \exp\left(-\frac{bt}{A}\right), \quad \dot{\psi}_0 > 0. \tag{4.99}$$

Thus, if we know the angle of nutation $\theta(t)$ then the angle of precession can be found from (4.99). Let us derive an equation for the angle θ. For this end let us calculate the right angular velocity from (4.19) – (4.20) and (4.97)

$$\boldsymbol{\Omega} = \frac{\dot{\theta}}{\theta}\boldsymbol{\theta} + \frac{\sin\theta}{\theta}\dot{\psi}\boldsymbol{e} \times \boldsymbol{\theta} - (1 - \cos\theta)\dot{\psi}\boldsymbol{e}. \tag{4.100}$$

Substituting expression (4.100) into Eq. (4.85) and projecting the obtained equation onto the vectors $\boldsymbol{\theta}$, $\boldsymbol{e}$ and $\boldsymbol{e} \times \boldsymbol{\theta}$, respectively, we get three scalar equations where two of them (projections on $\boldsymbol{e}$ and $\boldsymbol{e} \times \boldsymbol{\theta}$) will be identically fulfilled due to (4.99). Projection onto the vector $\boldsymbol{\theta}$ gives

$$A\ddot{\theta} + b\dot{\theta} + \left[c\left(\theta^2\right) - A\frac{\sin\theta}{\theta}\left(\frac{1 - \cos\theta_0}{1 - \cos\theta}\right)^2 \left(\dot{\psi}_0\right)^2 \exp\left(-\frac{2bt}{A}\right)\right]\theta = 0. \tag{4.101}$$

If the friction is absent $(b = 0)$, then this equation can be solved in terms of quadratures. The plane motions of the oscillator can be found from Eq. (4.101) when $\dot{\psi}_0 = 0$. In a general case, Eq. (4.101) can be studied by conventional methods of nonlinear mechanics. Let us note that even for small θ Eq. (4.101) is nonlinear

$$A\ddot{\theta} + b\dot{\theta} + \left[c(0) - A\left(\frac{\theta_0}{\theta}\right)^4 \dot{\psi}_0^2 \exp\left(-\frac{2bt}{A}\right)\right]\theta = 0. \tag{4.102}$$

In contrast with it, the system of Eqs. (4.85) – (4.86) can be linearized for small turns, i.e. for $|\boldsymbol{\theta}| << 1$, and we obtain the linear equation

$$A\ddot{\boldsymbol{\theta}} + b\dot{\boldsymbol{\theta}} + c(0)\boldsymbol{\theta} = \boldsymbol{0}. \tag{4.103}$$

Let us show that the nonlinear Eq. (4.102) follows from Eq. (4.103) if one takes into account $\theta = |\boldsymbol{\theta}|$. If the turn-vector is represented as $\boldsymbol{\theta} = \theta\boldsymbol{n}$, $\theta = |\boldsymbol{\theta}|$, $|\boldsymbol{n}| = 1$, then we have

$$\dot{\boldsymbol{\theta}} = \dot{\theta}\boldsymbol{n} + \theta\dot{\boldsymbol{n}}, \quad \ddot{\boldsymbol{\theta}} = \ddot{\theta}\boldsymbol{n} + 2\dot{\theta}\dot{\boldsymbol{n}} + \theta\ddot{\boldsymbol{n}}, \quad \boldsymbol{n} \cdot \dot{\boldsymbol{n}} = 0, \quad \ddot{\boldsymbol{n}} \cdot \boldsymbol{n} = -\dot{\boldsymbol{n}} \cdot \dot{\boldsymbol{n}}. \tag{4.104}$$

Substituting (4.104) into (4.103) gives

$$\left(A\ddot{\theta}+b\dot{\theta}+c\theta\right)\mathbf{n}+\left(2A\dot{\theta}+b\theta\right)\dot{\mathbf{n}}+A\theta\ddot{\mathbf{n}}=0.$$

Multiplying this equation by $\mathbf{n}$ and $\dot{\mathbf{n}}$ we obtain

$$A\ddot{\theta}+b\dot{\theta}+(c-Ax)\,\theta=0,\quad \frac{1}{2}A\theta\dot{x}+\left(2A\dot{\theta}+b\theta\right)x=0,\quad x=\dot{\mathbf{n}}\cdot\dot{\mathbf{n}}. \tag{4.105}$$

From the second equation of this system it follows

$$\frac{\dot{x}}{x}+4\frac{\dot{\theta}}{\theta}+2\frac{b}{A}=0\Rightarrow x\theta^4=x_o\theta_0^4\exp\left(-\frac{2bt}{A}\right),\quad x\,(0)=x_0,\quad \theta\,(0)=\theta_0.$$

Substituting this expression for x into the first Eq. (4.103) we obtain (4.102). Therefore, the solution of linear Eq. (4.103) determines the exact solution of nonlinear Eq. (4.102). In many other cases we have the same situation.

If friction is absent $(b=0)$, then Eq. (4.101) has an exact solution

$$\theta=\theta_0=\mathrm{const},\quad \dot{\psi}=\dot{\psi}_0=\mathrm{const},\quad \left(\dot{\psi}\right)^2=\frac{c\left(\theta_0^2\right)\theta_0}{A\sin\theta_0}. \tag{4.106}$$

This solution is called regular precession which will be considered in the next section. If the friction is present, then for the large times Eq. (4.101) transforms to Eq. (4.87).

4.6.3 Comparison of two approaches

Let us compare the two approaches described. The first approach is defined by representation (4.88) of the turn-tensor where $\mathbf{k}$ and $\mathbf{p}$ are arbitrary orthogonal unit vectors. For the angles ψ,ϑ,φ and the velocities $\dot{\psi},\dot{\vartheta},\dot{\varphi}$ we may provide arbitrary initial conditions if and only if we are able to find a general solution of Eqs. (4.91). It is not known, if this is possible.

In the second approach, the representation of the turn-tensor has a special form

$$\mathbf{P}=\mathbf{Q}\left(\boldsymbol{\theta}\right)=\mathbf{Q}\left[\theta\mathbf{Q}\left(\psi\boldsymbol{e}\right)\cdot\mathbf{m}\right]=\mathbf{Q}\left(\psi\boldsymbol{e}\right)\cdot\mathbf{Q}\left(\theta\mathbf{m}\right)\cdot\mathbf{Q}^T\left(\psi\boldsymbol{e}\right) \tag{4.107}$$

due to Eq. (4.97) for the turn-vector. The unit vectors **e** and **m** are very special and found in the solution process. The two angles θ and ψ can describe only the special solution we were looking for, and which is contained in the first approach. In order to see this, let us accept the relation $\varphi=-\psi$, i.e. $\beta=0$, in representation (4.88) resulting in $\vartheta=\theta$. In such a case, system (4.91) – (4.92) reduces to

$$A\left(\dot{\psi}\left(1-\cos\theta\right)\right)^{\cdot}+b\dot{\psi}\left(1-\cos\theta\right)=0,$$

$$A\left(\ddot{\theta}-\dot{\psi}^2\sin\theta\right)+b\dot{\theta}+c\left(\theta^2\right)\theta=0,$$
$$A\left(\left(\dot{\psi}\sin\theta\right)^{\cdot}+\dot{\psi}\dot{\theta}\right)+b\dot{\psi}\sin\theta=0.$$

The first equation of this system gives to us integral (4.99). The third equation is an identity, if we take into account the first equation. At last, the second equation coincides with Eq. (4.101). Thus, the system (4.91) has a particular solution coinciding with the solution found above. However, when using representation (4.88) this solution does not allow to satisfy all initial conditions since the vectors $\mathbf{k}$ and $\mathbf{p}$ have preassigned directions and $\mathbf{k}\neq\mathbf{e}$.

4.7 The regular precession of the rigid body oscillator

Let us consider a body with a transversally isotropic tensor of inertia. The elastic foundation is supposed to be transversally isotropic as well. The equations of motion are given by expressions (4.53), (4.54) and expression (4.47) for the elastic moment

$$\left[A_1\boldsymbol{\omega}+(A_3-A_1)\left(\mathbf{k}'\cdot\boldsymbol{\omega}\right)\mathbf{k}'\right]^{\cdot}+C\boldsymbol{\theta}+D\mathbf{Z}^T\cdot\mathbf{k}=0,\quad \mathbf{k}'=\mathbf{P}\cdot\mathbf{k};\quad (4.108)$$

$$\dot{\boldsymbol{\theta}}=\boldsymbol{\omega}-\frac{1}{2}\boldsymbol{\theta}\times\boldsymbol{\omega}+\frac{1-g}{\theta^2}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\omega}),\qquad (4.109)$$

where the functions C and D are defined by (4.48). We assume a particular solution of system (4.108), (4.109) to be represented in the form

$$\boldsymbol{\theta}=\vartheta\mathbf{p}',\quad \mathbf{p}'=\mathbf{Q}(\psi\mathbf{k})\cdot\mathbf{p},\quad \mathbf{p}\cdot\mathbf{k}=0,\quad \mathbf{P}=\mathbf{Q}(\vartheta\mathbf{p}'),\qquad (4.110)$$

where the motion (4.110) is called a regular precession if

$$\vartheta=\mathrm{const},\quad \dot{\psi}=\mathrm{const}\quad\Rightarrow\quad \dot{\boldsymbol{\theta}}=\dot{\psi}\mathbf{k}\times\boldsymbol{\theta}.\qquad (4.111)$$

The left angular velocity then follows from (4.19) and is given as

$$\boldsymbol{\omega}=\mathbf{Q}(\psi\mathbf{k})\cdot\boldsymbol{\omega}_0,\quad \boldsymbol{\omega}_0=\dot{\psi}\left[(1-\cos\vartheta)\,\mathbf{k}+\sin\vartheta\,\mathbf{k}\times\mathbf{p}\right]=\mathrm{const}.\quad (4.112)$$

We see that the angular velocity vector $\boldsymbol{\omega}$ is a precession of the vector $\boldsymbol{\omega}_0$ around the axis $\mathbf{k}$ orthogonal to the turn-vector

$$\boldsymbol{\theta}\cdot\boldsymbol{\omega}=\boldsymbol{\theta}\cdot\boldsymbol{\Omega}=0,\quad \mathbf{k}\cdot\boldsymbol{\theta}=0.$$

In addition, let us accept the restriction

$$D\left(\theta^2,\mathbf{k}\cdot\boldsymbol{\theta}\right)|_{\mathbf{k}\cdot\boldsymbol{\theta}=0}=\frac{\partial}{\partial\left(\mathbf{k}\cdot\boldsymbol{\theta}\right)}U\left(\theta^2,\mathbf{k}\cdot\boldsymbol{\theta}\right)|_{\mathbf{k}\cdot\boldsymbol{\theta}=0}=0,$$

which is satisfied for most kinds of elastic energy. Then we obtain from Eq. (4.108) for the assumed solution

$$\dot{\psi}^2 = \frac{C\left(\vartheta^2, 0\right)\vartheta}{\sin\vartheta\left[A_3\left(1-\cos\vartheta\right)+A_1\cos\vartheta\right]}. \tag{4.113}$$

Especially, if $A_1 = A_3 = A$, we find solution (4.106).

Now we have to investigate the stability of solution (4.110) – (4.113). In general, it is a rather cumbersome process. In order to simplify it, let us accept

$$A = A_1 = A_3, \quad D\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right) = 0, \quad C\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right) = c = \text{const}. \tag{4.114}$$

This means that the tensor of inertia and the elastic foundation are supposed to be isotropic. Under these assumptions the perturbed equations of motion (4.108) – (4.109) take the form

$$A\dot{\boldsymbol{\omega}}_\varepsilon + c\boldsymbol{\theta}_\varepsilon = 0, \quad \dot{\boldsymbol{\theta}}_\varepsilon = \boldsymbol{\omega}_\varepsilon - \frac{1}{2}\boldsymbol{\theta}_\varepsilon\times\boldsymbol{\omega}_\varepsilon + \frac{1-g_\varepsilon}{\theta_\varepsilon^2}\boldsymbol{\theta}_\varepsilon\times\left(\boldsymbol{\theta}_\varepsilon\times\boldsymbol{\omega}_\varepsilon\right). \tag{4.115}$$

The perturbed quantities $\boldsymbol{\omega}_\varepsilon$ and $\boldsymbol{\theta}_\varepsilon$ could be represented by

$$\boldsymbol{\omega}_\varepsilon = \boldsymbol{\omega} + \varepsilon\boldsymbol{\eta}, \quad \boldsymbol{\theta}_\varepsilon = \boldsymbol{\theta} + \varepsilon\boldsymbol{\varphi}, \quad |\varepsilon| \ll 1, \tag{4.116}$$

where $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ are defined by (4.110) – (4.113). Such a choice, however, would yield equations for $\boldsymbol{\eta}$ and $\boldsymbol{\varphi}$ with varying coefficients. Therefore, it will be better to represent the functions $\boldsymbol{\omega}_\varepsilon$ and $\boldsymbol{\theta}_\varepsilon$ by

$$\boldsymbol{\omega}_\varepsilon = \mathbf{Q}\left(\psi\mathbf{k}\right)\cdot\left(\boldsymbol{\omega}_0 + \varepsilon\boldsymbol{\eta}\right), \quad \boldsymbol{\theta}_\varepsilon = \mathbf{Q}\left(\psi\mathbf{k}\right)\cdot\left(\vartheta\mathbf{p} + \varepsilon\boldsymbol{\varphi}\right), \tag{4.117}$$

where the function ψ is defined by (4.113). It is easy to calculate

$$\dot{\boldsymbol{\omega}}_\varepsilon = \mathbf{Q}\left(\psi\mathbf{k}\right)\cdot\left(\dot{\psi}\mathbf{k}\times\boldsymbol{\omega}_0 + \varepsilon\left(\dot{\boldsymbol{\eta}} + \dot{\psi}\mathbf{k}\times\boldsymbol{\eta}\right)\right),$$

$$\dot{\boldsymbol{\theta}}_\varepsilon = \mathbf{Q}\left(\psi\mathbf{k}\right)\cdot\left(\dot{\psi}\vartheta\mathbf{k}\times\mathbf{p} + \varepsilon\left(\dot{\boldsymbol{\varphi}} + \dot{\psi}\mathbf{k}\times\boldsymbol{\varphi}\right)\right).$$

Taking into account these expressions, notation (4.35) and Eqs. (4.115) we have

$$A\left(\dot{\boldsymbol{\omega}}\right)^* + c\boldsymbol{\theta}^* = 0, \left(\dot{\boldsymbol{\omega}}\right)^* = \mathbf{Q}\left(\psi\mathbf{k}\right)\cdot\left(\dot{\boldsymbol{\eta}} + \dot{\psi}\mathbf{k}\times\boldsymbol{\eta}\right), \boldsymbol{\theta}^* = \mathbf{Q}\left(\psi\mathbf{k}\right)\cdot\boldsymbol{\varphi},$$

$$\left(\dot{\boldsymbol{\theta}}\right)^* = \boldsymbol{\omega}^* - \frac{1}{2}\left(\boldsymbol{\theta}^*\times\boldsymbol{\omega} + \boldsymbol{\theta}\times\boldsymbol{\omega}^*\right) + \left(\frac{1-g}{\theta^2}\right)^*\left[\boldsymbol{\theta}\times\left(\boldsymbol{\theta}\times\boldsymbol{\omega}\right)\right] + \\ + \left(\frac{1-g}{\theta^2}\right)\left[\boldsymbol{\theta}\times\left(\boldsymbol{\theta}\times\boldsymbol{\omega}\right)\right]^*, \tag{4.118}$$

where the vectors θ and ω are defined by (4.110) and (4.112) respectively. Further, for any orthogonal tensor $\mathbf{Q}$ and any pair of vectors $\mathbf{a}$ and $\mathbf{b}$ there exists an identity

$$(\mathbf{Q}\cdot\mathbf{a})\times(\mathbf{Q}\cdot\mathbf{b}) = (\det\mathbf{Q})\,\mathbf{Q}\cdot(\mathbf{a}\times\mathbf{b}), \quad \det\mathbf{Q}=1.$$

Using this identity, we have from Eqs. (4.118) the following equations for the variations η and φ

$$A\left(\dot{\eta}+\dot{\psi}\mathbf{k}\times\eta\right)+c\varphi=0,$$

$$\dot{\varphi}+\dot{\psi}\mathbf{k}\times\varphi=\frac{\vartheta\sin\vartheta}{2(1-\cos\vartheta)}\eta-\frac{\vartheta-\sin\vartheta}{2(1-\cos\vartheta)}(\mathbf{p}\cdot\varphi)\,\omega_0-\frac{1}{2}\varphi\times\omega_0-$$
$$-\frac{1}{2}\vartheta\mathbf{p}\times\eta+\frac{2(1-\cos\vartheta)-\vartheta\sin\vartheta}{2\vartheta(1-\cos\vartheta)}(\varphi\cdot\omega_0+\vartheta\mathbf{p}\cdot\eta)\,\mathbf{p},$$

where $\dot{\psi}$ is determined by (4.113) and $\vartheta=const$. This system of linear differential equations with constant coefficients can be further investigated by conventional methods. Our aim was only to show the derivation of the equations in terms of variations.

4.8 Conclusion

We have presented a general model of a rigid body oscillator given in terms of Eqs. (4.53) – (4.54) or Eqs. (4.55) – (4.56), respectively. Using the notation of a turn-vector for describing rotations of rigid bodies we could introduce the integrating tensor $\mathbf{Z}(\theta)$ and a potential moment $\mathbf{M}(\theta)$. In order to study the properties of a rigid body oscillator, we may accept $\mathbf{M}_{ext}=0$. In such a case, Eqs. (4.55) – (4.56) take the form

$$\mathbf{A}\cdot\dot{\Omega}+\Omega\times\mathbf{A}\cdot\Omega+\mathbf{Z}(\theta)\cdot\frac{dU}{d\theta}=0, \tag{4.119}$$

$$\dot{\theta}=\Omega+\frac{1}{2}\theta\times\Omega+\frac{1-g}{\theta^2}\theta\times(\theta\times\Omega). \tag{4.120}$$

In this form, the rigid body oscillator has many applications to important technical problems. Especially, it will be useful for investigations of the microscale phenomena on the basis of principles of classical mechanics. For example, in [12] the well-known Klein-Gordon and Schrödinger equations could be derived from an application of this model to the media by Lord Kelvin.

A Appendix. Notation and terminology

In the paper, the direct tensor calculus is used. This approach is introduced by J. Gibbs and may be found in the books [3], [5], [6], and in many modern books on differential geometry and linear algebra. The direct tensor calculus is widely used in continuum mechanics — see [6], [9]. In order to develop multipolar continuum mechanics, it is necessary to use the methods of rigid body dynamics. This is a sufficient reason to describe the rigid body dynamics in terms of the direct tensor calculus. For readers more familiar with matrices notations, the following analogies hold

$$\mathbf{a}\cdot\mathbf{n} \rightleftarrows \quad \mathbf{a}^{\mathsf{T}}\mathbf{n}, \quad \mathbf{a}\times\mathbf{n} \rightleftarrows \underline{\tilde{a}}\mathbf{n}, \quad \mathbf{n}\otimes\mathbf{n} \rightleftarrows \mathbf{n}\mathbf{n}^{\mathsf{T}},$$

$$\mathbf{n}\times\mathbf{E} \rightleftarrows \tilde{\mathbf{n}}\mathbf{n} \triangleq \begin{Vmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{Vmatrix}.$$

These analogies are valid if we choose some orthonormal basis. However, the left sides are valid in any basis while the right sides depend on a choice of the basis.

Now we have to discuss the terminology. In the paper, we use the terms “turn-vector” and “turn-tensor” instead of the conventional terms “rotation vector” and “rotation tensor”, respectively. For this, an explanation may be given. A rotation is a process, a turn is an instant action. The turn-tensor is turning a body from the reference position to the actual position at each moment of time. The same can be said with respect to a turn-vector. In contrast with this, the vector $\boldsymbol{\omega}dt \neq d\boldsymbol{\theta}$ is rotating a body from the position at a moment of time t to the body position at the moment of time $t + dt$. Really, let $\mathbf{d}_k$ be a triplet of vectors fixed with respect to the body in the reference position and $\mathbf{D}_k(t)$ is the triplet $\mathbf{d}_k$ turned to the actual position. Then we have

$$\mathbf{D}_k(t) = \mathbf{P}(t)\cdot\mathbf{d}_k, \quad \mathbf{D}_k(t+dt) = \mathbf{P}(t+dt)\cdot\mathbf{d}_k, \quad \mathbf{d}_k = \mathbf{P}^{\mathsf{T}}(t)\cdot\mathbf{D}_k(t).$$

From this follows

$$\mathbf{D}_k(t+dt) = \mathbf{P}(t+dt)\cdot\mathbf{P}^{\mathsf{T}}(t)\cdot\mathbf{D}_k(t) = \left(\mathbf{P}(t) + \dot{\mathbf{P}}(t)\,dt\right)\cdot\mathbf{P}^{\mathsf{T}}(t)\cdot\mathbf{D}(t).$$

Making use of the Poisson Eq. (4.9) we obtain

$$\mathbf{D}_k(t+dt) = \mathbf{D}_k(t) + \boldsymbol{\omega}(t)\,dt\times\mathbf{D}_k(t).$$

Thus, namely the vector $\boldsymbol{\omega}dt$ rotates the triplet $\mathbf{D}_k(t)$ into the triplet
$\mathbf{D}_k(t+dt)$ rather than the vector $d\boldsymbol{\theta}$. The infinitely small vector $\delta\boldsymbol{\chi} = \boldsymbol{\omega}dt$ may be named the rotation vector. Of course, we are not sure that we select the best terms. However, it is important that we need two different terms for a rotation, since a body is rotating around one axis and at the same time the body is turned around an another axis. The axis of rotation is the line spanned by the vector of the left angular velocity while the axis of turn is the line spanned by the turn-vector. For a regular precession, these two axes are orthogonal — see section 7.

B Appendix. The representation for the integrating tensor

The representation for the integrating tensor may be obtained by many approaches and all of them are rather long. From our point of view, the shortest way is given below.

Calculating the trace from both sides of the Poisson Eq. (4.9), one can obtain

$$(\mathrm{tr}\,\mathbf{Q})^{\cdot} = \mathrm{tr}\dot{\mathbf{Q}} = \mathrm{tr}\,(\boldsymbol{\omega} \times \mathbf{Q}) = -2\frac{\sin\theta}{\theta}\boldsymbol{\theta} \cdot \boldsymbol{\omega}, \quad \mathrm{tr}\,(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}.$$

Taking into account the equality

$$\mathrm{tr}\,\mathbf{Q} = 1 + 2\cos\theta,$$

from the previous equation it is easy to derive

$$\theta\dot{\theta} = \boldsymbol{\theta} \cdot \dot{\boldsymbol{\theta}} = \boldsymbol{\theta} \cdot \boldsymbol{\omega}. \qquad (4.121)$$

Multiplying Eq. (4.9) by the vector $\boldsymbol{\theta}$, one can get

$$\dot{\mathbf{Q}} \cdot \boldsymbol{\theta} = \boldsymbol{\omega} \times \boldsymbol{\theta} = -\mathbf{R} \cdot \boldsymbol{\omega}.$$

Making use of the identity

$$\dot{\mathbf{Q}} \cdot \boldsymbol{\theta} = (\mathbf{Q} \cdot \boldsymbol{\theta})^{\cdot} - \mathbf{Q} \cdot \dot{\boldsymbol{\theta}} = -(\mathbf{Q} - \mathbf{E}) \cdot \dot{\boldsymbol{\theta}}$$

and Eq. (4.3), the previous equation can be rewritten as

$$\left(\frac{\sin\theta}{\theta}\mathbf{R} + \frac{1-\cos\theta}{\theta^2}\mathbf{R}^2\right) \cdot \dot{\boldsymbol{\theta}} = \mathbf{R} \cdot \boldsymbol{\omega}.$$

A general solution of this equation has the form

$$\boldsymbol{\omega} = \lambda\boldsymbol{\theta} + \left(\frac{\sin\theta}{\theta}\mathbf{E} + \frac{1-\cos\theta}{\theta^2}\mathbf{R}\right) \cdot \dot{\boldsymbol{\theta}}, \qquad (4.122)$$

where the scalar function λ must be found. Multiplying Eq. (4.122) by $\boldsymbol{\theta}$ and taking into account equality (4.121) yields

$$\lambda = \frac{\theta - \sin\theta}{\theta^3}\boldsymbol{\theta} \cdot \dot{\boldsymbol{\theta}}.$$

Then Eq. (4.122) takes a form

$$\boldsymbol{\omega} = \left(\mathbf{E} + \frac{1-\cos\theta}{\theta^2}\mathbf{R} + \frac{\theta - \sin\theta}{\theta^3}\mathbf{R}^2\right) \cdot \dot{\boldsymbol{\theta}} = \mathbf{Z}^{-1} \cdot \dot{\boldsymbol{\theta}}, \qquad (4.123)$$

where we use the identity

$$\mathbf{R}^2 = \boldsymbol{\theta} \otimes \boldsymbol{\theta} - \theta^2 \mathbf{E}.$$

Expression (4.123) gives to us representation (4.20). Thus we had found the tensor $\mathbf{Z}^{-1}$. Since the tensor $\mathbf{Z}^{-1}$ is an isotropic function of the tensor $\mathbf{R}$, we conclude that tensor $\mathbf{Z}$ is the isotropic tensor function of the tensor $\mathbf{R}$ as well

$$\mathbf{Z} = \alpha \mathbf{E} + \beta \mathbf{R} + \gamma \mathbf{R}^2, \quad \mathbf{Z} \cdot \mathbf{Z}^{-1} = \mathbf{E}.$$

From this it follows

$$\alpha = 1, \quad \beta = -\frac{1}{2}, \quad \gamma = \frac{1-g}{\theta^2}, \quad g = \frac{\theta \sin \theta}{2(1-\cos\theta)},$$

which is expression (4.11).

C Appendix. Elastic energy of foundation

In the section 4.3, there was given the definition of an elastic energy in terms of a potential function $U(\boldsymbol{\theta})$. This function was interpreted as elastic energy of the foundation. However, in the nonlinear theory of elasticity the concept of elastic energy has a uniquely determined meaning. Thus, it is necessary to show that there is no contradiction between these two concepts.

The foundation is supposed to be an elastic body. The boundary of the foundation is the surface $S = S_1 \cup S_2 \cup S_3$, where the part S_1 of the surface S is fixed, the part S_2 is a free surface, and the part S_3 is the contact surface between the foundation and the rigid body. Let us formulate the energy balance for the system "foundation plus rigid body"

$$\dot{K} + \dot{U} = 0. \tag{4.124}$$

The function K is only the kinetic energy of the rigid body, since the foundation is supposed to be without inertia. U is the total intrinsic energy, i.e. elastic energy or energy of deformation which is confined to the elastic foundation, since the intrinsic energy of the rigid body has a constant value. The right side of (4.124) is equal to zero because the power of external forces is absent.

Now let us write the equation of energy balance for the rigid body only. The external forces acting on the body are generating stresses acting on the part S_3 of the boundary. Thus, we can write

$$\dot{K} = -\int\limits_{P \in S_3} \mathbf{N}(P) \cdot \boldsymbol{\tau}(P) \cdot \dot{\mathbf{R}}(P)\, dS(P), \tag{4.125}$$

where $\mathbf{R}(P)$ is the position vector of a contact point P on surface S_3, the vector $\mathbf{N}$ is an external unit normal to the surface S_3, and the tensor $\boldsymbol{\tau}$ is the Cauchy stress tensor.

According to the kinematics of a rigid body, we have

$$\mathbf{R}(P) = \mathbf{R}(Q) + \mathbf{P}(t) \cdot (\mathbf{r}(P) - \mathbf{r}(Q)), \tag{4.126}$$

where Q is a reference point, $\mathbf{r}(P)$ and $\mathbf{r}(Q)$ are the position vectors of points P and Q in the reference position. From Eq. (4.126) follows

$$\boldsymbol{\nu}(P) = \boldsymbol{\nu}(Q) + \boldsymbol{\omega}(t) \times [\mathbf{R}(P) - \mathbf{R}(Q)]. \tag{4.127}$$

Substituting $\dot{\mathbf{R}}(P) = \boldsymbol{\nu}(P)$ in Eq. (4.127) by (4.125) we obtain

$$\dot{K} = \mathbf{F} \cdot \boldsymbol{\nu}(Q) + \mathbf{M}_e \cdot \boldsymbol{\omega}, \tag{4.128}$$

where

$$\mathbf{F} = -\int\limits_{P\in S_3} \mathbf{N}(P) \cdot \boldsymbol{\tau}(P)\, dS(P),$$

$$\mathbf{M}_e = -\int\limits_{P\in S_3} [\mathbf{R}(P) - \mathbf{R}(Q)] \times \boldsymbol{\tau}(P) \cdot \mathbf{N}(P)\, dS(P).$$

Making use of (4.124), Eq. (4.128) can be rewritten as

$$\mathbf{F} \cdot \boldsymbol{\nu}(Q) + \mathbf{M}_e \cdot \boldsymbol{\omega} = -\dot{U}(\mathbf{R}(Q), \boldsymbol{\theta}), \tag{4.129}$$

where the vector $\boldsymbol{\theta}$ is the turn-vector of the rigid body and henceforth of the surface S_3. If the point Q is fixed, we have definition (4.21). Thus, the potential U in expression (4.46) is an elastic energy of the foundation.

D Appendix. Transversally isotropic potential

Let there be given a scalar function $U(\boldsymbol{\theta})$ of a vector argument. This function is called transversally isotropic with the symmetry axis $\mathbf{k}$ if for any turn-tensor $\mathbf{Q}(\alpha\mathbf{k})$ the equality

$$U(\boldsymbol{\theta}) = U[\mathbf{Q}(\alpha\mathbf{k}) \cdot \boldsymbol{\theta}] \tag{4.130}$$

is valid. Let us consider a continuous set of tensors $\mathbf{Q}(\alpha(\tau)\mathbf{k})$. For any of them Eq. (4.130) must be valid. Note that the left side of (4.130) is independent of τ. Thus, we have

$$\frac{d}{d\tau} U(\mathbf{Q} \cdot \boldsymbol{\theta}) = 0 \quad \Rightarrow \quad \frac{dU}{d\boldsymbol{\theta}'} \cdot \frac{d}{d\tau}\boldsymbol{\theta}' = 0, \quad \boldsymbol{\theta}' = \mathbf{Q}(\alpha(\tau)\mathbf{k}) \cdot \boldsymbol{\theta}. \tag{4.131}$$

Making use of the Poisson Eq. (4.9) we obtain

$$\frac{d}{d\tau}\theta' = \frac{d\alpha}{d\tau}\mathbf{k}\times\theta'.$$

Substituting this expression into (4.131) and accepting

$$\alpha\left(0\right)=0,\quad \left[d\alpha\left(\tau\right)/d\tau\right]_{\tau=0}\neq 0$$

we have

$$\frac{dU}{d\boldsymbol{\theta}}\cdot\left(\mathbf{k}\times\boldsymbol{\theta}\right)=0 \quad\Rightarrow\quad \frac{dU}{d\boldsymbol{\theta}}=\varphi\mathbf{k}+\psi\boldsymbol{\theta}, \tag{4.132}$$

where φ and ψ are some scalar functions. Multiplying this equation by the vector $d\boldsymbol{\theta}$ yields

$$dU=\frac{dU}{d\boldsymbol{\theta}}\cdot d\boldsymbol{\theta}=\varphi d\left(\mathbf{k}\cdot\boldsymbol{\theta}\right)+\frac{1}{2}\psi d\left(\theta^2\right).$$

From this expression we see that a general form of a transversally isotropic potential may be represented as

$$U\left(\boldsymbol{\theta}\right)=F\left(\mathbf{k}\cdot\boldsymbol{\theta},\theta^2\right).$$

This is expression (4.24).

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