Dynamic Forms of Equilibrium of a Bar Compressed by a Dead Force

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Abstract

The report presents the stationary solutions of nonlinear equations of motion of a console beam compressed by a dead force.

1 Introduction

The equilibrium configurations of a thin rod compressed by a dead force were found by Euler in 1744. It was a first investigation of nonlinear problem in mechanics. Up to now exact solutions of dynamic nonlinear equations for a console beam were not found, since, as a general rule, the nonlinear equations with particular derivatives does not allow the separation of variables by means of finite number of operations. However there are some special cases when it is possible. One such a case will be described in what follow.

2 The Statement of the Problem

Let us consider the thin flexible beam clamped on one end and compressed by a dead force on another end. The cross-section of a bar is supposed to be transversely isotropic. The classic equation can be represented in the form of the next system of equations.

The equations of motion

$$\underline{N}'(s,t) = g\ddot{\underline{R}}(s,t), \qquad \underline{M}'(s,t) + \underline{R}'(s,t)\times\underline{N}(s,t) = \underline{0}, \tag{1}$$

where $f' = \partial f/\partial s$, $\dot{f} = \partial f/\partial t$; s is a coordinate of a cross-section of a bar in undeformed state; the vector $\underline{R}(s,t)$ determines the position of a cross-section in deformed state; ρ is the mass of the bar per unit length; $\rho = \text{const}$; $\underline{N}(s,t)$ is the vector of intrinsic force; $\underline{M}(s,t)$ is the vector of intrinsic moments.

The geometrical relations expresses the condition of inextensibility of the bar axis

$$\underline{R}'(s,t) = \underline{\underline{P}}(s,t)\cdot\underline{\tau}, \qquad \underline{\tau}\cdot\underline{\tau} = 1, \quad \underline{R}'\cdot\underline{R}' = 1, \tag{2}$$

where $\underline{\tau}$ is an unit vector directed along undeformed axis of a bar; $\underline{R}'$ is a tangential vector to the deformed axis of a bar; the properly orthogonal tensor $\underline{\underline{P}}(s,t)$ describes the turn of a cross-section with a coordinate s.

$$\underline{\underline{P}}^T(s,t)\cdot\underline{\underline{P}}'(s,t) = \underline{\underline{E}}, \qquad \det\underline{\underline{P}}(s,t) = 1 \tag{3}$$

The constitutive equation has a form

$$\underline{M}(s,t) = C_1\underline{\Phi}(s,t) + (C_3 - C_1)(\underline{R}'\cdot\underline{\Phi})\underline{R}', \tag{4}$$

where C_1 is the bending stiffness; C_3 is the torsional stiffness; $\underline{\Phi}$ is the vector of bending-torsion.

The vector of bending-torsion $\underline{\Phi}(s,t)$ and the vector of angular velocity $\underline{\omega}(s,t)$ can be found from equations by Poisson [1].

$$\dot{\underline{\underline{P}}}(s,t) = \underline{\omega}(s,t)\times\underline{\underline{P}}(s,t), \qquad \underline{\underline{P}}'(s,t) = \underline{\Phi}(s,t)\times\underline{\underline{P}}(s,t) \tag{5}$$

From the equations (5) the next equation can be derived

$$\dot{\underline{\Phi}}(s,t) = \underline{\omega}'(s,t) + \underline{\omega}(s,t)\times\underline{\Phi}(s,t) \tag{6}$$

Thus we have a closed system of equations for unknown functions $\underline{R}(s,t)$, $\underline{N}(s,t)$ and $\underline{\underline{P}}(s,t)$. If the tensor of turn $\underline{\underline{P}}(s,t)$ is known then the vector of position $\underline{R}(s,t)$ can be found by means of integration of equation (2). So the basic unknowns are the vector $\underline{N}(s,t)$ and the tensor $\underline{\underline{P}}(s,t)$.

However for this we have to transform the first equation from system (1). Differentiating the equation (1) with respect to the coordinate s and making use of (2) and (5) we obtain

$$\underline{N}'' = \rho\ddot{\underline{R}}'(s,t) = \rho[\dot{\underline{\omega}}\times\underline{\underline{E}} + \underline{\omega}\times\underline{\underline{E}}\times\underline{\omega}]\cdot\underline{\underline{P}}\cdot\underline{\tau} \tag{7}$$

Also we have an additional boundary condition which follows from (1)

$$\underline{N}'(0,t) = \underline{0} \tag{8}$$

Let us formulate boundary conditions

$$s = 0: \qquad \underline{R}(0,t) = \underline{0}, \qquad \underline{\underline{P}}(0,t) = \underline{\underline{E}}; \tag{9}$$

$$s = l: \qquad \underline{N}(l,t) = -Q\underline{\tau}, \qquad \underline{M}(l,t) = \underline{0}, \qquad (10)$$

where $\underline{\underline{E}}$ is an unit tensor of second rank. When $Q > 0$ we have a bar compressed by a constant force Q.

The initial conditions have a standard form

$$\underline{R}(s,0) = \underline{f}(s), \qquad \dot{\underline{R}}(s,0) = \underline{v}(s) \qquad (11)$$

In what follows we accept the strong restrictions on the functions $\underline{f}(s)$ and $\underline{v}(s)$.

3 Alternative Statement of the Problem

Sometimes it is more convenient to use the alternative statement of the problem in terms of right quantities [1]. Let us introduce into consideration the right vector of bending-torsion $\underline{\phi}(s,t)$ and the right vector of angular velocity $\underline{\Omega}(s,t)$

$$\underline{\Phi} = \underline{\underline{P}}\cdot\underline{\phi}, \quad \underline{\omega} = \underline{\underline{P}}\cdot\underline{\Omega} \;\Rightarrow\; \dot{\underline{\underline{P}}} = \underline{\underline{P}}\times\underline{\Omega}, \quad \underline{\underline{P}}' = \underline{\underline{P}}\times\underline{\phi} \quad (12)$$

The equation (6) takes a form

$$\dot{\underline{\phi}} = \underline{\Omega}' - \underline{\Omega}\times\underline{\phi} \qquad (13)$$

Let us introduce the new variables

$$\underline{N} = \underline{\underline{P}}\cdot\underline{n}, \qquad \underline{M} = \underline{\underline{P}}\cdot\underline{m} \qquad (14)$$

In order to transform the equation of motion (1) we have to differentiate the first equation from (1) with respect to the coordinate s and to use the equation (2).

Now the equations of motion (1) and (7) take the form

$$(\underline{n}' + \underline{\phi}\times\underline{n})' + \underline{\phi}\times(\underline{n}'\times\underline{\phi}\times\underline{n}) = \rho[\dot{\underline{\Omega}}\times\underline{\tau} + \underline{\Omega}\times(\underline{\Omega}\times\underline{\tau})], \quad (15)$$

$$\underline{m}' + \underline{\phi}\times\underline{m} + \underline{\tau}\times\underline{n} = \underline{0} \qquad (16)$$

When deriving the equations (15) – (16) the next identity is useful

$$\underline{\underline{P}}\cdot(\underline{a}\times\underline{b}) = (\underline{\underline{P}}\cdot\underline{a})\times(\underline{\underline{P}}\cdot\underline{b}),$$
$$\forall \underline{a}, \underline{b}, \underline{\underline{P}}: \; \underline{\underline{P}}\cdot\underline{\underline{P}}^T = \underline{\underline{E}}, \; \det\underline{\underline{P}} = 1$$

The constitutive equation (4) can be rewritten in terms of new variables as

$$\underline{m} = C_1\underline{\phi} + (C_3 - C_1)(\underline{\phi}\cdot\underline{\tau})\underline{\tau} \qquad (17)$$

The boundary conditions for system (12) – (17) have the form

$$\begin{aligned} s = 0: &\quad \underline{n}' + \underline{\phi}\times\underline{n} = 0, \qquad \underline{\underline{P}}(0,t) = \underline{\underline{E}} \\ s = l: &\quad \underline{n} = -Q\underline{\underline{P}}^T\cdot\underline{\tau}, \qquad \underline{m}(0,t) = \underline{0} \end{aligned} \qquad (18)$$

In such a statement the basic system does not contain a tensor of turn and a vector of position.

4 Integral by Poisson

From the equations (2) and (5) it follows

$$\underline{R}'' = \underline{\Phi}\times\underline{R}' \quad \Longrightarrow \quad \underline{\Phi} = (\underline{\Phi}\cdot\underline{R}')\underline{R}' + \underline{R}'\times\underline{R}'' \qquad (19)$$

The second equation in (19) expresses the representation by Poisson (1833) for the vector of bending-torsion. Substituting (19) into (4) we have

$$\underline{M} = C_1\underline{R}'\times\underline{R}'' + C_3(\underline{R}'\cdot\underline{\Phi})\underline{R}' \qquad (20)$$

From this equation it follows

$$\underline{M}\cdot\underline{R}'' = 0, \qquad \underline{M}\cdot\underline{R}' = C_3\underline{\Phi}\cdot\underline{R}' \qquad (21)$$

The second equation from (1) shows that

$$\underline{M}'\cdot\underline{R}' = (\underline{M}\cdot\underline{R}')' - \underline{M}\cdot\underline{R}'' = 0$$

This equation and the equations (21) give to us

$$\underline{M}\cdot\underline{R}' = C_3\underline{R}'\cdot\underline{\Phi} = A(t) = \mathrm{const}(s) \qquad (22)$$

The integral (22) was derived by Poisson [2,p.627]. The conditions (10) shows that the function $A(t)$ is equal to zero. Thus we have

$$\underline{M} = C_1\underline{u}\times\underline{u}', \quad \underline{\Phi} = \underline{u}\times\underline{u}', \quad \underline{u}(s,t) \equiv \underline{R}'(s,t) \qquad (23)$$

Making use of (23) the second equation from (1) can be rewritten in the form

$$\underline{u}\times[C_1\underline{u}'' + \underline{N}] = 0$$

or in equivalent form

$$C_1\underline{u}'' + \underline{N} = \lambda(s,t)\underline{u} \quad \Rightarrow \quad \lambda = \underline{N}\cdot\underline{u} - C_1|\underline{u}'|^2 \quad (24)$$

For the vector $\underline{N}$ the next expression can be found from (7)

$$\underline{N} = -Q\underline{\tau} - \rho(l-s)\int_0^l \ddot{\underline{u}}(\xi,t)d\xi + \rho\int_s^l (\xi - s)\ddot{\underline{u}}(\xi,t)d\xi \qquad (25)$$

Now the equation (24) can be rewritten in the form

$$C_1\underline{R}^{IV} - (\lambda\underline{R}')' + \rho\ddot{\underline{R}} = \underline{0}, \qquad (26)$$

where λ is given by second expression in (24) and the expression (25) must be replaced by

$$\underline{N}(s,t) = Q\underline{\tau} - \rho\int_s^l \ddot{\underline{R}}(\xi,t)d\xi$$

The boundary conditions for the equation (26) have the form

$$\begin{aligned} s = 0: &\quad \underline{R} = \underline{0}, \qquad \underline{R}' = \underline{\tau}; \\ s = l: &\quad \underline{R}'' = \underline{0}, \quad C_1\underline{R}''' - \lambda\underline{R}' = Q\underline{\tau} \end{aligned} \qquad (27)$$

The problem (26) – (27) is very complicate in a general case for an exact analytical solution. Only static solution of this problem is well known. The problem (26) – (27) can be rewritten in terms of vector $\underline{u} = \underline{R}'$

$$C_1 \underline{u}^{VI} - (\lambda \underline{u})'' + \rho \ddot{\underline{u}} = 0, \tag{28}$$

$$\begin{aligned} s = 0: &\quad \underline{u} = \underline{\tau}, \quad C_1 \underline{u}''' - (\lambda \underline{u})'\Big|_{s=0} = \underline{0}; \\ s = l: &\quad \underline{u}' = \underline{0}, \quad C_1 \underline{u}'' - \lambda \underline{u} = Q \underline{\tau} \end{aligned} \tag{29}$$

5 Stationary Movements of the Bar

Let us consider the special kind of the movement which can be defined by tensor of turn

$$\begin{aligned} \underline{\underline{P}} = \underline{\underline{Q}}(\psi \underline{\tau}) \cdot \underline{\underline{Q}}(\vartheta \underline{e}_0) \cdot \underline{\underline{Q}}^T(\psi \underline{\tau}) = \underline{\underline{Q}}(\vartheta \underline{e}), \\ \underline{e} = \underline{\underline{Q}}(\psi \underline{\tau}) \cdot \underline{e}_0, \end{aligned} \tag{30}$$

where $\vartheta(s)$ is an angle of nutation and it does not depend on time; $\psi(t)$ is an angle of precession and it does not depend on space coordinate; $\phi(t) = -\psi(t)$ is an angle of own rotation.

Also we accept the relations

$$\begin{aligned} \dot{\psi}(t) = \omega = \text{const}, \quad \psi(0) = 0 \\ \underline{e}(t) \cdot \underline{\tau} = 0, \quad \underline{e}(0) = \underline{e}_0 \end{aligned} \tag{31}$$

The vector $\vartheta(s)\underline{e}(t)$ is the vector of turn. The left $\underline{\omega}$ and the right $\underline{\Omega}$ angular velocities can be found from the expressions [1]

$$\underline{\omega} = \omega[(1 - \cos\vartheta)\underline{\tau} + \sin\vartheta \underline{\tau} \times e], \quad \omega \equiv \dot{\psi}; \tag{32}$$

$$\underline{\Omega} = \omega[-(1 - \cos\vartheta)\underline{\tau} + \sin\vartheta \underline{\tau} \times e] \tag{33}$$

The left $\underline{\Phi}$ and the right $\underline{\phi}$ vectors of bending-torsion are defined by formulae

$$\underline{\Phi} = \underline{\phi} = \vartheta'(s)\underline{e}(t) \tag{34}$$

The next representations are valid

$$\begin{aligned} \underline{u}(s,t) = \underline{R}'(s,t) = \cos\vartheta \underline{\tau} - \sin\vartheta \underline{\tau} \times \underline{e}(t), \\ \ddot{\underline{u}}(s,t) = \omega^2 \sin\vartheta \underline{\tau} \times e \end{aligned} \tag{35}$$

Making use of (35) the expression (25) can be rewritten in the form

$$\underline{N} = -Q\underline{\tau} - T(s)\underline{\tau} \times \underline{e},$$

$$T(s) = \rho\omega^2 [(l - s) \int_0^l \sin\vartheta(\xi) d\xi - \int_s^l (\xi - s) \sin\vartheta(\xi) d\xi] \tag{36}$$

The function $\lambda(s,t)$ in such a case has the form

$$\lambda(s) = -\cos\vartheta Q + \sin\vartheta T(s) - C_1 v'^2, \tag{37}$$

i.e. the function λ does not depend on time. Let us introduce the vector of displacement

$$\underline{R} = s\underline{\tau} + \underline{w} = (s + u)\underline{\tau} + w\underline{\tau} \times \underline{e}$$

Making use of (35) we have

$$\begin{aligned} x \equiv \underline{u} \cdot \underline{\tau} = \underline{R}' \cdot \underline{\tau} = 1 + u' = \cos\vartheta, \\ y \equiv \underline{u} \cdot (\underline{\tau} \times \underline{e}) = \underline{R}' \cdot (\underline{\tau} \times \underline{e}) = w' = -\sin\vartheta \end{aligned} \tag{38}$$

If we take into account the relations (35) and (38) then the system (28) – (29) can be rewritten in such a manner

$$C_1 x'' - \lambda x = Q, \tag{39}$$

$$s = 0: \ x = 1; \qquad s = l: \ x' = 0$$

$$C_1 y^{IV} - (\lambda y)'' - \rho\omega^2 y = 0, \tag{40}$$

$$\begin{aligned} s = 0: \ y = 0, \ C_1 y''' - (\lambda y)'_{s=0} = 0 \\ s = l: \ y' = 0, \ C_1 y'' - \lambda y = 0 \end{aligned}$$

The problems (39) – (40) are connected since the function λ depends on functions x and y. May be it would be easy to consider another approach.

The equations (7), (23), (35) and (36) give us

$$\begin{aligned} \underline{N}'' = \rho \ddot{\underline{u}} \quad \Rightarrow \\ \Rightarrow \quad T''(s) = -\rho\omega^2 \sin\vartheta = \rho\omega^2 w' \quad \Rightarrow \\ \Rightarrow \quad T'(s) = \rho\omega^2 w(s) \end{aligned} \tag{41}$$

For the bending moment we have an expression

$$\underline{M} = C_1 \vartheta'(s) \underline{e}(t)$$

Let us calculate the vector product

$$\underline{R}' \times \underline{N} = (Q \sin\vartheta + T \cos\vartheta) \underline{e}(t)$$

Now the second equation from the system (1) takes the form

$$C_1 \vartheta'' + Q \sin\vartheta + T \cos\vartheta = 0 \tag{42}$$

From the equations (41) and (42) it follows

$$C_1 \left[\frac{1}{\sqrt{1 - w'^2}} \left(\frac{w''}{\sqrt{1 - w'^2}} \right)' \right]' + Q \left(\frac{w'}{\sqrt{1 - w'^2}} \right)' - \rho\omega^2 w = 0 \tag{43}$$

Boundary conditions to this equation have the form

$$\begin{aligned} s = 0: \ w = 0, \ w' = 0; \\ s = l: \ w'', \ C_1 \left(\frac{w''}{\sqrt{1 - w'^2}} \right)' + Q w' = 0 \end{aligned} \tag{44}$$

Thus we obtain nonlinear spectral problem. For any ω^2 we have a trivial solution $w = 0$. The rigorous existence proof of nonlinear solutions for the problem (43) – (44) is not known. Let us suppose that this problem has a real solution $w(s)$. In such a case it is not difficult to obtain the formula

$$\rho\omega^2 \int_0^l w^2 ds = C_1 \int_0^l \frac{w''^2 ds}{(1 - w'^2)} - Q \int_0^l \frac{w'^2 ds}{\sqrt{1 - w'^2}} \quad (45)$$

The spectral problem (43) – (44) is not usual one because of nonlinearity and the presence of two parameters Q and $\rho\omega^2$. Let us remind that the parameter ω has a meaning of the velocity of precession.

6 An Asymptotic Solution of the Basic Problem

Let us introduce the new variables

$$s = l\xi, \quad 0 \le \xi \le 1; \quad q = Ql^2/C_1; \quad \lambda^2 = \rho\omega^2 l^4/C_1;$$

$$w(s) = \mu l v(\xi), \quad \mu = \max_{s=[0,l]} \left| \frac{w(s)}{l} \right|,$$

$$|v(\xi)| \le 1, \quad \mu \ll 1$$

In such a case the problem (43) – (44) takes the form

$$\left[\frac{1}{\sqrt{1-\mu^2 v'^2}} \left(\frac{v''}{\sqrt{1-\mu^2 v'^2}} \right)' \right]' + q \left(\frac{v'}{\sqrt{1-\mu^2 v'^2}} \right)' - \lambda^2 v = 0,$$
$$f' = df/d\xi \quad (46)$$

$$\xi = 0: \quad v = 0, \; v' = 0,$$
$$\xi = 1: \quad v'' = 0, \; \left(\frac{v''}{\sqrt{1-\mu^2 v'^2}} \right)' + q v' = 0 \quad (47)$$

Let us look for the solution of the problem (46) – (47) in the form of asymptotic series

$$v = \sum_{k=0}^{\infty} \mu^{2k} v_k(\xi), \qquad \lambda^2 = \sum_{k=0}^{\infty} \mu^{2k} \lambda_k \quad (48)$$

Since we look for a solution on the finite interval then we can use the simplest form of asymptotics. Substituting (48) into the equations (46) – (47) we obtain the system for definition of functions $v_k(\xi)$. For the function $v_0(\xi)$ we have a problem

$$v_0^{IV} + q v_0'' - \lambda v_0 = 0, \quad (49)$$

$$\xi = 0: \quad v_0 = v_0' = 0;$$
$$\xi = 1: \quad v_0'' = 0, \quad v_0''' + q v_0' = 0 \quad (50)$$

It is not difficult to prove that all eigenvalues of the problem (49) – (50) are real numbers. For the problem (46) – (47) we are not able to prove such a fact. We have to remember that only nonnegative eigenvalues of the problem (49) – (50) have a physical sense.

7 Conclusions

The nontrivial solutions of the problem (46) – (47) can be named the dynamic forms of equilibrium of the bar compressed by a dead force. The usual forms of equilibrium correspond to the eigenvalue $\lambda^2 = 0$, whereas the dynamic forms correspond to the case $\lambda^2 \neq 0$. The eigenvalues $\lambda^2 \neq 0$ determine the angular velocities of rotation of the bar.

8 Acknowledgements

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References

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