

Short Communication

A note on transversely-isotropic invariants

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The problem of a functional basis for a system of vectors and second rank tensors with respect to the symmetry transformation $Q(\varphi m)$ is discussed. The invariants are found as integrals of a generic partial differential equation. This approach is illustrated for the case of a single symmetric tensor. Then a theorem on the number of functionally independent invariants for a system of vectors and second rank tensors is formulated.

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1 Introduction

An application of the theory of tensor functions is to find a basic set of scalar invariants for a given group of symmetry transformations, such that each invariant relative to the same group is expressible as a single-valued function of the basic set. The basic set of invariants is called functional basis. To obtain compact representation for invariants, it is required that the functional basis is irreducible in the sense that removing any one invariant from the basis will imply that a complete representation for all the invariants is no longer possible.

Such a problem arises in the formulation of constitutive equations for a given group of material symmetries. For example, the strain energy density of an elastic non-polar material is a scalar valued function of the second rank symmetric strain tensor. In the theory of the Cosserat continuum two strain measures are introduced, where the first strain measure is the polar tensor while the second one is the axial tensor, e.g. [1]. The strain energy density of a thin elastic shell is a function of two second rank tensors and one vector, e.g. [2]. In all cases the problem is to find a minimum set of functionally independent invariants for the considered tensorial arguments. Transverse isotropy is an important type of the symmetry transformation due to variety of applications. Transverse isotropy is usually assumed in constitutive modelling of fiber reinforced materials, e.g. [3], fiber suspensions, e.g. [4], directionally solidified alloys, e.g. [5], deep drawing sheets, e.g. [6, 7], and piezoelectric materials, e.g. [8].

For the theory of tensor functions we refer to [9]. Representations of tensor functions are reviewed in [10, 11]. An orthogonal transformation of a scalar α , a vector \mathbf{a} and a second rank tensor \mathbf{A} is defined by [2, 12]

$$\alpha' \equiv (\det \mathbf{Q})^\zeta \alpha, \quad \mathbf{a}' \equiv (\det \mathbf{Q})^\zeta \mathbf{Q} \cdot \mathbf{a}, \quad \mathbf{A}' \equiv (\det \mathbf{Q})^\zeta \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^T, \quad (1)$$

where \mathbf{Q} is an orthogonal tensor, i.e. $\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{I}$, $\det \mathbf{Q} = \pm 1$, \mathbf{I} is the second rank unit tensor, $\zeta = 0$ for absolute (polar) scalars, vectors, and tensors and $\zeta = 1$ for axial ones. A definition of polar and axial (pseudo-Euclidean) tensors of different rank may be found in [13]. An example of the axial scalar is the mixed product of three polar vectors, i.e. $\alpha = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. A typical example of the axial vector is the cross product of two polar vectors, i.e. $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. An example of the second rank axial tensor is the skew-symmetric tensor $\mathbf{W} = \mathbf{a} \times \mathbf{I}$, where \mathbf{a} is a polar vector. Consider a group of orthogonal transformations S (e.g., the material symmetry transformations) characterized by a set of orthogonal tensors \mathbf{Q} . A scalar-valued function of a second rank tensor $f = f(\mathbf{A})$ is called to be an orthogonal invariant under the group S if

$$\forall \mathbf{Q} \in S: \quad f(\mathbf{A}') = (\det \mathbf{Q})^\eta f(\mathbf{A}), \quad (2)$$

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where $\eta = 0$ if values of f are absolute scalars and $\eta = 1$ if values of f are axial scalars.

Any second rank tensor \mathbf{B} can be decomposed into the symmetric and the skew-symmetric part, i.e. $\mathbf{B} = \mathbf{A} + \mathbf{a} \times \mathbf{I}$, where \mathbf{A} is the symmetric tensor and \mathbf{a} is the associated vector. Therefore $f(\mathbf{B}) = f(\mathbf{A}, \mathbf{a})$. If \mathbf{B} is a polar (axial) tensor, then \mathbf{a} is an axial (polar) vector. For the set of second rank tensors and vectors the definition of an orthogonal invariant (2) can be generalized as follows

$$\forall \mathbf{Q} \in S : f(\mathbf{A}'_1, \mathbf{A}'_2, \dots, \mathbf{A}'_n, \mathbf{a}'_1, \mathbf{a}'_2, \dots, \mathbf{a}'_k) = (\det \mathbf{Q})^\eta f(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k), \quad \mathbf{A}_i = \mathbf{A}'_i{}^T. \quad (3)$$

The invariants and generating sets for tensor-valued functions with respect to different cases of transverse isotropy are discussed in [14, 15] (see also relevant references therein). The aim of this paper is to analyze the problem of functional basis within the theory of linear first order partial differential equations rather than the algebra of polynomials. We develop the idea proposed in [16] for the invariants with respect to the full orthogonal group to the case of transverse isotropy. The invariants will be found as integrals of the generic partial differential equations. Although a functional basis formed by these invariants does not include any redundant element, functional relations between them may exist. It may be therefore useful to find out simple forms of such relations. We show that the proposed approach may supply results in a direct, natural manner.

2 Invariants for a second rank symmetric tensor

Consider the proper orthogonal tensor which represents a rotation about a fixed axis, i.e.

$$\mathbf{Q}(\varphi \mathbf{m}) = \mathbf{m} \otimes \mathbf{m} + \cos \varphi (\mathbf{I} - \mathbf{m} \otimes \mathbf{m}) + \sin \varphi \mathbf{m} \times \mathbf{I}, \quad \det \mathbf{Q}(\varphi \mathbf{m}) = 1, \quad (4)$$

where \mathbf{m} is assumed to be a constant unit vector (axis of rotation) and φ denotes the angle of rotation about \mathbf{m} . The symmetry transformation defined by this tensor corresponds to the transverse isotropy, whereby five different cases are possible, e.g. [17, 18]. Let us find scalar-valued functions of a second rank symmetric tensor \mathbf{A} satisfying the condition

$$f(\mathbf{A}'(\varphi)) = f(\mathbf{Q}(\varphi \mathbf{m}) \cdot \mathbf{A} \cdot \mathbf{Q}^T(\varphi \mathbf{m})) = f(\mathbf{A}), \quad \mathbf{A}'(\varphi) \equiv \mathbf{Q}(\varphi \mathbf{m}) \cdot \mathbf{A} \cdot \mathbf{Q}^T(\varphi \mathbf{m}). \quad (5)$$

Eq. (5) must be valid for any angle of rotation φ . In (5) only the left-hand side depends on φ . Therefore its derivative with respect to φ can be set to zero, i.e.

$$\frac{df}{d\varphi} = \frac{d\mathbf{A}'}{d\varphi} \cdot \left(\frac{\partial f}{\partial \mathbf{A}'} \right)^T = 0. \quad (6)$$

The derivative of \mathbf{A}' with respect to φ can be calculated by the following rules

$$\begin{aligned} d\mathbf{A}'(\varphi) &= d\mathbf{Q}(\varphi \mathbf{m}) \cdot \mathbf{A} \cdot \mathbf{Q}^T(\varphi \mathbf{m}) + \mathbf{Q}(\varphi \mathbf{m}) \cdot \mathbf{A} \cdot d\mathbf{Q}^T(\varphi \mathbf{m}), \\ d\mathbf{Q}(\varphi \mathbf{m}) &= \mathbf{m} \times \mathbf{Q}(\varphi \mathbf{m}) d\varphi \Rightarrow d\mathbf{Q}^T(\varphi \mathbf{m}) = -\mathbf{Q}^T(\varphi \mathbf{m}) \times \mathbf{m} d\varphi. \end{aligned} \quad (7)$$

By inserting the above equations into (6) we obtain

$$(\mathbf{m} \times \mathbf{A} - \mathbf{A} \times \mathbf{m}) \cdot \left(\frac{\partial f}{\partial \mathbf{A}} \right)^T = 0. \quad (8)$$

Eq. (8) is classified in [19] to be the linear homogeneous first order partial differential equation. The characteristic system of (8) is

$$\frac{d\mathbf{A}}{ds} = (\mathbf{m} \times \mathbf{A} - \mathbf{A} \times \mathbf{m}). \quad (9)$$

Any system of n linear ordinary differential equations has not more than $n - 1$ functionally independent integrals [19]. By introducing a basis \mathbf{e}_i the tensor \mathbf{A} can be written down in the form $\mathbf{A} = A^{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ and (9) is a system of six ordinary differential equations with respect to coordinates A^{ij} . The five integrals of (9) may be written down as follows

$$g_i(\mathbf{A}) = c_i, \quad i = 1, 2, \dots, 5,$$

where c_i are integration constants. Any function of the five integrals g_i is the solution of the partial differential equation (8). Therefore the five integrals g_i represent the invariants of the symmetric tensor \mathbf{A} with respect to the symmetry transformation (4). The solutions of (9) are

$$\mathbf{A}^k(s) = \mathbf{Q}(s\mathbf{m}) \cdot \mathbf{A}_0^k \cdot \mathbf{Q}^T(s\mathbf{m}), \quad k = 1, 2, 3, \quad (10)$$

where \mathbf{A}_0 is the initial condition. In order to find the integrals, the variable s must be eliminated from (10). Taking into account the following identities

$$\text{tr}(\mathbf{Q} \cdot \mathbf{A}^k \cdot \mathbf{Q}^T) = \text{tr}(\mathbf{Q}^T \cdot \mathbf{Q} \cdot \mathbf{A}^k) = \text{tr} \mathbf{A}^k, \quad \mathbf{m} \cdot \mathbf{Q}(sm) = \mathbf{m}, \quad (\mathbf{Q} \cdot \mathbf{a}) \times (\mathbf{Q} \cdot \mathbf{b}) = (\det \mathbf{Q}) \mathbf{Q} \cdot (\mathbf{a} \times \mathbf{b}) \quad (11)$$

and using the notation $\mathbf{Q}_m \equiv \mathbf{Q}(sm)$ the integrals can be found as follows

$$\begin{aligned} \text{tr}(\mathbf{A}^k) &= \text{tr}(\mathbf{A}_0^k), \quad k = 1, 2, 3, \\ \mathbf{m} \cdot \mathbf{A}^l \cdot \mathbf{m} &= \mathbf{m} \cdot \mathbf{Q}_m \cdot \mathbf{A}_0^l \cdot \mathbf{Q}_m^T \cdot \mathbf{m} \\ &= \mathbf{m} \cdot \mathbf{A}_0^l \cdot \mathbf{m}, \quad l = 1, 2, \\ \mathbf{m} \cdot \mathbf{A}^2 \cdot (\mathbf{m} \times \mathbf{A} \cdot \mathbf{m}) &= \mathbf{m} \cdot \mathbf{Q}_m^T \cdot \mathbf{A}_0^2 \cdot \mathbf{Q}_m \cdot (\mathbf{m} \times \mathbf{Q}_m^T \cdot \mathbf{A}_0 \cdot \mathbf{Q}_m \cdot \mathbf{m}) \\ &= \mathbf{m} \cdot \mathbf{A}_0^2 \cdot \mathbf{Q}_m \cdot [(\mathbf{Q}_m^T \cdot \mathbf{m}) \times (\mathbf{Q}_m^T \cdot \mathbf{A}_0 \cdot \mathbf{m})] \\ &= \mathbf{m} \cdot \mathbf{A}_0^2 \cdot (\mathbf{m} \times \mathbf{A}_0 \cdot \mathbf{m}). \end{aligned} \quad (12)$$

As a result we can formulate the six invariants of the tensor \mathbf{A} with respect to the symmetry transformation (4) as follows

$$I_k = \text{tr}(\mathbf{A}^k), \quad k = 1, 2, 3, \quad I_4 = \mathbf{m} \cdot \mathbf{A} \cdot \mathbf{m}, \quad I_5 = \mathbf{m} \cdot \mathbf{A}^2 \cdot \mathbf{m}, \quad I_6 = \mathbf{m} \cdot \mathbf{A}^2 \cdot (\mathbf{m} \times \mathbf{A} \cdot \mathbf{m}). \quad (13)$$

The invariants with respect to different symmetry transformations are discussed in [14]. For the case of the transverse isotropy the authors derived six invariants applying another approach. In this sense our result coincides with the result given in [14]. However, from our derivations follows that only five invariants listed in (13) are functionally independent. Taking into account that I_6 is the mixed product of vectors \mathbf{m} , $\mathbf{A} \cdot \mathbf{m}$ and $\mathbf{A}^2 \cdot \mathbf{m}$ the relation between the invariants can be written down as follows

$$I_6^2 = \det \begin{bmatrix} 1 & I_4 & I_5 \\ I_4 & I_5 & \mathbf{m} \cdot \mathbf{A}^3 \cdot \mathbf{m} \\ I_5 & \mathbf{m} \cdot \mathbf{A}^3 \cdot \mathbf{m} & \mathbf{m} \cdot \mathbf{A}^4 \cdot \mathbf{m} \end{bmatrix}. \quad (14)$$

One can verify that $\mathbf{m} \cdot \mathbf{A}^3 \cdot \mathbf{m}$ and $\mathbf{m} \cdot \mathbf{A}^4 \cdot \mathbf{m}$ are transversely isotropic invariants too. However, applying the Cayley-Hamilton theorem, e.g. [9], they can be uniquely expressed by I_1, I_2, \dots, I_5 in the following way [20]

$$\mathbf{m} \cdot \mathbf{A}^3 \cdot \mathbf{m} = J_1 I_5 + J_2 I_4 + J_3, \quad \mathbf{m} \cdot \mathbf{A}^4 \cdot \mathbf{m} = (J_1^2 + J_2) I_5 + (J_1 J_2 + J_3) I_4 + J_1 J_3,$$

where J_1, J_2 and J_3 are the principal invariants of \mathbf{A} . They are defined as follows [20]

$$J_1 \equiv I_1, \quad J_2 \equiv (I_2^2 - I_1^2)/2, \quad J_3 \equiv (2I_3 - 3I_2 I_1 + I_1^3)/6. \quad (15)$$

Let us note that the invariant I_6 cannot be dropped. In order to verify this, it is enough to consider two different tensors

$$\mathbf{A} \text{ and } \mathbf{B} = \mathbf{Q}_n \cdot \mathbf{A} \cdot \mathbf{Q}_n^T, \quad \mathbf{Q}_n \equiv \mathbf{Q}(\pi \mathbf{n}) = 2\mathbf{n} \otimes \mathbf{n} - \mathbf{I}, \quad \mathbf{n} \cdot \mathbf{n} = 1, \quad \mathbf{n} \cdot \mathbf{m} = 0, \quad \det \mathbf{Q}_n = 1.$$

One can prove that the tensor \mathbf{A} and the tensor \mathbf{B} have the same invariants I_1, I_2, \dots, I_5 . Taking into account that $\mathbf{m} \cdot \mathbf{Q}_n = -\mathbf{m}$ and applying the last identity in (11) we may write

$$I_6(\mathbf{B}) = \mathbf{m} \cdot \mathbf{B}^2 \cdot (\mathbf{m} \times \mathbf{B} \cdot \mathbf{m}) = \mathbf{m} \cdot \mathbf{A}^2 \cdot \mathbf{Q}_n^T \cdot (\mathbf{m} \times \mathbf{Q}_n \cdot \mathbf{A} \cdot \mathbf{m}) = -\mathbf{m} \cdot \mathbf{A}^2 \cdot (\mathbf{m} \times \mathbf{A} \cdot \mathbf{m}) = -I_6(\mathbf{A}).$$

We observe that the only difference between two considered tensors is the sign of I_6 . Therefore, the triples of vectors \mathbf{m} , $\mathbf{A} \cdot \mathbf{m}$, $\mathbf{A}^2 \cdot \mathbf{m}$ and \mathbf{m} , $\mathbf{B} \cdot \mathbf{m}$, $\mathbf{B}^2 \cdot \mathbf{m}$ have different orientations and cannot be combined by a rotation. It should be noted that the functional relation (14) would in no way imply that the invariant I_6 should be "dependent" and hence "redundant", namely should be removed from the basis (13). In fact, the relation (14) determines the magnitude but not the sign of I_6 .

To describe yielding and failure of oriented solids a dyad $\mathbf{M} = \mathbf{v} \otimes \mathbf{v}$ has been used in [21,22], where the vector \mathbf{v} specifies a privileged direction. A plastic potential is assumed to be an isotropic function of the symmetric Cauchy stress tensor and the tensor generator \mathbf{M} . Applying the representation of isotropic functions the integrity basis including ten invariants was found. In the special case $\mathbf{v} = \mathbf{m}$ the number of invariants reduces to the five I_1, I_2, \dots, I_5 defined by (13). Further details

of this approach and applications in continuum mechanics are given in [9,23]. However, the problem statement to find an integrity basis of a symmetric tensor \mathbf{A} and a dyad \mathbf{M} , i.e. to find scalar valued functions $f(\mathbf{A}, \mathbf{M})$ satisfying the condition

$$f(\mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^T, \mathbf{Q} \cdot \mathbf{M} \cdot \mathbf{Q}^T) = (\det \mathbf{Q})^\eta f(\mathbf{A}, \mathbf{M}), \quad \forall \mathbf{Q}, \quad \mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{I}, \quad \det \mathbf{Q} = \pm 1 \tag{16}$$

essentially differs from the problem statement (5). In order to show this we take into account that the symmetry group of a dyad \mathbf{M} , i.e. the set of orthogonal solutions of the equation $\mathbf{Q} \cdot \mathbf{M} \cdot \mathbf{Q}^T = \mathbf{M}$ includes the following elements

$$\begin{aligned} \mathbf{Q}_{1,2} &= \pm \mathbf{I}, \\ \mathbf{Q}_3 &= \mathbf{Q}(\varphi \mathbf{m}), \quad \mathbf{m} = \frac{\mathbf{v}}{|\mathbf{v}|}, \\ \mathbf{Q}_4 &= \mathbf{Q}(\pi \mathbf{n}) = 2\mathbf{n} \otimes \mathbf{n} - \mathbf{I}, \quad \mathbf{n} \cdot \mathbf{n} = 1, \quad \mathbf{n} \cdot \mathbf{v} = 0, \end{aligned} \tag{17}$$

where $\mathbf{Q}(\varphi \mathbf{m})$ is defined by (4). The solutions of the problem (16) are automatically the solutions of the following problem

$$f(\mathbf{Q}_i \cdot \mathbf{A} \cdot \mathbf{Q}_i^T, \mathbf{M}) = (\det \mathbf{Q}_i)^\eta f(\mathbf{A}, \mathbf{M}), \quad i = 1, 2, 3, 4,$$

i.e. the problem to find the invariants of \mathbf{A} relative to the symmetry group (17). However, (17) includes much more symmetry elements if compared to the problem statement (5).

An alternative set of transversely isotropic invariants can be formulated by use of the following decomposition

$$\mathbf{A} = \alpha \mathbf{m} \otimes \mathbf{m} + \beta (\mathbf{I} - \mathbf{m} \otimes \mathbf{m}) + \mathbf{A}_{pD} + \mathbf{t} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{t}, \tag{18}$$

where $\alpha, \beta, \mathbf{A}_{pD}$ and \mathbf{t} are projections of \mathbf{A} . With the projectors $\mathbf{P}_1 = \mathbf{m} \otimes \mathbf{m}$ and $\mathbf{P}_2 = \mathbf{I} - \mathbf{m} \otimes \mathbf{m}$ we may write

$$\begin{aligned} \alpha &= \mathbf{m} \cdot \mathbf{A} \cdot \mathbf{m} = \text{tr}(\mathbf{A} \cdot \mathbf{P}_1), \\ \beta &= \frac{1}{2} (\text{tr} \mathbf{A} - \mathbf{m} \cdot \mathbf{A} \cdot \mathbf{m}) = \frac{1}{2} \text{tr}(\mathbf{A} \cdot \mathbf{P}_2), \\ \mathbf{A}_{pD} &= \mathbf{P}_2 \cdot \mathbf{A} \cdot \mathbf{P}_2 - \beta \mathbf{P}_2, \\ \mathbf{t} &= \mathbf{m} \cdot \mathbf{A} \cdot \mathbf{P}_2. \end{aligned} \tag{19}$$

The above decomposition is the analogue to the following representation of a vector \mathbf{a}

$$\mathbf{a} = \mathbf{I} \cdot \mathbf{a} = \mathbf{m} \otimes \mathbf{m} \cdot \mathbf{a} + (\mathbf{I} - \mathbf{m} \otimes \mathbf{m}) \cdot \mathbf{a} = \psi \mathbf{m} + \boldsymbol{\tau}, \quad \psi = \mathbf{a} \cdot \mathbf{m}, \quad \boldsymbol{\tau} = \mathbf{P}_2 \cdot \mathbf{a}. \tag{20}$$

The decompositions of the type (18) are applied in [14,24]. The projections introduced in (19) have the following properties

$$\text{tr}(\mathbf{A}_{pD}) = 0, \quad \mathbf{A}_{pD} \cdot \mathbf{m} = \mathbf{m} \cdot \mathbf{A}_{pD} = \mathbf{0}, \quad \mathbf{t} \cdot \mathbf{m} = 0. \tag{21}$$

With (18) and (21) the tensor equation (9) can be transformed to the following system of equations

$$\left\{ \begin{aligned} \frac{d\alpha}{ds} &= 0, \\ \frac{d\beta}{ds} &= 0, \\ \frac{d\mathbf{A}_{pD}}{ds} &= \mathbf{m} \times \mathbf{A}_{pD} - \mathbf{A}_{pD} \times \mathbf{m}, \\ \frac{d\mathbf{t}}{ds} &= \mathbf{m} \times \mathbf{t}. \end{aligned} \right. \tag{22}$$

From the first two equations we observe that α and β are transversely isotropic invariants. The third equation can be transformed to one scalar and one vector equation as follows

$$\frac{d\mathbf{A}_{pD}}{ds} \cdot \mathbf{A}_{pD} = 0 \quad \Rightarrow \quad \frac{d(\mathbf{A}_{pD} \cdot \mathbf{A}_{pD})}{ds} = 0, \quad \frac{d\mathbf{b}}{ds} = \mathbf{m} \times \mathbf{b}$$

with $\mathbf{b} \equiv \mathbf{A}_{pD} \cdot \mathbf{t}$. We observe that $\text{tr}(\mathbf{A}_{pD}^2) = \mathbf{A}_{pD} \cdot \mathbf{A}_{pD}$ is the transversely isotropic invariant too. Finally, we have to find the integrals of the following system

$$\left\{ \begin{aligned} \frac{d\mathbf{t}}{ds} &= \mathbf{t} \times \mathbf{m}, \\ \frac{d\mathbf{b}}{ds} &= \mathbf{b} \times \mathbf{m}. \end{aligned} \right. \tag{23}$$

The solutions of (23) are

$$\mathbf{t}(s) = \mathbf{Q}(sm) \cdot \mathbf{t}_0, \quad \mathbf{b}(s) = \mathbf{Q}(sm) \cdot \mathbf{b}_0,$$

where \mathbf{t}_0 and \mathbf{b}_0 are initial conditions. The vectors \mathbf{t} and \mathbf{b} belong to the plane of isotropy, i.e. $\mathbf{t} \cdot \mathbf{m} = 0$ and $\mathbf{b} \cdot \mathbf{m} = 0$. Therefore, one can verify the following integrals

$$\mathbf{t} \cdot \mathbf{t} = \mathbf{t}_0 \cdot \mathbf{t}_0, \quad \mathbf{b} \cdot \mathbf{b} = \mathbf{b}_0 \cdot \mathbf{b}_0, \quad \mathbf{t} \cdot \mathbf{b} = \mathbf{t}_0 \cdot \mathbf{b}_0, \quad (\mathbf{t} \times \mathbf{b}) \cdot \mathbf{m} = (\mathbf{t}_0 \times \mathbf{b}_0) \cdot \mathbf{m}. \quad (24)$$

We found seven integrals, but only five of them are functionally independent. In order to formulate the relation between the integrals we compute

$$\mathbf{b} \cdot \mathbf{b} = \mathbf{t} \cdot \mathbf{A}_{pD}^2 \cdot \mathbf{t}, \quad \mathbf{t} \cdot \mathbf{b} = \mathbf{t} \cdot \mathbf{A}_{pD} \cdot \mathbf{t}.$$

Applying the Cayley-Hamilton theorem we obtain

$$2\mathbf{A}_{pD}^2 = \text{tr}(\mathbf{A}_{pD}^2)(\mathbf{I} - \mathbf{m} \otimes \mathbf{m}), \quad \mathbf{t} \cdot \mathbf{A}_{pD}^2 \cdot \mathbf{t} = \frac{1}{2} \text{tr}(\mathbf{A}_{pD}^2)(\mathbf{t} \cdot \mathbf{t}).$$

Because $\text{tr}(\mathbf{A}_{pD}^2)$ and $\mathbf{t} \cdot \mathbf{t}$ are already defined, the invariant $\mathbf{b} \cdot \mathbf{b}$ can be omitted. The vector $\mathbf{t} \times \mathbf{b}$ is spanned on the axis \mathbf{m} . Therefore

$$\mathbf{t} \times \mathbf{b} = \gamma \mathbf{m}, \quad \gamma = (\mathbf{t} \times \mathbf{b}_m) \cdot \mathbf{m}, \quad \gamma^2 = (\mathbf{t} \times \mathbf{b}) \cdot (\mathbf{t} \times \mathbf{b}) = (\mathbf{t} \cdot \mathbf{t})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{t} \cdot \mathbf{b})^2.$$

Now we can summarize six invariants and one relation between them as follows

$$\begin{aligned} \bar{I}_1 &= \alpha, \quad \bar{I}_2 = \beta, \quad \bar{I}_3 = \frac{1}{2} \text{tr}(\mathbf{A}_{pD}^2), \quad \bar{I}_4 = \mathbf{t} \cdot \mathbf{t} = \mathbf{t} \cdot \mathbf{A} \cdot \mathbf{m}, \quad \bar{I}_5 = \mathbf{t} \cdot \mathbf{A}_{pD} \cdot \mathbf{t}, \quad \bar{I}_6 = (\mathbf{t} \times \mathbf{A}_{pD} \cdot \mathbf{t}) \cdot \mathbf{m}, \\ \bar{I}_6^2 &= \bar{I}_4^2 \bar{I}_3 - \bar{I}_5^2. \end{aligned} \quad (25)$$

Let us assume that the symmetry transformation $\mathbf{Q}_n \equiv \mathbf{Q}(\pi \mathbf{n})$ belongs to the symmetry group of the transverse isotropy, as it made in [9,23]. In this case $f(\mathbf{A}') = f(\mathbf{Q}_n \cdot \mathbf{A} \cdot \mathbf{Q}_n^T) = f(\mathbf{A})$ must be valid. With $\mathbf{Q}_n \cdot \mathbf{m} = -\mathbf{m}$ we can write

$$\alpha' = \alpha, \quad \beta' = \beta, \quad \mathbf{A}'_{pD} = \mathbf{A}_{pD}, \quad \mathbf{t}' = -\mathbf{Q}_n \cdot \mathbf{t}.$$

Therefore in (25) $\bar{I}'_k = \bar{I}_k$, $k = 1, 2, \dots, 5$ and

$$\begin{aligned} \bar{I}'_6 &= (\mathbf{t}' \times \mathbf{A}'_{pD} \cdot \mathbf{t}') \cdot \mathbf{m} = ((\mathbf{Q}_n \cdot \mathbf{t}) \times \mathbf{Q}_n \cdot \mathbf{A}_{pD} \cdot \mathbf{t}) \cdot \mathbf{m} \\ &= (\mathbf{t} \times \mathbf{A}_{pD} \cdot \mathbf{t}) \cdot \mathbf{Q}_n \cdot \mathbf{m} = -(\mathbf{t} \times \mathbf{A}_{pD} \cdot \mathbf{t}) \cdot \mathbf{m} = -\bar{I}_6. \end{aligned}$$

Consequently

$$f(\mathbf{A}') = f(\bar{I}'_1, \bar{I}'_2, \dots, \bar{I}'_5, \bar{I}'_6) = f(\bar{I}_1, \bar{I}_2, \dots, \bar{I}_5, -\bar{I}_6) \Rightarrow f(\mathbf{A}) = f(\bar{I}_1, \bar{I}_2, \dots, \bar{I}_5, \bar{I}_6^2)$$

and \bar{I}_6^2 can be omitted due to the last relation in (25).

3 A number of functionally independent invariants for a set of vectors and second rank symmetric tensors

Setting $\mathbf{Q} = \mathbf{Q}(\varphi \mathbf{m})$ in (3) and taking the derivative of (3) with respect to φ results in the following generic partial differential equation

$$\sum_{i=1}^n \left(\frac{\partial f}{\partial \mathbf{A}_i} \right)^T \cdot (\mathbf{m} \times \mathbf{A}_i - \mathbf{A}_i \times \mathbf{m}) + \sum_{j=1}^k \frac{\partial f}{\partial \mathbf{a}_j} \cdot (\mathbf{m} \times \mathbf{a}_j) = 0. \quad (26)$$

The characteristic system of (26) is

$$\begin{cases} \frac{d\mathbf{A}_i}{ds} = (\mathbf{m} \times \mathbf{A}_i - \mathbf{A}_i \times \mathbf{m}), & i = 1, 2, \dots, n, \\ \frac{d\mathbf{a}_j}{ds} = \mathbf{m} \times \mathbf{a}_j, & j = 1, 2, \dots, k. \end{cases} \quad (27)$$

The above system is a system of N ordinary differential equations, where $N = 6n + 3k$ is the total number of coordinates of \mathbf{A}_i and \mathbf{a}_j for a selected basis. The system (27) has not more than $N - 1$ functionally independent integrals. Therefore we can formulate:

Theorem 3.1 *A set of n symmetric second rank tensors and k vectors with $N = 6n + 3k$ independent coordinates for a given basis has not more than $N - 1$ functionally independent invariants for $N > 1$ and one invariant for $N = 1$ with respect to the symmetry transformation $Q(\varphi\mathbf{m})$.*

In essence, the proof of this theorem is given within the theory of linear first order partial differential equations [19].

As an example let us consider the set of a symmetric second rank tensor \mathbf{A} and a vector \mathbf{a} . This set has eight independent invariants. For a visual perception it is useful to keep in mind that the considered set is equivalent to

$$\mathbf{A}, \mathbf{a}, \mathbf{A} \cdot \mathbf{a}, \mathbf{A}^2 \cdot \mathbf{a}.$$

Therefore it is necessary to find the list of invariants, whose fixation determines this set as a rigid whole. The generic equation (26) takes the form

$$\left(\frac{\partial f}{\partial \mathbf{A}}\right)^T \cdot (\mathbf{m} \times \mathbf{A} - \mathbf{A} \times \mathbf{m}) + \frac{\partial f}{\partial \mathbf{a}} \cdot (\mathbf{m} \times \mathbf{a}) = 0. \tag{28}$$

The characteristic system of (28) is

$$\frac{d\mathbf{A}}{ds} = \mathbf{m} \times \mathbf{A} - \mathbf{A} \times \mathbf{m}, \quad \frac{d\mathbf{a}}{ds} = \mathbf{m} \times \mathbf{a}.$$

This system of ninth order has eight independent integrals. Six of them are invariants of \mathbf{A} and \mathbf{a} with respect to the full orthogonal group. They fix the considered set as a rigid whole. The orthogonal invariants are [16]

$$I_k = \text{tr}(\mathbf{A}^k), \quad k = 1, 2, 3, \quad \tilde{I}_4 = \mathbf{a} \cdot \mathbf{a}, \quad \tilde{I}_5 = \mathbf{a} \cdot \mathbf{A} \cdot \mathbf{a}, \quad \tilde{I}_6 = \mathbf{a} \cdot \mathbf{A}^2 \cdot \mathbf{a}, \quad \tilde{I}_7 = \mathbf{a} \cdot \mathbf{A}^2 \cdot (\mathbf{a} \times \mathbf{A} \cdot \mathbf{a}). \tag{29}$$

The list (29) contains seven integrals but the following relation between them exists

$$\tilde{I}_7^2 = \det \begin{bmatrix} \tilde{I}_4 & \tilde{I}_5 & \tilde{I}_6 \\ \tilde{I}_5 & \tilde{I}_6 & \mathbf{a} \cdot \mathbf{A}^3 \cdot \mathbf{a} \\ \tilde{I}_6 & \mathbf{a} \cdot \mathbf{A}^3 \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{A}^4 \cdot \mathbf{a} \end{bmatrix}. \tag{30}$$

The invariants $\mathbf{a} \cdot \mathbf{A}^3 \cdot \mathbf{a}$ and $\mathbf{a} \cdot \mathbf{A}^4 \cdot \mathbf{a}$ in (30) can be expressed in terms of invariants $I_1, I_2, \dots, \tilde{I}_6$ with the help of the Cayley-Hamilton theorem as follows

$$\mathbf{a} \cdot \mathbf{A}^3 \cdot \mathbf{a} = J_1 \tilde{I}_6 + J_2 \tilde{I}_5 + J_3 \tilde{I}_4, \quad \mathbf{a} \cdot \mathbf{A}^4 \cdot \mathbf{a} = (J_1^2 + J_2) \tilde{I}_6 + (J_3 + J_1 J_2) \tilde{I}_5 + J_1 J_3 \tilde{I}_4,$$

where J_1, J_2, J_3 are principal invariants of \mathbf{A} defined by (15).

Let us note that the invariant \tilde{I}_7 cannot be ignored. In order to verify this fact it is enough to consider two different sets

$$\mathbf{A}, \mathbf{a} \quad \text{and} \quad \mathbf{B} = \mathbf{Q}_p \cdot \mathbf{A} \cdot \mathbf{Q}_p^T, \mathbf{a},$$

where $\mathbf{Q}_p = \mathbf{I} - 2\mathbf{p} \otimes \mathbf{p}$, $\mathbf{p} \cdot \mathbf{p} = 1$, $\mathbf{p} \cdot \mathbf{a} = 0$. One can prove that invariants $I_1, I_2, \dots, \tilde{I}_6$ are the same for these two sets. The only difference is the invariant \tilde{I}_7 , i.e. $\mathbf{a} \cdot \mathbf{B}^2 \cdot (\mathbf{a} \times \mathbf{B} \cdot \mathbf{a}) = -\mathbf{a} \cdot \mathbf{A}^2 \cdot (\mathbf{a} \times \mathbf{A} \cdot \mathbf{a})$. Therefore the triples of vectors $\mathbf{a}, \mathbf{A} \cdot \mathbf{a}, \mathbf{A}^2 \cdot \mathbf{a}$ and $\mathbf{a}, \mathbf{B} \cdot \mathbf{a}, \mathbf{B}^2 \cdot \mathbf{a}$ have different orientations and cannot be combined by a rotation. In order to fix the considered set with respect to the unit vector \mathbf{m} it is enough to fix the next two invariants

$$\tilde{I}_8 = \mathbf{m} \cdot \mathbf{A} \cdot \mathbf{m}, \quad \tilde{I}_9 = \mathbf{m} \cdot \mathbf{a}. \tag{31}$$

The invariants (29), (31), and the restriction (30) are eight independent transversely isotropic invariants.

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