The Theory of Simple Elastic Shells

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Abstract. In the report the main aspects of the shell theory based on the direct approach are presented. The main attention will be focussed on the establishment of the constitutive equations. It is shown that the deformation energy must be an integral of a system of partial differential equations. Thus it can be expressed as a function of integrals of the characteristic system of the ordinary differential equations. These integrals are called the reduced deformation tensors. In order to found the structure of the elasticity tensors a new theory of symmetry is introduced.

1 Introduction

At present the shell theory find out the new branches of applications. As an example, biological membranes, thin polymeric films and thin structures made from shape memory materials may be pointed out. In addition, the manufacture technology for some shells results in significant changes of the material properties. As a result the conventional versions of the shell theory, based on the derivation of the basic equations from the 3D-theory of elasticity, can not be used. In these situations the effective elastic modulus of the shell must be found directly for the shell structure. That means that we have to use the direct method of the construction of the shell theory. The main idea of the direct approach is the introduction of an elastic 2D-continuum with some physical properties. The basic laws of mechanics and thermodynamics are applied directly to this 2D-continuum. The main advantage of the direct approach is the possibility to obtain quite strict equations.

At present many variants of the shell theory exist. Most of them can be characterized by two facts: a) they are based on two-dimensional equations and b) they operate with forces and moments (moments of the higher order are ignored). These two facts may be used for the following definition:

A simple shell is a 2D-continuum in which the interaction between different parts of the shell is due to forces and moments.

A simple shell is a model for the description of the mechanical behavior of shell-type structures. The theory of simple shells allows to make a correct plane photo of three-dimensional phenomena. The main advantage of the theory of simple shells is that this theory can be applied for shells with a complex inner structure - for multilayered, for stiffened, etc. In addition, such a theory can be used in the analysis of biological membranes, etc. In this sense the theory of simple shells allows the formalization of an old engineering problem - the built up of the shell theory with effective stiffness.

Below only some part of the report is presented. The results are based on works of authors (Zhilin, 1976; Zhilin, 1982; Altenbach, 1988; Altenbach and Zhilin, 1988) with some new modifications.

2 General Nonlinear Theory

2.1 Kinematics of Simple Shells

The kinematical model of a simple shell is based on the introduction of a *directed material surface*, i.e. the carrying surface each point of which is connected with a orthonormal triad of vectors. In what follows the direct tensor notation – see, for example, Lurie (1990) – will be used. In the reference configuration ($t = 0$) the directed surface is determined by

$$\{\mathbf{r}(q^1, q^2); \mathbf{d}_1(q^1, q^2), \mathbf{d}_2(q^1, q^2), \mathbf{d}_3(q^1, q^2)\} \equiv \{\mathbf{r}(q); \mathbf{d}_k(q)\}$$

with $\mathbf{r}(q) \equiv \mathbf{r}(q^1, q^2)$ - the position vector defining the geometry of the surface, $q^1, q^2 \in \Omega$, $\mathbf{d}_k(q)$ with $k = 1, 2, 3$ denote a triad of orthonormal vectors obeying the condition $\mathbf{d}_k \cdot \mathbf{d}_m = \delta_{km}$. In the actual configuration ($t \neq 0$) we have

$$\{\mathbf{R}(q, t); \mathbf{D}_k(q, t)\}, \qquad \mathbf{D}_k \cdot \mathbf{D}_m = \delta_{km}.$$

Note that $\mathbf{R}(q, 0) = \mathbf{r}(q), \mathbf{D}_k(q, 0) = \mathbf{d}_k(q)$. In addition, a natural basis can be introduced

$$\mathbf{R}_\alpha(q, t) = \frac{\partial \mathbf{R}}{\partial q^\alpha} \equiv \partial_\alpha \mathbf{R}, \quad \mathbf{R}_3 \equiv \mathbf{N} : \quad \mathbf{N} \cdot \mathbf{R}_\alpha = 0, \quad \mathbf{N} \cdot \mathbf{N} = 1.$$

In what follows we use the relation $\mathbf{d}_3 = \mathbf{n}$. For further derivations we will introduce the dual basis

$$\mathbf{R}^i : \quad \mathbf{R}^i \cdot \mathbf{R}_k = \delta^i_k, \quad \mathbf{R}^3 \equiv \mathbf{N}.$$

Then we can define the following two-dimensional HAMILTON operators

$$\widetilde{\boldsymbol{\nabla}} \equiv \mathbf{R}^\alpha(q, t) \frac{\partial}{\partial q^\alpha}, \quad \boldsymbol{\nabla} = \mathbf{r}^\alpha(q) \frac{\partial}{\partial q^\alpha}.$$

Let us introduce the first $\mathbf{A}$ and the second $\mathbf{B}$ fundamental tensors of the surface

$$\mathbf{A} = \widetilde{\boldsymbol{\nabla}} \mathbf{R} = \mathbf{R}^\alpha(q, t) \otimes \mathbf{R}_\alpha(q, t) = \mathbf{R}_\alpha \otimes \mathbf{R}^\alpha = A^{\alpha\beta} \mathbf{R}_\alpha \otimes \mathbf{R}_\beta = A_{\alpha\beta} \mathbf{R}^\alpha \otimes \mathbf{R}^\beta,$$

$$\mathbf{1} = \mathbf{A} + \mathbf{N} \otimes \mathbf{N}, \quad \mathbf{B} = -\widetilde{\boldsymbol{\nabla}} \mathbf{N} = -\mathbf{R}^\alpha \otimes \partial_\alpha \mathbf{N} = B^{\alpha\beta} \mathbf{R}_\alpha \otimes \mathbf{R}_\beta = B_{\alpha\beta} \mathbf{R}^\alpha \otimes \mathbf{R}^\beta,$$

where $\mathbf{1}$ is the unity second rank tensor. In what follows we accept $\mathbf{a} = \mathbf{A}(q, 0) = \nabla\mathbf{r}$, $\mathbf{b} = \mathbf{B}(q, 0) = -\nabla\mathbf{n}$ and $\mathbf{n} = \mathbf{N}(q, 0)$. Note that all tensors here and below are introduced as quantities of the 3D-space defined on the surface.

The motion of the directed surface can be defined as

$$\mathbf{R}(q, t), \quad \mathbf{P}(q, t) \equiv \mathbf{D}^k(q, t) \otimes \mathbf{d}_k(q), \quad \mathbf{D}^k = \delta^{km}\mathbf{D}_m.$$

Here $\mathbf{P}(q, t)$ is an orthogonal turn-tensor, $\operatorname{Det}\mathbf{P} = +1$, $\mathbf{P}(q, 0) = \mathbf{1}$. Let us introduce the linear and the angular velocities $\mathbf{v}$, $\boldsymbol{\omega}$ of the body-points

$$\mathbf{v}(q, t) = \dot{\mathbf{R}}(q, t), \quad \dot{\mathbf{P}}(q, t) = \boldsymbol{\omega}(q, t) \times \mathbf{P}(q, t), \quad \mathbf{P}(q, 0) = \mathbf{1}, \quad \dot{f} \equiv \frac{df}{dt}.$$

For the further discussion we need a vector $\boldsymbol{\Phi}_\alpha$ characterizing the change of $\mathbf{P}(q, t)$ along the surface

$$\partial_\alpha\mathbf{P} = \boldsymbol{\Phi}_\alpha(q, t) \times \mathbf{P}(q, t), \quad \Rightarrow \quad \boldsymbol{\Phi}_\alpha = -\frac{1}{2}[\partial_\alpha\mathbf{P} \cdot \mathbf{P}^T]_\times.$$

It is easy to show that

$$\dot{\boldsymbol{\Phi}}_\alpha = \partial_\alpha\boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{\Phi}_\alpha, \quad \partial_\alpha\boldsymbol{\Phi}_\beta - \partial_\beta\boldsymbol{\Phi}_\alpha - \boldsymbol{\Phi}_\alpha \times \boldsymbol{\Phi}_\beta = \mathbf{0}.$$

Note that $\boldsymbol{\Phi}_\alpha = \mathbf{0}$ if we have only a rigid body motion.

2.2 Equations of Motion

The local form of equations of motion can be written as

$$\widetilde{\nabla}\cdot\mathbf{T} + \rho\mathbf{F}_* = \rho(\mathbf{v} + \boldsymbol{\Theta}_1^T\cdot\boldsymbol{\omega})^{\cdot}, \quad \widetilde{\nabla}\cdot\mathbf{M} + \mathbf{T}_\times + \rho\mathbf{L} = \rho(\boldsymbol{\Theta}_1\cdot\mathbf{v} + \boldsymbol{\Theta}_2\cdot\boldsymbol{\omega})^{\cdot} + \rho\mathbf{v}\times\boldsymbol{\Theta}_1^T\cdot\boldsymbol{\omega},$$

where $\mathbf{T} = \mathbf{R}_\alpha \otimes \mathbf{T}^\alpha$ denotes the force tensor, $\mathbf{M} = \mathbf{R}_\alpha \otimes \mathbf{M}^\alpha$ – the moment tensor and $\mathbf{T}_\times \equiv \mathbf{R}_\alpha \times \mathbf{T}^\alpha$. The vectors $\mathbf{F}_*$ and $\mathbf{L}$ are the mass density of the external forces and moments respectively.

Let the vectors $\mathbf{T}_{(\nu)}$ and $\mathbf{M}_{(\nu)}$ be respectively the vectors of external force and moment acting on the boundary curve with the external normal $\boldsymbol{\nu}$. Then the formulae of Cauchy are valid

$$\mathbf{T}_{(\nu)} = \boldsymbol{\nu} \cdot \mathbf{T}, \quad \mathbf{M}_{(\nu)} = \boldsymbol{\nu} \cdot \mathbf{M}.$$

Introducing the Piola-Kirchhoff tensors

$$\mathbf{T}_\Pi = \sqrt{\frac{A}{a}}(\widetilde{\nabla}\mathbf{r})^T \cdot \mathbf{T}, \quad \mathbf{M}_\Pi = \sqrt{\frac{A}{a}}(\widetilde{\nabla}\mathbf{r})^T \cdot \mathbf{M},$$

the local form of the equations of motion can be rewritten in the reference configuration

$$\nabla \cdot \mathbf{T}_\Pi + \rho_0\mathbf{F} = \rho_0(\mathbf{v} + \boldsymbol{\Theta}_1^T \cdot \boldsymbol{\omega})^{\cdot},$$

$$\nabla \cdot \mathbf{M}_\Pi + (\nabla\mathbf{R}^T \cdot \mathbf{T}_\Pi)_\times + \rho_0\mathbf{L} = \rho_0(\boldsymbol{\Theta}_1 \cdot \mathbf{v} + \boldsymbol{\Theta}_2 \cdot \boldsymbol{\omega})^{\cdot} + \rho_0\mathbf{v} \times \boldsymbol{\Theta}_1^T \cdot \boldsymbol{\omega}.$$

Note that the last one equations are very convenient for formulating equations of shell stability problems.

2.3 Equation of the Balance of Energy

Let us formulate the balance of energy for the two-dimensional continuum

$$\frac{d}{dt} \int\limits_{(\Delta\Sigma)} \rho(\mathcal{K} + \mathcal{U}) d\Sigma = \int\limits_{(\Delta\Sigma)} \rho(\mathbf{F} \cdot \mathbf{v} + \mathbf{L} \cdot \boldsymbol{\omega}) d\Sigma + \int\limits_{C} (\mathbf{T}_{(\nu)} \cdot \mathbf{v} + \mathbf{M}_{(\nu)} \cdot \boldsymbol{\omega}) dC,$$

where $\mathcal{U}$ is the mass density of the intrinsic energy. For isothermal processes $\mathcal{U}$ is called the deformation energy. The local form can be expressed as

$$\rho\dot{\mathcal{U}} = \mathbf{T}^{\mathsf{T}} \cdot\cdot\, \tilde{\boldsymbol{\nabla}}\mathbf{v} - (\mathbf{R}_{\alpha} \times \mathbf{T}^{\alpha}) \cdot \boldsymbol{\omega} - \mathbf{M}^{\mathsf{T}} \cdot\cdot\, \tilde{\boldsymbol{\nabla}}\boldsymbol{\omega}.$$

Introducing the energetic tensors (Lurie, 1990)

$$\mathbf{T}_e = (\tilde{\boldsymbol{\nabla}}\mathbf{r})^{\mathsf{T}} \cdot \mathbf{T} \cdot \mathbf{P}, \quad \mathbf{M}_e = (\tilde{\boldsymbol{\nabla}}\mathbf{r})^{\mathsf{T}} \cdot \mathbf{M} \cdot \mathbf{P},$$

finally we get

$$\rho\dot{\mathcal{U}} = \mathbf{T}_e^{\mathsf{T}} \cdot\cdot\, \dot{\mathbf{E}} + \mathbf{M}_e^{\mathsf{T}} \cdot\cdot\, \dot{\mathbf{F}}, \tag{1}$$

where $\mathbf{E}, \mathbf{F}$ denote the first and the second deformation tensors

$$\mathbf{E} = \boldsymbol{\nabla}\mathbf{R} \cdot \mathbf{P} - \mathbf{a}, \quad \mathbf{F} = (\boldsymbol{\Phi}_{\alpha} \cdot \mathbf{D}_k)\mathbf{r}^{\alpha} \otimes \mathbf{d}^k. \tag{2}$$

For elastic simple shells from the energy balance equation (2) the Cauchy-Green relations follow

$$\mathbf{T}_e = \sqrt{\frac{A}{a}} \frac{\partial \rho_0 \mathcal{U}}{\partial \mathbf{E}}, \quad \mathbf{M}_e = \sqrt{\frac{A}{a}} \frac{\partial \rho_0 \mathcal{U}}{\partial \mathbf{F}}. \tag{3}$$

2.4 Definition of the Tensors of Inertia. Restrictions on the Tensors of Forces and Moments

The tensors of inertia $\rho\boldsymbol{\Theta}_{\alpha}$ define the distribution of the mass in the material body-points in the actual configuration. The following equation describes the relation between the tensors of inertia in the actual and the initial configurations

$$\rho\boldsymbol{\Theta}_{\alpha}(q, t) = \mathbf{P}(q, t) \cdot \rho_0 \boldsymbol{\Theta}_{\alpha}^0(q) \cdot \mathbf{P}^{\mathsf{T}}(q, t).$$

The next representations may be proved for the density, the first and the second tensor of inertia of the directed surface

$$\rho_0 = \langle \tilde{\rho}_0 \rangle, \quad \rho_0 \boldsymbol{\Theta}_1^0 = -\langle \tilde{\rho}_0 z \rangle \mathbf{c}, \quad \rho_0 \boldsymbol{\Theta}_2^0 = -\langle \tilde{\rho}_0 z^2 \rangle \mathbf{a}, \quad \langle \mathbf{f} \rangle = \int\limits_{-h_1}^{h_2} \mathbf{f} \mu d\Sigma,$$

with $\mathbf{c} = -\mathbf{a} \times \mathbf{n}$, $\mu = 1 - 2zH + z^2 G$, H and G are the mean and Gaussian curvatures, $\tilde{\rho}_0(q, z)$ denotes the 3D-density of mass.

It can be shown that the following relations are valid

$$\rho\boldsymbol{\Theta}_2 = \frac{\rho}{\rho_0} \langle \tilde{\rho}_0 z^2 \rangle (\mathbf{1} - \mathbf{D}_3 \mathbf{D}_3), \quad \rho\boldsymbol{\Theta}_1^{\mathsf{T}} = \frac{\rho}{\rho_0} \langle \tilde{\rho}_0 z^2 \rangle (\mathbf{1} \times \mathbf{D}_3). \tag{4}$$

Besides, for simple shells of constant thickness and made from non-polar material the restrictions

$$\mathbf{L} \cdot \mathbf{D}_3 = \mathbf{0}, \quad \mathbf{M} \cdot \mathbf{D}_3 = \mathbf{0}, \quad \mathbf{M}_e^{\mathsf{T}} \cdot\cdot\, [(\mathbf{F} - \mathbf{b} \cdot \mathbf{c}) \cdot \mathbf{c}] + \mathbf{T}_e^{\mathsf{T}} \cdot\cdot\, [(\mathbf{E} + \mathbf{a}) \cdot \mathbf{c}] = \mathbf{0} \tag{5}$$

are valid.

2.5 Reduced Deformation Tensors

The specific deformation energy $\mathcal{U} = \mathcal{U}(\mathbf{E}, \mathbf{F})$ contains 12 scalar arguments. The number of arguments can be reduced because we have to satisfy the restrictions (5). Making use of (3) and (5) one may obtain the next system of equations for the specific energy $\mathcal{U}(\mathbf{E}, \mathbf{F})$

$$\left(\frac{\partial \mathcal{U}}{\partial \mathbf{E}}\right)^{\mathsf{T}} \cdot\cdot \left[(\mathbf{E}+\mathbf{a})\cdot\mathbf{c}\right] + \left(\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right)^{\mathsf{T}} \cdot\cdot \left[(\mathbf{F}-\mathbf{b}\cdot\mathbf{c})\cdot\mathbf{c}\right] = 0, \quad \frac{\partial \rho_0 \mathcal{U}}{\partial(\mathbf{F}\cdot\mathbf{n})} = \mathbf{0}. \quad (6)$$

For the first equation of this system we have the characteristic system

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E} = (\mathbf{E}+\mathbf{a})\cdot\mathbf{c}, \quad \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{F} = (\mathbf{F}-\mathbf{b}\cdot\mathbf{c})\cdot\mathbf{c}, \quad (7)$$

This is a system of an order 12, which has only 11 independent integrals. One can choose the next 11 integrals

$$\boldsymbol{\mathcal{E}} = \frac{1}{2}\left[(\mathbf{E}+\mathbf{a})\cdot\mathbf{a}\cdot(\mathbf{E}+\mathbf{a})^{\mathsf{T}} - \mathbf{a}\right], \quad \boldsymbol{\Phi} = (\mathbf{F}-\mathbf{b}\cdot\mathbf{c})\cdot\mathbf{a}\cdot(\mathbf{E}+\mathbf{a})^{\mathsf{T}} + \mathbf{b}\cdot\mathbf{c}\cdot\boldsymbol{\mathcal{E}} + \mathbf{b}\cdot\mathbf{c},$$
$$\boldsymbol{\gamma} = \mathbf{E}\cdot\mathbf{n}, \quad \boldsymbol{\gamma}_* = \mathbf{F}\cdot\mathbf{n}. \quad (8)$$

Of course, it is possible to choose another set of integrals instead of integrals (8), but all of them may be expressed in terms of integrals given in (8). Arbitrary function $\mathcal{U}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \boldsymbol{\gamma}, \boldsymbol{\gamma}_*)$ of the integrals (8) satisfies the first equation of the system (6). However the second equation in (6) shows that the specific energy does not depend of the vector $\boldsymbol{\gamma}_*$. Thus we finally have $\mathcal{U} = \mathcal{U}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \boldsymbol{\gamma})$. Tensors $\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \boldsymbol{\gamma}$ are called the reduced deformation tensors. Here $\boldsymbol{\mathcal{E}}$ denote tensile and plane shear strains, $\boldsymbol{\Phi}$ - bending and torsional strains and $\boldsymbol{\gamma}$ - transverse shear.

Up to here all results are the exact ones. They are valid for shells made from arbitrary materials. Note that all physical properties of a shell are contained in the specific deformation energy. The above described theory of shell is called the Reissner-type theory.

2.6 Special cases

Let us briefly discuss the popular versions of the shell theory.

Love-type theory. This is the most popular case in the applied mechanics. In such a case the deformation of the transversal shear is supposed to be zero

$$\boldsymbol{\gamma} \equiv (\mathbf{E}+\mathbf{a})\cdot\mathbf{n} = \boldsymbol{\nabla}\mathbf{R}\cdot\mathbf{D}_3 = \mathbf{r}^{\alpha}(\mathbf{R}_{\alpha}\cdot\mathbf{D}_3) = \mathbf{0} \quad \Rightarrow \quad \mathbf{D}_3 = \mathbf{N},$$

where $\mathbf{N}$ is the unit normal to the deformed reference surface. Besides, the equalities (2) and $\mathbf{D}_3 = \mathbf{P}\cdot\mathbf{n}$ were used. The inertia tensors (4) and the deformation tensors must be replaced by

$$\rho\boldsymbol{\Theta}_2 = \frac{\rho}{\rho_0}\langle\tilde{\rho}_0 z^2\rangle\mathbf{A}, \quad \rho\boldsymbol{\Theta}_1^{\mathsf{T}} = -\frac{\rho}{\rho_0}\langle\tilde{\rho}_0 z^2\rangle\mathbf{C}, \quad \mathbf{C} = -\mathbf{A}\times\mathbf{N}$$

and

$$\mathcal{E} = \frac{1}{2}\left[(\mathbf{E}+\mathbf{a})\cdot\mathbf{a}\cdot(\mathbf{E}+\mathbf{a})^{\mathsf{T}} - \mathbf{a}\right], \quad \boldsymbol{\Phi} = -\boldsymbol{\nabla}\mathbf{R}\cdot\mathbf{B}\cdot\mathbf{C}\cdot\boldsymbol{\nabla}\mathbf{R}^{\mathsf{T}} + \mathbf{b}\cdot\mathbf{c}\cdot(\mathbf{1}+\mathcal{E}) \quad (9)$$

respectively.

Moment-free (membrane) shell theory. This case follows from the Love-type theory when the specific deformation energy depends on tensor $\boldsymbol{\mathcal{E}}$ only. We have

$$\mathbf{T}_e = \sqrt{\frac{A}{a}}\frac{\partial\rho_0\mathcal{U}}{\partial\mathcal{E}}, \quad \mathbf{M}_e = \mathbf{0}, \quad \boldsymbol{\Theta}_2 = \boldsymbol{\Theta}_1 = \mathbf{0} \quad \Rightarrow \quad \mathbf{T} = \mathbf{T}^{\mathsf{T}},\ \mathbf{T}\cdot\mathbf{N} = \mathbf{0}.$$

Soft shells. Soft shells are made from a material like textile. In addition to the previous case we have to accept

$$\mathbf{a}\cdot\frac{\partial\mathcal{U}}{\partial\mathcal{E}}\cdot\mathbf{a} \geqslant 0 \quad \forall\mathbf{a}:\ |\mathbf{a}| \neq 0,\ \mathbf{a}\cdot\mathbf{n} = 0.$$

Besides, some additional restrictions must be accepted. Soft shells are very important for applications and rather difficult for solving. As far as we know the general theory of soft shell is absent in the literature.

2.7 Deformation Energy of Solid Shell

For a shell made from the solid material the deformations are relatively small while the displacements and rotations can be relatively large. In such a case the following quadratic approximation can be introduced

$$\rho_0\mathcal{U} = \mathbf{T}_0 \cdot\cdot\, \mathbf{E} + \mathbf{M}_0^{\mathsf{T}} \cdot\cdot\, \boldsymbol{\Phi} + \mathbf{N}_0\cdot\boldsymbol{\gamma} + \frac{1}{2}\mathcal{E} \cdot\cdot\, {}^{(4)}\mathbf{C}_1 \cdot\cdot\, \mathcal{E} + \mathcal{E} \cdot\cdot\, {}^{(4)}\mathbf{C}_2 \cdot\cdot\, \boldsymbol{\Phi} + \frac{1}{2}\boldsymbol{\Phi} \cdot\cdot\, {}^{(4)}\mathbf{C}_3 \cdot\cdot\, \boldsymbol{\Phi} + \frac{1}{2}\boldsymbol{\gamma}\cdot\boldsymbol{\Gamma}\cdot\boldsymbol{\gamma} + \boldsymbol{\gamma}\cdot({}^{(3)}\boldsymbol{\Gamma}_1 \cdot\cdot\, \mathcal{E} + {}^{(3)}\boldsymbol{\Gamma}_2 \cdot\cdot\, \boldsymbol{\Phi}). \quad (10)$$

Here

$\mathbf{T}_0, \mathbf{M}_0, \mathbf{N}_0,\ {}^{(4)}\mathbf{C}_1, {}^{(4)}\mathbf{C}_2, {}^{(4)}\mathbf{C}_3,\ {}^{(3)}\boldsymbol{\Gamma}_1, {}^{(3)}\boldsymbol{\Gamma}_2,\ \boldsymbol{\Gamma}$ denote stiffness tensors of different rank. They express the effective elastic properties of the simple shell. The differences between various classes of simple shells are connected with different expressions of the stiffness tensors, the tensors of inertia and the two-dimensional density. Approximations like (10) are very popular in the nonlinear theory of elasticity. The stiffness tensors in (10) do not depend of the deformations. Thus they may be found from the experiments with the linear shell theory.

2.8 Linearized Basic Equations

In the linear theory there are no differences between the actual and the reference configurations. This is equivalent to the condition that the energetic and the true tensors of forces and moments are the same. The displacements and rotations are supposed to be small. In such a case instead of (8) we have

$$\mathcal{E} \simeq \boldsymbol{\epsilon} \equiv \frac{1}{2}(\mathbf{e}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{e}^{\mathsf{T}}), \quad \boldsymbol{\Phi} \simeq \mathbf{k} \equiv \boldsymbol{\kappa}\cdot\mathbf{a}+\frac{1}{2}(\mathbf{e}\cdot\cdot\mathbf{c})\mathbf{b}, \quad \boldsymbol{\gamma} = \mathbf{e}\cdot\mathbf{n} = \nabla\mathbf{u}\cdot\mathbf{n}+\mathbf{c}\cdot\boldsymbol{\varphi}, \tag{11}$$

with

$$\mathbf{e} = \nabla\mathbf{u} + \mathbf{a}\times\boldsymbol{\varphi}, \quad \boldsymbol{\kappa} = \nabla\boldsymbol{\varphi}. \tag{12}$$

The linearized equations of motion (2.2) take a form

$$\boldsymbol{\nabla}\cdot\mathbf{T} + \rho\mathbf{F}_* = \rho(\ddot{\mathbf{u}} + \boldsymbol{\Theta}_1^{\mathsf{T}}\cdot\ddot{\boldsymbol{\varphi}}), \quad \boldsymbol{\nabla}\cdot\mathbf{M} + \mathbf{T}_{\times} + \rho\mathbf{L} = \rho(\boldsymbol{\Theta}_1\cdot\ddot{\mathbf{u}} + \boldsymbol{\Theta}_2\cdot\ddot{\boldsymbol{\varphi}}). \tag{13}$$

The Cauchy–Green relations can be rewritten as

$$\mathbf{T}\cdot\mathbf{a} + \frac{1}{2}(\mathbf{M}\cdot\cdot\mathbf{b})\mathbf{c} = \frac{\partial\rho\mathcal{U}}{\partial\boldsymbol{\epsilon}}, \quad \mathbf{T}\cdot\mathbf{n} = \frac{\partial\rho\mathcal{U}}{\partial\boldsymbol{\gamma}}, \quad \mathbf{M} = \frac{\partial\rho\mathcal{U}}{\partial\mathbf{k}}.$$

Then from (10) we get

$$\mathbf{T}\cdot\mathbf{a} + \frac{1}{2}(\mathbf{M}\cdot\cdot\mathbf{b})\mathbf{c} = \mathbf{T}_0 + {}^{(4)}\mathbf{C}_1\cdot\cdot\boldsymbol{\epsilon} + {}^{(4)}\mathbf{C}_2\cdot\cdot\mathbf{k} + \boldsymbol{\gamma}\cdot{}^{(3)}\boldsymbol{\Gamma}_1, \tag{14}$$

$$\mathbf{T}\cdot\mathbf{n} = \mathbf{N}_0+\boldsymbol{\Gamma}\cdot\boldsymbol{\gamma}+{}^{(3)}\boldsymbol{\Gamma}_1\cdot\cdot\boldsymbol{\epsilon}+{}^{(3)}\boldsymbol{\Gamma}_2\cdot\cdot\mathbf{k}, \quad \mathbf{M}^{\mathsf{T}} = \mathbf{M}_0^{\mathsf{T}}+\boldsymbol{\epsilon}\cdot\cdot{}^{(4)}\mathbf{C}_2+{}^{(4)}\mathbf{C}_3\cdot\cdot\mathbf{k}+\boldsymbol{\gamma}\cdot{}^{(3)}\boldsymbol{\Gamma}_2.$$

3 Determination of the Elastic Stiffness Tensors

Now we have to realize the most complicated part of the direct approach to the shell theory and to construct the stiffness tensors. The solution of the problem was given by Zhilin (1982). It is clear that we have to use the properties of the symmetry. For this we have to solve two problems. First one: the classical theory of symmetry can not be used because it is valid for Euclidean tensors only. In the shell theory the Noneuclidean tensors are used. Second one: the stiffness tensors depend on the symmetry of the material of the shell, symmetry of the surface shape at the point under consideration and symmetry of the intrinsic structure of the shell. Below only the main idea of the approach will be given.

3.1 General Remarks and Restrictions

It is necessary to specified the quantities: $\mathbf{T}_0$, $\mathbf{M}_0$, $\mathbf{N}_0$, ${}^{(4)}\mathbf{C}_1$, ${}^{(4)}\mathbf{C}_2$, ${}^{(4)}\mathbf{C}_3$, $\boldsymbol{\Gamma}$, ${}^{3}\boldsymbol{\Gamma}_1, {}^{3}\boldsymbol{\Gamma}_2$. The following constraints are obvious: $\mathbf{d}\cdot\cdot{}^{(4)}\mathbf{C}_1 = {}^{(4)}\mathbf{C}_1\cdot\cdot\mathbf{d}$, $\mathbf{d}\cdot\cdot{}^{(4)}\mathbf{C}_3 = {}^{(4)}\mathbf{C}_3\cdot\cdot\mathbf{d}$, $\mathbf{c}\cdot\cdot\boldsymbol{\Gamma} = 0$, $\mathbf{c}\cdot\cdot{}^{(4)}\mathbf{C}_1 = \mathbf{0}$, $\mathbf{c}\cdot\cdot{}^{(4)}\mathbf{C}_2 = \mathbf{0}$, ${}^{(3)}\boldsymbol{\Gamma}\cdot\cdot\mathbf{c} = \mathbf{0}$, $\mathbf{T}_0\cdot\cdot\mathbf{c} = 0$, where $\mathbf{d}$ is an arbitrary tensor and $\mathbf{c}$ denotes an antisymmetric tensor (both of second rank). In the Euclidean space $\mathbb{R}^3$ we have only polar vectors. In the oriented Euclidean space $\mathbb{R}^3_0$ - polar and axial vectors. For the shell theory it is convenient to use the representation $\mathbb{R}^3_{0,n} = \mathbb{R}^2_0 \oplus \mathbb{R}^1_n$. Here we have three orientations, but only two of them are independent. We will use orientation of 3D-space and orientation on the line spanned on the normal $\mathbf{n}$. In the space $\mathbb{R}^3_{0,n}$ four types of tensors can be introduced: 1. polar tensors, which are independent from the orientation in $\mathbb{R}^3$ and in the subspaces, 2. axial tensors, which change sign if the orientation in $\mathbb{R}^3$ is changing, but not if the orientation changes in $\mathbb{R}^1$, 3. n-oriented tensors, which change sign if the orientation in $\mathbb{R}^1_n$ changes, but independent of the orientation in $\mathbb{R}^3$, and 4. axial n-oriented tensors, which change sign if the orientation in $\mathbb{R}^3$ is changing, and if the orientation changes in $\mathbb{R}^1$. In the shell theory the next objects are introduced. Polar tensors: $\rho, \mathcal{U}, \mathcal{W}, \mathbf{u}, \dot{\mathbf{u}}, \mathbf{E}, \mathbf{T}_0, \mathbf{T}, \mathbf{a}, {}^{(4)}\mathbf{C}_1, {}^{(4)}\mathbf{C}_3, \boldsymbol{\Gamma}, \rho\boldsymbol{\Theta}_2$. Axial tensors: $\rho\boldsymbol{\Theta}_1, \boldsymbol{\varphi}, \boldsymbol{\omega}, \mathbf{F}, \boldsymbol{\Phi}, \mathbf{b}\cdot\mathbf{c}, \mathbf{M}_0, {}^{(4)}\mathbf{C}_2$. The tensors $\mathbf{b}, \mathbf{B}, \boldsymbol{\gamma}, \mathbf{Q} = \mathbf{T}\cdot\mathbf{n}, {}^{(3)}\boldsymbol{\Gamma}_1, \mathbf{Q}_0$ are n-oriented objects. Axial n-oriented tensors: $\mathbf{c} = -\mathbf{a}\times\mathbf{n}, {}^{(3)}\boldsymbol{\Gamma}_2$.

Let $\mathbf{Q}$ be an orthogonal tensor. Let us introduce the orthogonal transformation of the tensor ${}^{(p)}\mathbf{S} = S^{i_1 i_2 \dots i_p}\, \mathbf{g}_{i_1} \otimes \mathbf{g}_{i_2} \otimes \dots \otimes \mathbf{g}_{i_p}$, where $\mathbf{g}_i$ is a basis in $\mathbb{R}^3$. We shall use the notation

$$\otimes_1^p \mathbf{Q} \cdot {}^{(p)}\mathbf{S} \equiv S^{i_1 i_2 \dots i_p} \mathbf{Q}\cdot\mathbf{g}_{i_1} \otimes \mathbf{Q}\cdot\mathbf{g}_{i_2} \otimes \dots \otimes \mathbf{Q}\cdot\mathbf{g}_{i_p},$$

$${}^{(p)}\mathbf{S}' \equiv (\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n})^{\beta} (\mathrm{Det}\mathbf{Q})^{\alpha} \otimes_1^p \mathbf{Q} \cdot {}^{(p)}\mathbf{S},$$

where $\alpha = \beta = 0$, if ${}^{(p)}\mathbf{S}$ is polar; $\alpha = 1, \beta = 0$, if ${}^{(p)}\mathbf{S}$ is axial; $\alpha = 0, \beta = 1$, if ${}^{(p)}\mathbf{S}$ is n-oriented; $\alpha = \beta = 1$, if ${}^{(p)}\mathbf{S}$ - is axial n-oriented. Note that $\mathbf{Q}\cdot\mathbf{n} = \pm\mathbf{n}$.

The group of symmetry (GS) for a tensor ${}^{(p)}\mathbf{S}$ is called a set of the orthogonal solutions of the equation

$${}^{(p)}\mathbf{S}' = {}^{(p)}\mathbf{S},$$

where ${}^{(p)}\mathbf{S}$ is given and $\mathbf{Q}$ must be found.

In what is followed we shall use the conventional relation

$$\mathbf{T} = \langle \mu^{-1} \cdot \boldsymbol{\sigma} \rangle, \quad \mathbf{M} = \langle \mu^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{c} z \rangle,$$

where $\boldsymbol{\sigma}$ is the the stress tensor of the classical theory of elasticity.

The relations

$$\rho_0(\dot{\mathbf{u}} + \boldsymbol{\Theta}_1^{\mathsf{T}} \cdot \dot{\boldsymbol{\varphi}}) = \langle \tilde{\rho}_0 \dot{\mathbf{u}}_* \rangle, \quad \rho_0(\boldsymbol{\Theta}_1 \cdot \dot{\mathbf{u}} + \boldsymbol{\Theta}_2^{\mathsf{T}} \cdot \dot{\boldsymbol{\varphi}}) = \langle \tilde{\rho}_0 \dot{\mathbf{u}}_* \cdot \mathbf{c} z \rangle$$

may be obtained in order to find the displacement and rotations in terms of the vector $\mathbf{u}_*$ of displacement of 3D-theory of elasticity.

The external force $\rho\mathbf{F}_*$ and moments $\rho_0\mathbf{L}$ may be found as

$$\rho_0\mathbf{F}_* = \langle \tilde{\rho}_0 \tilde{\mathbf{F}} \rangle + \mu^+ \boldsymbol{\sigma}_n^+ + \mu^- \boldsymbol{\sigma}_n^-, \quad \rho_0\mathbf{L} = \mathbf{n} \times \langle \tilde{\rho}_0 \tilde{\mathbf{F}} z \rangle + (h/2)\mathbf{n} \times (\mu^+ \boldsymbol{\sigma}_n^+ - \mu^- \boldsymbol{\sigma}_n^-),$$

where $\mu^{+(-)} = 1 - (+)hH + (h^2/4)G$, $\boldsymbol{\sigma}_n^{+(-)}$ are the stress vectors on the upper and lower face surfaces of the shell.

3.2 Local Symmetry Groups of Simple Shells

The local group of symmetry (LGS) is a set of the orthogonal solutions of the following system

$$\otimes_1^4 \mathbf{Q} \cdot \mathbf{C}_1 = \mathbf{C}_1, \quad (\mathrm{Det})\mathbf{Q} \otimes_1^4 \mathbf{Q} \cdot \mathbf{C}_2 = \mathbf{C}_2, \quad \otimes_1^4 \mathbf{Q} \cdot \mathbf{C}_3 = \mathbf{C}_3, \quad \mathbf{Q} \cdot \boldsymbol{\Gamma} \cdot \mathbf{Q}^{\mathsf{T}} = \boldsymbol{\Gamma},$$
$$(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n}) \otimes_1^3 \mathbf{Q} \cdot \boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_1, \quad (\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n})(\mathrm{Det}\mathbf{Q}) \otimes_1^3 \mathbf{Q} \cdot \boldsymbol{\Gamma}_2 = \boldsymbol{\Gamma}_2.$$

If we know the stiffness tensors, then we are able to find LGS of the shell. However it is much more important to solve the inverse task and to find the structure of the stiffness tensors, if we know some elements of the shell symmetry. To this end let us introduce Curie-Neumann's Principle:

GS *of the consequence contains* GS *of the reason.*

The GS of the reason is the intersection of the next groups of symmetry: 1. GS of the material at given point of the shell, 2. LGS of the surface and 3. LGS of the intrinsic structure of the shell. For the surface LGS is determined as a set of the orthogonal solutions of the system

$$\mathbf{Q}\cdot\mathbf{a}\cdot\mathbf{Q}^{\mathsf{T}}=\mathbf{a},\quad (\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n})\mathbf{Q}\cdot\mathbf{b}\cdot\mathbf{Q}^{\mathsf{T}}=\mathbf{b}. \tag{15}$$

It is easy to see that LGS of the surface in a general case contains only three irreducible elements: $\mathbf{1}$, $\mathbf{n}\otimes\mathbf{n}-\mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_2\otimes\mathbf{e}_2$, $\mathbf{n}\otimes\mathbf{n}+\mathbf{e}_1\otimes\mathbf{e}_1-\mathbf{e}_2\otimes\mathbf{e}_2$, where $\mathbf{e}_\alpha$ are eigenvectors of $\mathbf{b}$. For the plates ($\mathbf{b}=\mathbf{0}$) the LGS is much more rich. In a general case the GS of a shell can not be richer than (15). That means that we are not able to simplify the structure of the stiffness tensors without additional assumptions. But up to here we do not use the fact that the shell has small thickness.

3.3 Dimension Analysis

Let the material of a shell be isotropic. In such a case the stiffness tensors depend on the following items: $h(q)$ - thickness of the shell, $E(q)$, $\nu(q)$ - isotropic elastic properties, $\mathbf{a}$, $\mathbf{b}$ - first and second metric tensors. Making use of standard analysis of dimensions one may prove the next representations

$$\mathbf{C}_1=\frac{Eh}{12(1-\nu^2)}\,\mathbf{C}_1^*(h\mathbf{b}\cdot\mathbf{c},\nu),\quad \mathbf{C}_2=\frac{Eh^2}{12(1-\nu^2)}\,\mathbf{C}_2^*(h\mathbf{b}\cdot\mathbf{c},\nu),$$
$$\mathbf{C}_3=\frac{Eh^3}{12(1-\nu^2)}\mathbf{C}_3^*(h\mathbf{b}\cdot\mathbf{c},\nu),\quad \boldsymbol{\Gamma}=Gh\boldsymbol{\Gamma}^*(h\mathbf{b}\cdot\mathbf{c},\nu),\quad G=\frac{E}{2(1+\nu)}. \tag{16}$$

Here all quantities with a star depend on the dimensionless tensor which is small

$$\|h\mathbf{b}\cdot\mathbf{c}\|^2=(h\mathbf{b}\cdot\mathbf{c})\cdot\cdot(h\mathbf{b}\cdot\mathbf{c})^{\mathsf{T}}=h^2/R_1^2+h^2/R_2^2\ll 1.$$

Thus one can use the following representation

$$\mathbf{C}_s=\frac{Eh^s}{12(1-\nu^2)}\left[\mathbf{C}_s^{(0)}+\mathbf{C}_s^{(1)}\cdot\cdot(h\mathbf{b}\cdot\mathbf{c})+(h\mathbf{b}\cdot\mathbf{c})\cdot\cdot\mathbf{C}_s^{(2)}\cdot\cdot(h\mathbf{b}\cdot\mathbf{c})+0(h^3)\right],$$
$$\boldsymbol{\Gamma}=Gh\left[\boldsymbol{\Gamma}^{(0)}+\boldsymbol{\Gamma}^{(1)}\cdot\cdot(h\mathbf{b}\cdot\mathbf{c})+0(h^2)\right],\quad 0(h^p)\equiv 0(\|h\mathbf{b}\cdot\mathbf{c}\|^P) \tag{17}$$

with $s=1,2,3$. In what follows we ignore the terms of an order $0(h^2)$ with respect to 1. Only in some situations it is necessary to take into account higher order terms, e.g. for the positive definiteness of the deformation energy. Instead of the tensors $\mathbf{C}_i$ and $\boldsymbol{\Gamma}$ we have to consider the tensors $\mathbf{C}_i^{(p)}$ and $\boldsymbol{\Gamma}^{(p)}$. It seems that the representations for the stiffness tensors does not simplify our discussions. But the $\mathbf{C}_i^{(p)}$ did not depend on the geometrical shape of the surface. In this case the group of symmetry of the tensors $\mathbf{C}_i^{(p)}$ does not include the GS of the second metric tensor $\mathbf{b}$.

3.4 Homogeneous thin shells

Let us assume that the shell is made from transversally isotropic material. If $\mathbf{n}$ is the axis of isotropy, and the structure of the shell is of such kind that for $\mathbf{b} \to \mathbf{0}$ we have instead of the shell a plate with a reference (middle) surface which is a surface of symmetry. In this case the tensors $\mathbf{C}_i^{(k)}$, $\boldsymbol{\Gamma}^{(k)}$ (not the tensors $\mathbf{C}_i$, $\boldsymbol{\Gamma}$ contains the following elements of symmetry

$$\mathbf{Q} = \pm\, \mathbf{n} \otimes \mathbf{n} + \mathbf{q}, \quad \mathbf{q} \cdot \mathbf{q}^{\mathsf{T}} = \mathbf{a}, \quad \mathbf{q} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{q} = \mathbf{0}. \tag{18}$$

In such a case it is easy to show that

$$\mathbf{C}_1^{(1)} = \mathbf{0}, \quad \mathbf{C}_2^{(0)} = \mathbf{0}, \quad \mathbf{C}_2^{(2)} = \mathbf{0}, \quad \mathbf{C}_3^{(1)} = \mathbf{0}, \quad \boldsymbol{\Gamma}^{(1)} = \mathbf{0} \tag{19}$$

We see that tensors $\mathbf{C}_1$, $\mathbf{C}_3$ and $\boldsymbol{\Gamma}$ with an error of $0(h^2)$ can be found from the plate tests. Tensor $\mathbf{C}_2$ may be found only from the shell tests. Let the tensor (18) belongs to GS of tensors $\mathbf{C}_i^{(k)}$, $\boldsymbol{\Gamma}^{(k)}$. That means that these tensors must be transversally isotropic. It is not difficult to find such tensors and after that we get

$$\begin{aligned}
\mathbf{C}_{(1)} &= \frac{Eh}{1-\nu^2}\left[A_1 \mathbf{a} \otimes \mathbf{a} + A_2(\mathbf{r}^\alpha \otimes \mathbf{r}^\beta \otimes \mathbf{r}_\alpha \otimes \mathbf{r}_\beta + \mathbf{r}^\alpha \otimes \mathbf{a} \otimes \mathbf{r}_\alpha - \mathbf{a} \otimes \mathbf{a})\right], \\
\mathbf{C}_{(2)} &= \frac{Eh^2}{12(1-\nu^2)}\left[B_1 hH\, \mathbf{a} \otimes \mathbf{c} + B_2 hH(\mathbf{r}^\alpha \otimes \mathbf{c} \otimes \mathbf{r}_\alpha + c^{\alpha\beta}\mathbf{r}_\alpha \otimes \mathbf{a} \otimes \mathbf{r}_\beta) + \right.\\
&\qquad \left. + B_3 \mathbf{a} \otimes (h\mathbf{b} \cdot \mathbf{c} - hH\mathbf{c}) + B_4 h(\mathbf{b} \cdot \mathbf{c} - H\mathbf{c}) \otimes \mathbf{a} + B_5 h(\mathbf{b} - H\mathbf{a}) \otimes \mathbf{c}\right], \\
\mathbf{C}_{(3)} &= \frac{Eh^3}{12(1-\nu^2)}\left[C_1 \mathbf{c} \otimes \mathbf{c} + C_2(\mathbf{r}^\alpha \otimes \mathbf{r}^\beta \otimes \mathbf{r}_\alpha \otimes \mathbf{r}_\beta + \mathbf{r}^\alpha \otimes \mathbf{a} \otimes \mathbf{r}_\alpha - \mathbf{a} \otimes \mathbf{a}) + \right.\\
&\qquad \left. + h^2 H_1^2 C_4 \mathbf{a} \otimes \mathbf{a}\right], \qquad \boldsymbol{\Gamma} = Gh\Gamma_0 \mathbf{a}
\end{aligned} \tag{20}$$

with $2H_1 = -(1/R_1) + (1/R_2)$. All results are valid for non-polar material and have an error $0(h^2)$. The modulus $A_1, A_2, C_1, C_2, C_4, \Gamma_0, B_1, \ldots, B_5$ depends only on the POISSON's ratio.

Making use of the solutions of some test problems one can obtain the following elastic modulus

$$A_1 = C_1 = \frac{1+\nu}{2}, \quad A_2 = C_2 = \frac{1-\nu}{2}, \quad \Gamma_0 = \frac{\pi^2}{12}, \quad C_4 = \frac{1-\nu}{24}, \tag{21}$$

$$B_1 = \frac{\nu(1+\nu)}{2(1-\nu)}, \quad B_2 = 0, \quad B_3 = \frac{1+\nu}{2}, \quad B_4 = -\frac{1-\nu}{4}, \quad B_5 = -\frac{1}{2}. \tag{22}$$

All modulus in (21) and (22), excluding C_4, were found from the tasks in which they determine the main terms of asymptotic expansions. Modulus C_4 is needed for the deformation energy to be positively defined. Some comments must be made with respect to the coefficient of transverse shear Γ_0. It may be shown that the inequality $\pi^2/12 \leqslant \Gamma_0 < 1$ must be valid always. If we are not interested in the high frequencies, then it would be better to accept $\Gamma_0 = 5/(6-\nu)$. In such a case the low frequencies can be found more exactly.

The tensors $\mathbf{T}_0$, $\mathbf{M}_0$, $\mathbf{Q}_0$ are defined by the expressions

$$\mathbf{T}_0 = \frac{\nu h}{2(1-\nu)}\mathbf{a}\left[\mathbf{n}\cdot(\boldsymbol{\sigma}_n^+ - \boldsymbol{\sigma}_n^-)\right], \quad \mathbf{M}_0 = \frac{\nu h^2}{12(1-\nu)}\mathbf{c}\left[\mathbf{n}\cdot(\boldsymbol{\sigma}_n^+ - \boldsymbol{\sigma}_n^-)\right],$$
$$\mathbf{Q}_0 = h(1-\Gamma_0)\mathbf{a}\left[\mathbf{n}\cdot(\boldsymbol{\sigma}_n^+ - \boldsymbol{\sigma}_n^-)\right]. \quad (23)$$

Note that the representations (20) are valid for many cases of the nonhomogeneous shell.

3.5 Simplest shell theory

In a general case the theory given above can not be simplified. If someone make it, then there exists the task in which the mistake will be present in the main term. However there are many practical problems when it is possible to use much more simple theory. The most popular theory of such a kind was given by L. Balabuh (1946) and V. Novozhilov (1946). Practically the same theory was given by W. Koiter (1959) and J. Sanders (1959). It was shown by K. Chernyh (1964) that the Koiter-Sanders theory differs from the Balabuch-Novozhilov theory in small terms. Let us note that there exist problems for arbitrary thin shell when the Balabuch-Novozhilov-Koiter-Sanders theory gives the mistakes in the main terms. Thus in general this theory can not be named the first-approximation theory. Shell theory may be called simplest if it is described by means of minimal number of the elastic modulus. However the deformation energy in the simplest theory must be positively defined for any type of deformations. In such a case we have to ignore the tensors $\mathbf{T}_0$, $\mathbf{M}_0$ and $\mathbf{Q}_0$. Besides, we have to accept $\Gamma_0 \to \infty \Rightarrow \boldsymbol{\gamma} = \mathbf{0}$ and $B_1 = B_2 = B_3 = B_4 = B_5 = C_4 = 0$. From the restriction that the deformation energy must be positive, we obtain $A_1 > 0, A_2 > 0$, $C_1 > 0$, $C_2 > 0$, $\Gamma_0 > 0$. The deformation energy takes a form

$$\rho_0 \mathcal{U} = \frac{Gh}{2}\left[2\boldsymbol{\mathcal{E}}\cdot\cdot\boldsymbol{\mathcal{E}} + \frac{2\nu}{1-\nu}(\mathrm{tr}\boldsymbol{\mathcal{E}})^2\right] +$$
$$+ \frac{Gh^3}{24}\left[\frac{1}{2}(\boldsymbol{\Phi}+\boldsymbol{\Phi}^T)\cdot\cdot(\boldsymbol{\Phi}+\boldsymbol{\Phi}^T) - (\mathrm{tr}\,\boldsymbol{\Phi})^2 + \frac{1+\nu}{1-\nu}(\mathbf{c}\cdot\cdot\boldsymbol{\Phi})^2\right], \quad (24)$$

where tensors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$ are defined by expressions (9). The deformation energy (24) may be used for small deformation and large rotations. For small rotations one can use linear shell theory. The vector of small rotation $\boldsymbol{\varphi}$ may be found in terms of the displacement vector

$$\Gamma_0 \to \infty, \quad |\mathbf{T}\cdot\mathbf{n}| < \infty \quad \Rightarrow \quad \boldsymbol{\gamma} = \mathbf{0}, \quad \Rightarrow \quad \mathbf{a}\cdot\boldsymbol{\varphi} = \mathbf{c}\cdot(\boldsymbol{\nabla} w + \mathbf{b}\cdot\mathbf{u}),$$

where $w = \mathbf{u} \cdot \mathbf{n}$. The Cauchy–Green relations (14) takes a form

$$\mathbf{T} \cdot \mathbf{a} + \frac{1}{2}(\mathbf{M} \cdot\cdot\, \mathbf{b})\mathbf{c} = \frac{Eh}{1-\nu^2}\left[(1-\nu)\boldsymbol{\epsilon} + \nu(\operatorname{tr}\boldsymbol{\epsilon})\mathbf{a}\right],$$
$$\mathbf{M} = \frac{Gh^3}{12}\left[\mathbf{k} + \mathbf{k}^T - (\operatorname{tr}\mathbf{k})\mathbf{a} - \frac{1+\nu}{1-\nu}(\mathbf{c} \cdot\cdot\, \mathbf{k})\mathbf{c}\right].$$

The vector $\mathbf{T} \cdot \mathbf{n}$ of the transverse force is defined by the equations of motion.

4 Conclusion

It was above shown that the direct approach to the shell theory is based on some new ideas of physical nature. As a result, this approach allows to build up the shell theory which can not be improved without introduction of new dynamics quantities like moments of higher orders. It is important that the theory does not need any hypothesis and may be applied for all possible cases. Of course, the elastic modulus must be found for these cases by means of special considerations. For example, the elastic modulus of the three-layers shells may be found from some transcendental equations which do not contain any small parameters and therefore can not be solved by means of asymptotic methods. The elastic modulus may be found for case when the friction between layers is present. In such a case the constitutive equations must be slightly modified.

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