3. A new approach to the analysis of free rotations of rigid bodies

3.1 Introduction

Free rotation of rigid bodies was the first problem that was completely solved in dynamics of rigid bodies. Originally it was studied by Leonard Euler. Later Poinsot offered the famous geometrical interpretation to show real rotation of the body. In modern literature this problem is called the case of the integrability by Euler – Poinsot. Up to now no theoretical adaptations were made to the classical solution, the description of which is presented in all books on dynamics of rigid bodies. The classical solution allows perfectly to find the rotations, i.e. angular velocities, of the body. However, the determination of the turns, i.e. angles, does not impress so much. Moreover, it may be shown that the application of the Euler angles to this problem is not the best way because of several reasons. Firstly, the Euler angles, as a rule, give a representation which is rather difficult for interpretation. Secondly, this representation generates difficulties for the numerical realization on computers. By these reasons it seems to be useful to give an alternative approach to the analysis of the Euler – Poinsot problem, that is based on the concept of the tensor of turn called in the sequel turn-tensor. Some main facts concerning the turn-tensor are presented in the introduction, where the new theorem on the representation of the turn-tensor is given. The theorem allows to simplify the solution of problems of the dynamics of rigid bodies.

In Euler-Poinsot's problem it is not difficult to find four first integrals of the basic equations. Three of them are well known. They express that the angular momentum vector of the body is constant. The fourth integral is that of energy which is directly expressed in terms of turns rather than of angular velocities. The energy integral in such a form allows to construct the most suitable representation of the turn-tensor to make the picture of turns of the body clear. It is found that there exist three and only three different types of rotations. Two of them give stable rotations, and the third type describes an unstable rotation. The type of rotation is determined for the given body by initial conditions only. In fact, the third type of rotation is the separatrix between two stable types of rotations. Under some conditions the stable rotations at certain moments of time can be very close to each other. Thus it is possible for the body to jump from one stable solution to another stable rotation. For example, the body can be rotating around the axis with minimal moment of inertia and there upon it can change the rotation to begin the rotation around the axis with maximal moment of inertia. Of course, small perturbations acting on the body are needed to provoke such a situation. As final result the problem is reduced to the integration of the simple differential equation of first order, the solution of which is a monotonically increasing function. All required quantities can be expressed in term of this function. It is shown how to see the turns of the body without integration of the equations if initial conditions are given.

In the rest part of the introduction certain aspects of the tensor of turn (turn-tensor) will be briefly presented. Partly they are known, some of them seem to be new.

The turn-tensor is the most suitable tool for the description of turns and rotations of rigid bodies (see, for example, Lagally [4]). In spite of this, up to now the turn-tensor has no applications in dynamics of rigid bodies. The main aim of the paper is to show the usefulness of the turn-tensor. As an illustration the classical Euler – Poinsot problem was chosen. This problem is described in all details by many authors (see, for example, Macmillan [6], Goldstein [2], Golubev [3]. So it is possible to see advantages and shortages of the approach, that is based on an application of the turn-tensor. However, it seems to be possible that a new solution of the old problem will be interesting and useful by itself.

Below the direct tensor calculus is used (see Lagally [4] or appendix to the book by Lurie [5]). In those books one can find the initial information about the turn-tensor. From the formal point of view our presentation is rather close to that in the book by Arnold [1].

A properly orthogonal tensor $\mathbf{P}$ is called turn-tensor and can be defined as a solution of the equations

$$\mathbf{P} \cdot \mathbf{P}^{\mathsf{T}} = \mathbf{P}^{\mathsf{T}} \cdot \mathbf{P} = \mathbf{E}, \quad \det \mathbf{P} = +1,$$

where $\mathbf{E}$ is the unit tensor.

Vectors of the position of points A and B of a rigid body will be denoted by $\mathbf{R}_A(t)$ and $\mathbf{R}_B(t)$. The basic equation of the kinematics of a rigid body has the form

$$\mathbf{R}_A(t) = \mathbf{R}_B(t) + \mathbf{P}(t) \cdot (\mathbf{r}_A - \mathbf{r}_B), \quad \mathbf{r} \equiv \mathbf{R}(0), \quad \mathbf{P}(0) = \mathbf{E}\,,$$

where tensor $\mathbf{P}(t)$ is called the turn-tensor of a rigid body.

The turn-tensor $\mathbf{P}(t)$ of the body does not depend on the choice of any points of the body, it describes the turns of the body and can be studied by itself.

Let us remind the **Euler theorem:** *An arbitrary turn-tensor* $\mathbf{P}(t) \neq \mathbf{E}$ *is represented in unique manner in the form*

$$\mathbf{P}(t) = (1 - \cos\varphi(t))\mathbf{m}(t) \otimes \mathbf{m}(t) + \cos\varphi(t)\mathbf{E} + \sin\varphi(t)\mathbf{m} \times \mathbf{E}, \qquad (3.1)$$

where the unit vector $\mathbf{m}(t)$ *is the fixed vector of* $\mathbf{P}(t)$

$$\mathbf{P}(t) \cdot \mathbf{m}(t) = \mathbf{m}(t) \cdot \mathbf{P}(t) = \mathbf{m}(t)\,,$$

the angle of turn, $\varphi(t)$*, is supposed to be positive if the turn goes counter-clockwise.*

The proof of the Euler theorem is based on the theorem of spectral decomposition of unsymmetrical second-rank tensor (see, for example, Goldstein [2]). A much more simple proof can be found in the paper by Zhilin [9]. The fixed vector $\mathbf{m}(t)$ and the angle of turn can be found from the formulae

$$1 + 2\cos\varphi = \mathrm{tr}\mathbf{P}, \quad 2\sin\varphi\,\mathbf{m}(t) = -\mathbf{P}_\times, \quad (\mathbf{a} \otimes \mathbf{b})_\times \equiv \mathbf{a} \times \mathbf{b},$$

where tensor $\mathbf{P}$ on the right-hand side can be taken in any given form. Let us consider the vector of turn, $\boldsymbol{\varphi}(t)$, and the logarithmic tensor of turn, $\mathbf{R}(t)$

$$\boldsymbol{\varphi}(t) = \varphi(t)\mathbf{m}(t), \quad \mathbf{R}(t) = \boldsymbol{\varphi}(t) \times \mathbf{E}.$$

Theorem: *The turn-tensor* $\mathbf{P}(t)$ *can be expressed in the form*

$$\mathbf{P}(t) = \exp\mathbf{R}, \quad \mathbf{R} = \boldsymbol{\varphi}(t) \times \mathbf{E}. \qquad (3.2)$$

The expression (3.2) is one of the forms of Euler's theorem (3.1).

Definition: *The straight line spanned by the fixed vector* $\mathbf{m}(t)$ *of turn-tensor* $\mathbf{P}(t)$ *is called an axis on turn of the body.*

It is seen that there are infinitely many parallel lines and any of them can be called axis of turn. This seems to be strange. Really, if a cylinder is rotating around its own axis, then, namely, this axis appears to be named axis of turn. However, this is impossible, because only the definition given above is in accordance with Galilei's principle of relativity.

Let us consider the left, $\mathbf{S}(t)$, and the right, $\mathbf{S}_r(t)$, tensors of spin ($\dot{\varphi} \equiv d\varphi/dt$)

$$\mathbf{S}(t) \equiv \dot{\mathbf{P}}(t) \cdot \mathbf{P}^T(t), \quad \mathbf{S}_r(t) \equiv \mathbf{P}^T(t) \cdot \dot{\mathbf{P}}(t), \quad (\mathbf{S} = \mathbf{P} \cdot \mathbf{S}_r \cdot \mathbf{P}^T).$$

Spin-tensors $\mathbf{S}$ and $\mathbf{S}_r$ are antisymmetric tensors and have accompanying vectors.

Definition: *The accompanying vector* $\boldsymbol{\omega}(t)$ *(*$\boldsymbol{\Omega}(t)$*) of the left (right) spin-tensor* $\mathbf{S}(t)$ *(*$\mathbf{S}_r(t)$*) is called the left (right) vector of angular velocity*

$$\mathbf{S}(t) \equiv \dot{\mathbf{P}}(t) \cdot \mathbf{P}^T(t) = \boldsymbol{\omega}(t) \times \mathbf{E}, \quad (\mathbf{S}_r \equiv \mathbf{P}^T \cdot \dot{\mathbf{P}} = \boldsymbol{\Omega} \times \mathbf{E} = \mathbf{E} \times \boldsymbol{\Omega}).$$

These equations can be rewritten in equivalent form:

$$\dot{\mathbf{P}}(t) = \boldsymbol{\omega}(t) \times \mathbf{P}(t), \quad \dot{\mathbf{P}}(t) = \mathbf{P}(t) \times \boldsymbol{\Omega}(t). \tag{3.3}$$

The first of these equations is called the left Poisson equation, and the second one the right Poisson equation. There exists the relation

$$\boldsymbol{\omega}(t) = \mathbf{P}(t) \cdot \boldsymbol{\Omega}, \quad \Rightarrow \quad \boldsymbol{\Omega} = \mathbf{P}^{T} \cdot \boldsymbol{\omega}(t). \tag{3.4}$$

In dynamics of rigid bodies the vector $\boldsymbol{\omega}(t)$ is called angular velocity in the space, whereas the vector $\boldsymbol{\Omega}(t)$ is called angular velocity in the body. Of course, both of them are vectors in the space. If the turn-tensor $\mathbf{P}(t)$ is known, then it is easy to find vectors $\boldsymbol{\omega}(t)$ and $\boldsymbol{\Omega}(t)$ in the form

$$\boldsymbol{\omega}(t) = -\frac{1}{2}(\dot{\mathbf{P}} \cdot \mathbf{P}^{T})_{\times}, \qquad \boldsymbol{\Omega}(t) = -\frac{1}{2}(\mathbf{P}^{T} \cdot \dot{\mathbf{P}})_{\times}. \tag{3.5}$$

The inverse problem, i.e. the determination of the turn-tensor, when angular velocities are known, is called the Darboux problem. If the left vector of angular velocity is known we get *the left Darboux problem*

$$\dot{\mathbf{P}}(t) = \boldsymbol{\omega}(t) \times \mathbf{P}(t), \quad \mathbf{P}(0) = \mathbf{P}_0, \quad \mathbf{P}_0 \cdot \mathbf{P}_0^{T} = \mathbf{E}, \quad \det \mathbf{P}_0 = 1.$$

If the right vector of angular velocity is known, we get *the right Darboux problem*

$$\dot{\mathbf{P}}(t) = \mathbf{P}(t) \times \boldsymbol{\Omega}(t), \quad \mathbf{P}(0) = \mathbf{P}_0, \quad \mathbf{P}_0 \cdot \mathbf{P}_0^{T} = \mathbf{E}, \quad \det \mathbf{P}_0 = 1.$$

How to solve the Darboux problem is known, however it is not simple task in a general case. So it will be much better to avoid the solution of the problem in full extent. In practice it is possible.

Proposition: *Let the tensor $\mathbf{P}_l(t)$ (the tensor $\mathbf{P}_r(t)$) be a particular solution of the left (right) Poisson equation (3.3). Then the general solution of the left (right) Darboux problem has the form*

$$\mathbf{P}(t) = \mathbf{P}_l(t) \cdot \mathbf{P}_l^{T}(0) \cdot \mathbf{P}_0, \quad \left(\mathbf{P}(t) = \mathbf{P}_0 \cdot \mathbf{P}_r^{T}(0) \cdot \mathbf{P}_r(t)\right).$$

Sometimes this fact can help to find the general solution.

Making use of Euler's theorem (3.1) and the first expression of (3.5) it is easy to get

$$\boldsymbol{\omega}(t) = \dot{\varphi}\mathbf{m}(t) + \sin\varphi(t)\dot{\mathbf{m}}(t) + (1 - \cos\varphi)\mathbf{m} \times \dot{\mathbf{m}}(t).$$

In many books on mechanics this expression is presented in a wrong form (see, for example, Golubev [3])

$$\boldsymbol{\omega} = \dot{\varphi}\mathbf{m}(t). \tag{3.6}$$

Definition: *The straight line spanned on the vector of angular velocity* $\omega(t)$ *is called an axis of rotation of the body.*

Theorem:

— *If the fixed vector of the turn-tensor does not depend on time, then the axis of turn coincides with the axis of rotation*;

— If vector $\omega(t)$ has constant direction, then the axis of turn coincides with axis of rotation if, and only if the vector ω is the fixed vector of turn-tensor $\mathbf{P}_0 = \mathbf{P}(0)$.

Therefore, it is well possible that the body turns around one axis, but at the same time the body rotates around another axis, which, for example, is orthogonal to the axis of turn. So when working with interpretation by Poinsot one must be very careful to avoid mistakes.

Let us consider the composition of turns, which are defined by turn-tensors $\mathbf{P}_2(t)$ and $\mathbf{P}_1(t)$

$$\mathbf{P}(t) = \mathbf{P}_2(t) \cdot \mathbf{P}_1(t). \tag{3.7}$$

Let the angular velocities $\omega(t)$, $\omega_1(t)$ and $\omega_2(t)$ correspond to the turn-tensors $\mathbf{P}(t)$, $\mathbf{P}_1(t)$ and $\mathbf{P}_2(t)$, respectively,

$$\dot{\mathbf{P}}(t) = \omega(t) \times \mathbf{P}(t),\ \dot{\mathbf{P}}_1(t) = \omega_1(t) \times \mathbf{P}_1(t),\ \dot{\mathbf{P}}_2(t) = \omega_2(t) \times \mathbf{P}_2(t). \tag{3.8}$$

The following identity will be useful:

$$(\mathbf{A} \cdot \mathbf{a}) \times (\mathbf{A} \cdot \mathbf{b}) = (\det \mathbf{A})(\mathbf{A}^{-\mathsf{T}}) \cdot (\mathbf{a} \times \mathbf{b}) \tag{3.9}$$

which is valid for any nonsingular tensor $\mathbf{A}$ and any vectors $\mathbf{a}$ and $\mathbf{b}$. If tensor $\mathbf{A}$ is a turn-tensor $\mathbf{Q}$, then identity (3.9) can be simplified as follows:

$$\mathbf{Q} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{Q} \cdot \mathbf{a}) \times (\mathbf{Q} \cdot \mathbf{b}) \Rightarrow \mathbf{Q} \cdot (\mathbf{a} \times \mathbf{E}) \cdot \mathbf{Q}^{\mathsf{T}} = (\mathbf{Q} \cdot \mathbf{a}) \times \mathbf{E}. \tag{3.10}$$

Theorem: *The left vector of an angular velocity of the composition (3.7) can be expressed in term of vectors (3.8) as follows:*

$$\omega(t) = \omega_2(t) + \mathbf{P}_2(t) \cdot \omega_1(t). \tag{3.11}$$

The proof follows directly from (3.7):

$$\begin{aligned}\dot{\mathbf{P}} = \dot{\mathbf{P}}_2 \cdot \mathbf{P}_1 + \mathbf{P}_2 \cdot \dot{\mathbf{P}}_1 \quad \Rightarrow \quad \omega \times \mathbf{P} = \omega_2 \times \mathbf{P} + \mathbf{P}_2 \cdot (\omega_1 \times \mathbf{P}_1) \quad \Rightarrow \\ \omega \times \mathbf{E} = \omega_2 \times \mathbf{E} + \mathbf{P}_2 \cdot (\omega_1 \times \mathbf{E}) \cdot \mathbf{P}_2^{\mathsf{T}}.\end{aligned}$$

Making use of the second identity (3.10) this equality can be rewritten in the form (3.11).

In textbooks on theoretical mechanics (see Golubev [3]) the equality (1.21) takes the form

$$\omega(t) = \omega_2(t) + \omega_1(t). \tag{3.12}$$

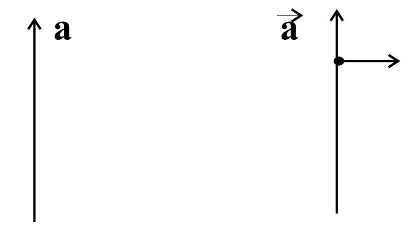

Figure 1: Vector and double vector

It is difficult to say who was the author of the formulae (3.6) and (3.12). In any case they were known at the end of the first quarter of this century.

Let us accept the notation

$$\mathbf{Q}(\varphi\mathbf{e}) \equiv (1-\cos\varphi)\mathbf{e}\otimes\mathbf{e} + \cos\varphi\mathbf{E} + \sin\varphi\mathbf{e}\times\mathbf{E} \tag{3.13}$$

for a turn on the angle φ around the vector $\mathbf{e}$: $|\mathbf{e}| = 1$. For any turn-tensor $\mathbf{T}$ the formula

$$\mathbf{T}\cdot\mathbf{Q}(\varphi\mathbf{e})\cdot\mathbf{T}^{T} = \mathbf{Q}(\varphi\mathbf{e}'), \quad \mathbf{e}' \equiv \mathbf{T}\cdot\mathbf{e} \tag{3.14}$$

is valid. It can be readily proved, that

$$\mathbf{Q}(\varphi\mathbf{m})\cdot\mathbf{Q}(\psi\mathbf{n}) = \mathbf{Q}(\psi\mathbf{n}')\cdot\mathbf{Q}(\varphi\mathbf{m}), \quad \mathbf{n}' \equiv \mathbf{Q}(\varphi\mathbf{m})\cdot\mathbf{n}.$$

This rule of the pseudo-permutability is important in many cases.

The main merit of a turn-tensor is that it can be represented in many equivalent forms, a choice of which depends on special properties of the problem under consideration. For example, Euler's theorem (3.1) is efficient if the axis of a turn is a priori known and does not change in time. If it is not so then Euler's theorem is practically useless. However, in the latter case one can represent the general turn as the composition of the simplest turn-tensors of the form (3.13), where vector $\mathbf{e}$ is a constant vector.

Let there be given a turn-tensor $Q(\varphi e)$, where $\mathbf{e} = \text{const}$. The left and the right vectors of angular velocity in such a case coincide and can be found by making use of the simplest formula (3.6),

$$\boldsymbol{\omega}(t) = \boldsymbol{\Omega}(t) = \dot{\varphi}\mathbf{e}, \quad \mathbf{Q}(\varphi\mathbf{e})\cdot\boldsymbol{\omega} = \boldsymbol{\omega}.$$

In order to study the rotations of the body it is useful to take the double vector into consideration, what can be done as follows. Let us take the vector $\mathbf{a}$, which will be called a basis. Let us connect to the vector $\mathbf{a}$ another vector $\mathbf{b}$, which will be called a cross-vector (see Fig.1). The rigid construction of two vectors $\mathbf{a}$ and $\mathbf{b}$ will be called a double vector and denoted as $\overrightarrow{\mathbf{a}}$. Let us introduce the double material "vector" $\overrightarrow{\mathbf{AB}}$ which is made from the points of the body. The position of the "vector" $\overrightarrow{\mathbf{AB}}$ determines the position of the body in a unique manner. Thus it is possible to observe the movement of the material "vector" $\overrightarrow{\mathbf{AB}}$ in order to know the movement of the body. The basis of the double vector $\overrightarrow{\mathbf{AB}}$ will be denoted by **AB**, where A and B are points of the body.

Let us formulate the **representation theorem of the turn-tensor:** *Let there be given two arbitrary unit vectors* **m** *and* **n**. *Any turn-tensor can be represented in the form of a composition of turns around vectors* **m** *and* **n***:*

$$\mathbf{P}(t) = \mathbf{Q}\left(\psi(t)\mathbf{m}\right) \cdot \mathbf{Q}\left(\vartheta(t)\mathbf{e}\right) \cdot \mathbf{Q}\left(\varphi(t)\mathbf{n}\right), \quad \mathbf{e} = \mathbf{m} \times \mathbf{n}/|\mathbf{m} \times \mathbf{n}|, \qquad (3.15)$$

where $\psi(t)$, $\vartheta(t)$, *and* $\varphi(t)$ *are called angle of precession, angle of a nutation, and angle of own rotation, respectively. If the vector* **m** *coincides with the vector* **n**, *then a vector* **e** *is any unit vector orthogonal to the vector* **m**; *in that case the angles* φ, ψ, *and* ϑ *are called Eulerian angles.*

Proof: Let the vectors **m** and **n** be placed in the plane of Fig.2.

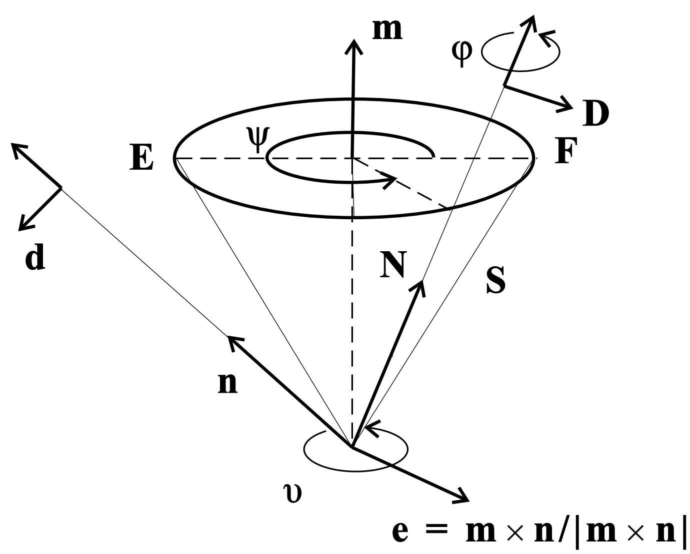

Figure 2: Representation of the turn-tensor

Let us choose the double vector $\overrightarrow{\mathbf{d}}$ and the double material vector $\overrightarrow{\mathbf{AB}}$ in such a manner, that at the moment of time $t = 0$ the vector $\overrightarrow{\mathbf{AB}}$ coincides with the double vector $\overrightarrow{\mathbf{d}}$ and in addition the basis **d** of the double vector $\overrightarrow{\mathbf{d}}$ coincides with a unit vector **n**: $\mathbf{d} = \mathbf{n}$. At the instant $t > 0$ the double material vector $\overrightarrow{\mathbf{AB}}$ coincides with the double vector $\overrightarrow{\mathbf{D}}(t)$. The conic surface S on Fig.2 is made up by a rotation of the basis **D** of the double vector $\overrightarrow{\mathbf{D}}$ around the unit vector **m**. The turn-tensor $\mathbf{P}(t)$ turns the double material vector $\overrightarrow{\mathbf{AB}}$ in such a way, that at the instant $t = 0$ the vector $\overrightarrow{\mathbf{AB}}$ coincides with the double vector $\overrightarrow{\mathbf{d}}$ and at the moment of time $t > 0$ the vector $\overrightarrow{\mathbf{AB}}$ coincides with the double vector $\overrightarrow{\mathbf{D}}(t)$. We want to construct the turn-tensor $\mathbf{P}(t)$ as a composition of the simplest turns around the axis **m** and **n**. Let us accept, that $\mathbf{P}(0) = \mathbf{E}$.

The first turn is by the angle of nutation $\vartheta(t)$ around the unit vector **e** (see Fig.2),

$$\mathbf{Q}(\vartheta\mathbf{e}) = (1-\cos\vartheta)\mathbf{e}\otimes\mathbf{e} + \cos\vartheta\mathbf{E} + \sin\vartheta\,\mathbf{e}\times\mathbf{E}. \tag{3.16}$$

As a result of this turn the basis **AB** of the double material vector $\overrightarrow{\mathbf{AB}}$ will be placed on the surface S and the vector **AB** coincides with the vector $\mathbf{d}'$,

$$\mathbf{d}' = \mathbf{Q}(\vartheta\mathbf{e})\cdot\mathbf{d} = \mathbf{Q}(\vartheta\mathbf{e})\cdot\mathbf{n} \equiv \mathbf{n}'.$$

The second turn is by the angle of precession ψ around the axis **m**:

$$\mathbf{Q}(\psi\mathbf{m}) = (1-\cos\psi)\mathbf{m}\otimes\mathbf{m} + \cos\psi\mathbf{E} + \sin\psi\,\mathbf{m}\times\mathbf{E}.$$

In this case the basis **AB** of the double material vector $\overrightarrow{\mathbf{AB}}$ is sliding on the surface S (see Fig.2) and after all it coincides with the basis **D** of the double vector $\overrightarrow{\mathbf{D}}$. It is clear that

$$\mathbf{D}(t) = \mathbf{Q}(\psi\mathbf{m})\cdot\mathbf{n}' = \mathbf{Q}(\psi\mathbf{m})\cdot\mathbf{Q}(\vartheta\mathbf{e})\cdot\mathbf{n}. \tag{3.17}$$

Finally, the third turn is by the angle of own rotation φ around the basis **D** of the double vector $\overrightarrow{\mathbf{D}}$,

$$\mathbf{Q}(\varphi\mathbf{D}) = (1-\cos\varphi)\mathbf{D}\otimes\mathbf{D} + \cos\varphi\mathbf{E} + \sin\varphi\,\mathbf{D}\times\mathbf{E}.$$

After that turn the cross-vector of the double material vector $\overrightarrow{\mathbf{AB}}$ coincides with the cross-vector of the double vector $\overrightarrow{\mathbf{D}}$ and therefore the vector $\overrightarrow{\mathbf{AB}}$ coincides with the double vector $\overrightarrow{\mathbf{D}}$. Now we get the total turn-tensor $\mathbf{P}(t)$ in the form

$$\mathbf{P}(t) = \mathbf{Q}(\varphi\mathbf{D})\cdot\mathbf{Q}(\psi\mathbf{m})\cdot\mathbf{Q}(\vartheta\mathbf{e}). \tag{3.18}$$

It seems that this representation does not coincide with the equation (3.15). However, (3.18) can be written in the form (3.15). Indeed, making use of (3.14) and (3.17) we can write down

$$\mathbf{P}(t) = \underbrace{\mathbf{Q}(\psi\mathbf{m})\cdot\mathbf{Q}(\vartheta\mathbf{e})\cdot\mathbf{Q}(\varphi\mathbf{n})\cdot\mathbf{Q}^{\mathsf{T}}(\vartheta\mathbf{e})\cdot\mathbf{Q}^{\mathsf{T}}(\psi\mathbf{m})}_{\mathbf{Q}(\varphi\mathbf{D})}\cdot\mathbf{Q}(\psi\mathbf{m})\cdot\mathbf{Q}(\vartheta\mathbf{e}) =$$
$$= \mathbf{Q}(\psi\mathbf{m})\cdot\mathbf{Q}(\vartheta\mathbf{e})\cdot\mathbf{Q}(\varphi\mathbf{n}).$$

This expression is identical to (3.15). The turn-tensor (3.18) can be taken in another form,

$$\mathbf{P}(t) = \mathbf{Q}(\varphi\mathbf{D})\cdot\mathbf{Q}(\psi\mathbf{m})\cdot\mathbf{Q}(\vartheta\mathbf{e})\cdot\underbrace{\mathbf{Q}^{\mathsf{T}}(\psi\mathbf{m})\cdot\mathbf{Q}(\psi\mathbf{m})}_{\mathbf{E}} =$$
$$= \mathbf{Q}(\varphi\mathbf{D})\cdot\mathbf{Q}(\vartheta\mathbf{e}')\cdot\mathbf{Q}(\psi\mathbf{m}), \tag{3.19}$$

where $\mathbf{e}' = \mathbf{Q}(\psi\mathbf{m}) \cdot \mathbf{e}$.

Accepting $\mathbf{m} = \mathbf{n}$ in (3.19) we have the classical representation of the turn-tensor expressed in terms of Eulerian angles. The full proof of the latter case can be found in the paper by Zhilin [9].

Remark: *As it can be readily seen the angles ψ, ϑ, φ are not uniquely defined. For example, the turn on the angle ϑ in (3.16) can be positive $(\vartheta > 0)$ or negative one $(\vartheta < 0)$. Of course, the value of the angle ϑ will be different in these cases. As a matter of fact this ambiguity is of fundamental importance. Really, the direction of the rotation is determined by initial conditions of the problem under consideration. Thus the representation of the turn-tensor must admit the freedom of choice of the direction of the rotation.*

The kinematic Euler equation immediately follows from representations (3.15) and the theorem (3.11),

$$\boldsymbol{\omega}(t) = \dot{\psi}\mathbf{m} + \dot{\vartheta}\mathbf{Q}(\psi\mathbf{m}) \cdot \mathbf{e} + \dot{\varphi}\mathbf{Q}(\psi\mathbf{m}) \cdot \mathbf{Q}(\vartheta\mathbf{e}) \cdot \mathbf{n} = \dot{\psi}\mathbf{m} + \dot{\vartheta}\mathbf{e}' + \dot{\varphi}\mathbf{D}, \quad (3.20)$$

since the vectors $\mathbf{m}$, $\mathbf{n}$, $\mathbf{e}$ are constant. For the right vector of angular velocity we have the result

$$\boldsymbol{\Omega}(t) = \mathbf{P}^{\mathsf{T}} \cdot \boldsymbol{\omega}(t) = \dot{\varphi}\mathbf{n} + \dot{\vartheta}\mathbf{Q}^{\mathsf{T}}(\varphi\mathbf{n}) \cdot \mathbf{e} + \dot{\psi}\mathbf{Q}^{\mathsf{T}}(\varphi\mathbf{n}) \cdot \mathbf{Q}^{\mathsf{T}}(\vartheta\mathbf{e}) \cdot \mathbf{m}.$$

It is curious, that the result (3.20) can be found by a wrong way. Let us take the representation of a turn-tensor in the form (3.19) and use the formulae (3.6) and (3.12). We shall get the correct result (3.20). This is so because of a superposition of two mistakes. Let us remove one of the mistakes. Let us take the representations of a turn-tensor in the form (3.15). In this case the formula (3.6) is correct for the turn-tensors $\mathbf{Q}(\psi\mathbf{m})$, $\mathbf{Q}(\vartheta\mathbf{e})$, and $\mathbf{Q}(\varphi\mathbf{n})$. The angular velocity $\boldsymbol{\omega}(t)$ calculated through the formula (1.22) is

$$\boldsymbol{\omega}(t) = \dot{\psi}\mathbf{m} + \dot{\vartheta}\mathbf{e} + \dot{\varphi}\mathbf{n},$$

which is obviously wrong. The formulae (3.6) and (3.12) are very popular in the textbooks on theoretical mechanics. However, it does not matter since when solving the problems the correct result (3.20) is used. It should be mentioned that in the books on dynamics of rigid bodies the particular cases of (3.20) are used. The theorem (3.15) plays an important role in solving of many problems. The vectors $\mathbf{m}$ and $\mathbf{n}$ are supposed to be unknown at the beginning. They must be chosen in the process of solving in order to find the simplest form of the solution. We have to point out that the use of Eulerian angles, as a general rule, leads to inconvenient solutions.

3.2 Free rotation of rigid bodies

This classical problem allows to see in all details how the turn-tensor is working. The solution given below seems to be too long. However, it can be seen that the large part of a solution is devoted to an answer on the question why we have to choose a turn-tensor in the offered form.

3.2.1 Statement of the problem

Let us consider a rigid body the centre of mass of which rests in some inertial frame of reference. External forces and moments do not act on the body. The position of a body at the instant $t = 0$ will be called a reference position. The position of a body at the given instant $t > 0$ will be called an actual position. The latter can be determined by a turn-tensor $\mathbf{P}(t)$: $\mathbf{P}(0) = \mathbf{E}$. The central tensor of inertia of a body at $t = 0$ is specified by its spectral representation,

$$\boldsymbol{\Theta} = \theta_1 \mathbf{d}_1 \otimes \mathbf{d}_1 + \theta_2 \mathbf{d}_2 \otimes \mathbf{d}_2 + \theta_3 \mathbf{d}_3 \otimes \mathbf{d}_3,$$

where the quantities

$$0 < \theta_1 \leq \theta_2 \leq \theta_3, \quad \theta_1 < \theta_3, \quad \theta_3 \leq \theta_1 + \theta_2$$

are called principle moments of inertia, the vectors $\mathbf{d}_k$ are eigenvectors of the tensor $\boldsymbol{\Theta}$ at $t = 0$. In the actual position the central tensor of inertia can be represented in the form

$$\boldsymbol{\Theta}^{(t)} = \mathbf{P}(t) \cdot \boldsymbol{\Theta} \cdot \mathbf{P}^{\mathsf{T}}(t) = \sum_{k=1}^{3} \theta_k \mathbf{D}_k \otimes \mathbf{D}_k, \quad \mathbf{D}_k = \mathbf{P}(t) \cdot \mathbf{d}_k.$$

The kinetic moment (moment of momentum, angular momentum) $\mathbf{L}$ of a body and its kinetic energy K are defined by the formulae

$$\mathbf{L}(t) = \mathbf{P}(t) \cdot \boldsymbol{\Theta} \cdot \mathbf{P}^{\mathsf{T}}(t) \cdot \boldsymbol{\omega}(t); \quad 2K = \boldsymbol{\omega} \cdot \mathbf{P}(t) \cdot \boldsymbol{\Theta} \cdot \mathbf{P}(t) \cdot \boldsymbol{\omega} = \boldsymbol{\omega}(t) \cdot \mathbf{L}(t),$$

where the vector $\boldsymbol{\omega}(t)$ is the left vector of angular velocity. Making use of the right vector of angular velocity, $\boldsymbol{\Omega}(t)$, we can write down

$$\mathbf{L}(t) = \mathbf{P}(t) \cdot \boldsymbol{\Theta} \cdot \boldsymbol{\Omega}(t); \quad h \equiv 2K = \boldsymbol{\Omega}(t) \cdot \boldsymbol{\Theta} \cdot \boldsymbol{\Omega}(t).$$

Euler's second law of dynamics gives

$$\dot{\mathbf{L}} = \mathbf{0} \quad \Rightarrow \quad \mathbf{L} = \text{const} \quad \Rightarrow \quad \mathbf{P} \cdot \boldsymbol{\Theta} \cdot \mathbf{P}^{\mathsf{T}} \cdot \boldsymbol{\omega} = \mathbf{L} = \text{const}. \tag{3.21}$$

Thus we have three first integrals. Usually they are not used since they contain the turns in addition to angular velocity. However, we will work with these integrals. The equation (3.21) can be rewritten in the form

$$\boldsymbol{\omega}(t) = \mathbf{P}(t) \cdot \boldsymbol{\Theta}^{-1} \cdot \mathbf{P}^{\mathsf{T}}(t) \cdot \mathbf{L}, \quad \mathbf{L} = \text{const}, \tag{3.22}$$

where $\boldsymbol{\omega}(t)$ and $\mathbf{P}(t)$ are related by the left Poisson equation (3.3). Making use of the representation theorem of the turn-tensor (3.15) we can look for a tensor $\mathbf{P}(t)$ in the form

$$\mathbf{P}(t) = \mathbf{Q}(\psi\mathbf{p}) \cdot \mathbf{Q}(\vartheta\mathbf{e}) \cdot \mathbf{Q}(\varphi\mathbf{n}), \quad \mathbf{e} = (\mathbf{p} \times \mathbf{n})/|\mathbf{p} \times \mathbf{n}|, \tag{3.23}$$

where the unit vectors $\mathbf{p}$ and $\mathbf{n}$ are not a priory known, but they do not depend on time. The vector $\boldsymbol{\omega}(t)$ corresponding to the tensor (3.23) is defined by

$$\boldsymbol{\omega}(t) = \dot{\psi}\mathbf{p} + \dot{\vartheta}\mathbf{Q}(\psi\mathbf{p}) \cdot \mathbf{e} + \dot{\varphi}\mathbf{Q}(\psi\mathbf{p}) \cdot \mathbf{Q}(\vartheta\mathbf{e}) \cdot \mathbf{n}. \tag{3.24}$$

After substituting expressions (3.23) and (3.24) into the equation (3.22) we will get an equation for finding the angles ψ, ϑ, and φ. In order to find the angles ψ, ϑ and φ we have to choose unit vectors $\mathbf{p}$ and $\mathbf{n}$. This choice is important to arrive at a solution of simplest form. Thus, first of all we have to point out unit vectors $\mathbf{p}$ and $\mathbf{n}$. For this let us note that from the equation (3.22) follows

$$h \equiv 2K = \mathbf{L} \cdot \boldsymbol{\omega} = \mathbf{L} \cdot \mathbf{P}(t) \cdot \boldsymbol{\Theta}^{-1} \cdot \mathbf{P}^{\mathsf{T}}(t) \cdot \mathbf{L}. \tag{3.25}$$

Let us show that the kinetic energy has a constant value, i.e. the quantity h, determined by (3.25), is the first integral of the equation (3.22). We have

$$\begin{aligned}
\dot{h} = \mathbf{L} \cdot \dot{\mathbf{P}} \cdot \boldsymbol{\Theta}^{-1} \cdot \mathbf{P}^{\mathsf{T}} \cdot \mathbf{L} + \mathbf{L} \cdot \mathbf{P} \cdot \boldsymbol{\Theta}^{-1} \cdot \dot{\mathbf{P}}^{\mathsf{T}} \cdot \mathbf{L} = \\
= \mathbf{L} \cdot (\boldsymbol{\omega} \times \mathbf{P}) \cdot \boldsymbol{\Theta}^{-1} \cdot \mathbf{P}^{\mathsf{T}} \cdot \mathbf{L} - \mathbf{L} \cdot \mathbf{P} \cdot \boldsymbol{\Theta}^{-1} \cdot (\mathbf{P}^{\mathsf{T}} \times \boldsymbol{\omega}) \cdot \mathbf{L} = \\
= \mathbf{L} \cdot (\boldsymbol{\omega} \times \boldsymbol{\omega}) - (\boldsymbol{\omega} \times \boldsymbol{\omega}) \cdot \mathbf{L} = 0 \quad \Rightarrow \quad h = \text{const}.
\end{aligned}$$

The energy integral (3.25) contains the turns only. In such a form this integral was never used. However, especially this form of an energy integral allows to find the most suitable form of a turn-tensor. In addition, the integral (3.25) has a clear geometrical sense: free rotations of a body go by such a manner, that the inverse central moment of inertia of a body with respect to the axis spanned on the vector $\mathbf{L}$ and passing through the centre of mass has constant magnitude.

3.2.2 Transformation of the energy integral

In general the turn-tensor can be expressed through a set of three parameters. The energy integral gives the relation, superposed on these parameters. Therefore, only two of them are independent variables, and free rotations of the body are two-parameter movements. Thus it is necessary to find the general form of a two-parameter turn-tensor conserving the energy. Let us introduce the unit vector $\mathbf{m}(t)$

$$l\mathbf{m}(t) \equiv \mathbf{L} \cdot \mathbf{P}(t) = \mathbf{P}^T(t) \cdot \mathbf{L} = \mathbf{\Theta} \cdot \mathbf{\Omega}(t), \quad l \equiv |\mathbf{L}| > 0. \tag{3.26}$$

The energy integral can be rewritten in the form

$$\mathbf{m}(t) \cdot \mathbf{\Theta}^{-1} \cdot \mathbf{m}(t) = \frac{h}{l^2} \equiv \frac{h}{l^2}\mathbf{m} \cdot \mathbf{E} \cdot \mathbf{m} \quad \Rightarrow$$
$$\mathbf{m}(t) \cdot \left(\mathbf{\Theta}^{-1} - \frac{h}{l^2}\mathbf{E}\right) \cdot \mathbf{m}(t) = 0. \tag{3.27}$$

The equation (3.27) determines the bundle of the straight lines passing through the centre of mass of the body. If these straight lines are known then we are able to establish the structure of the turn-tensor by making use of (3.26). The geometrical sense of these straight lines is obvious: it is the set of those axes, passing through the centre of mass, relative to which the inverse moments of inertia of the body are the same. The bundle of the straight lines (3.27) is the ruled surface fixed in the space. It is easy to point out the equation of this surface. The vector $\mathbf{m}(t)$ can be represented as

$$\mathbf{m}(t) = x_m\mathbf{d}_1 + y_m\mathbf{d}_2 + z_m\mathbf{d}_3, \quad x_m^2 + y_m^2 + z_m^2 = 1, \tag{3.28}$$

where x, y, z are the axes of coordinates spanned by the vectors $\mathbf{d}_k$ that are fixed in the space. Let us introduce the spectral representation

$$\mathbf{\Theta}^{-1} - \frac{h}{l^2}\mathbf{E} = \left(\frac{1}{\theta_1} - \frac{h}{l^2}\right)\mathbf{d}_1 \otimes \mathbf{d}_1 + \left(\frac{1}{\theta_2} - \frac{h}{l^2}\right)\mathbf{d}_2 \otimes \mathbf{d}_2 + \left(\frac{1}{\theta_3} - \frac{h}{l^2}\right)\mathbf{d}_3 \otimes \mathbf{d}_3. \tag{3.29}$$

Let us suppose that the inequalities

$$l^2 - h\theta_1 > 0, \quad l^2 - h\theta_3 < 0 \tag{3.30}$$

are valid. It means that we eliminate the permanent rotations from the analysis for the sake of brevity. Making use of (3.28) – (3.30) the equation (3.27) can be rewritten as

$$\left(\frac{h}{l^2} - \frac{1}{\theta_3}\right)z_m^2 = \left(\frac{1}{\theta_1} - \frac{h}{l^2}\right)x_m^2 + \left(\frac{1}{\theta_2} - \frac{h}{l^2}\right)y_m^2. \tag{3.31}$$

The type of the surface (3.31) essentially depends on the signs of coefficients in the equation (3.31). The coefficients $h\theta_3 - l^2$ and $l^2 - h\theta_1$ are always positive. The sign of the coefficient at the variable y_m^2,

$$\sigma \equiv l^2 - \theta_2 h, \tag{3.32}$$

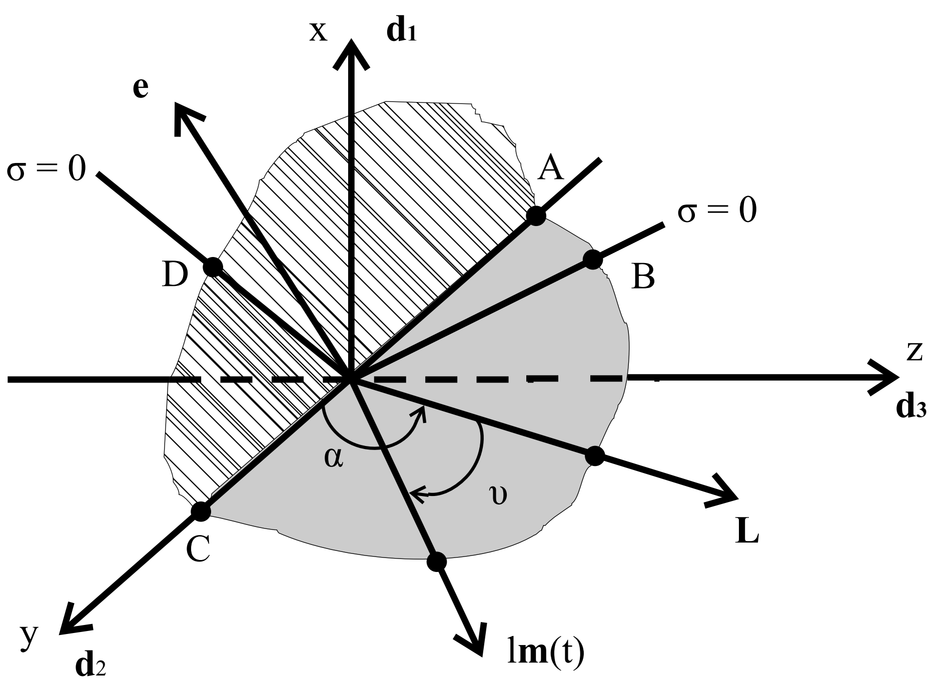

Figure 3: The case $\sigma > 0$

depends on initial conditions and can be different. There exist three different cases,

$$a)\ \sigma = 0, \quad b)\ \sigma > 0, \quad c)\ \sigma < 0. \tag{3.33}$$

These cases must be studied separately. The parameter σ defined by (3.32) can be represented in the form

$$\sigma = \left(1 - \frac{\theta_2}{\theta_3}\right) l_3^2 - \left(\frac{\theta_2}{\theta_1} - 1\right) l_1^2, \quad l_k \equiv \mathbf{L} \cdot \mathbf{d}_k. \tag{3.34}$$

As it will be seen below, rotations of a body in cases (3.33) essentially differ from each other.

3.2.3 Rotations of a body in the case $\sigma = 0$

If the parameter σ is equal to zero, then the surface (3.31) decomposes in two planes:

$$x_m = \sqrt{\frac{\theta_1}{\theta_3}\,\frac{\theta_3 h - l^2}{l^2 - \theta_1 h}}\, z_m \equiv \sqrt{\frac{\theta_1}{\theta_3}\,\frac{\theta_3 - \theta_2}{\theta_2 - \theta_1}}\, z_m; \quad x_m = -\sqrt{\frac{\theta_1}{\theta_3}\,\frac{\theta_3 - \theta_2}{\theta_2 - \theta_1}}\, z_m. \tag{3.35}$$

These planes are the characteristics of a body. The vector $\mathbf{m}(t)$ must belong to one of these planes in any instant — see Fig.3, where the parts of the planes (3.35) are shown. The vector $\mathbf{L}$ is supposed to belong to the first of planes (3.35). At the instant $t > 0$ the vector $\mathbf{m}(t)$ is a result of the turn of the vector $\mathbf{L}$ by the turn-tensor

$\mathbf{P}^{\mathsf{T}}(t)$. It is obvious, that the vector $\mathbf{L}$ must be turned around the normal $\mathbf{e}$ to the plane in which it is placed. The normal vector $\mathbf{e}$ can be found as

$$\mathbf{e} = (\mathbf{d}_2 \times \mathbf{L})/|\mathbf{d}_2 \times \mathbf{L}| \equiv (\mathbf{d}_2 \times \mathbf{L})/\sqrt{l^2 - l_2^2}.$$

The turn by the angle ϑ around a vector $\mathbf{e}$ is determined by the turn-tensor $\mathbf{Q}(\vartheta\mathbf{e})$, where the notation (3.13) is used. It is always possible to add to this turn the turn around the vector $\hat{\mathbf{L}} \equiv \mathbf{L}/l$, since $\mathbf{Q}(\psi\hat{\mathbf{L}}) \cdot \mathbf{L} = \mathbf{L}$. Thus, if the parameter σ is equal to zero, then the turn-tensor is forced to admit the representation

$$\mathbf{P}(t) = \mathbf{Q}(\psi\hat{\mathbf{L}}) \cdot \mathbf{Q}(\vartheta\mathbf{e}), \quad \mathbf{e} = \mathbf{d}_2 \times \mathbf{L}/\sqrt{l^2 - l_2^2}, \quad \hat{\mathbf{L}} \equiv \mathbf{L}/l. \tag{3.36}$$

The vector $\mathbf{m}(t)$ takes the form

$$l\mathbf{m}(t) = \mathbf{P}^{\mathsf{T}} \cdot \mathbf{L} = \mathbf{Q}^{\mathsf{T}}(\vartheta\mathbf{e}) \cdot \mathbf{L} = \frac{1}{\sin\alpha}\left(l \sin\vartheta\, \mathbf{d}_2 + \sin(\alpha - \vartheta)\mathbf{L}\right), \tag{3.37}$$

where the notations

$$\sin\alpha = \sqrt{l^2 - l_2^2}\,/\,l, \quad \cos\alpha = l_2/l$$

are used — see Fig.3.

The left vector of an angular velocity corresponding to the turn-tensor (3.36) is calculated in accordance with (3.11)

$$\boldsymbol{\omega}(t) = \dot{\psi}\hat{\mathbf{L}} + \dot{\vartheta}\mathbf{Q}(\psi\hat{\mathbf{L}}) \cdot \mathbf{e}.$$

Making use of this expression the equation (3.22) can be written in the form

$$\dot{\psi}\hat{\mathbf{L}} + \dot{\vartheta}\mathbf{Q}(\psi\hat{\mathbf{L}}) \cdot \mathbf{e} = \mathbf{Q}(\psi\hat{\mathbf{L}}) \cdot \mathbf{Q}(\vartheta\mathbf{e}) \cdot \boldsymbol{\Theta}^{-1} \cdot l\mathbf{m}(t).$$

Multiplying this equation by the tensor $\mathbf{P}^{\mathsf{T}}$ we will get

$$\dot{\psi}\mathbf{m}(t) + \dot{\vartheta}\mathbf{e} = l\boldsymbol{\Theta}^{-1} \cdot \mathbf{m}(t).$$

Since the vectors $\mathbf{m}$ and $\mathbf{e}$ are orthogonal, from the equation (3.36) follows

$$\dot{\psi} = l\mathbf{m} \cdot \boldsymbol{\Theta}^{-1} \cdot \mathbf{m} = h/l, \quad \dot{\vartheta} = l\mathbf{e} \cdot \boldsymbol{\Theta}^{-1} \cdot \mathbf{m}, \tag{3.38}$$

where the energy integral was used. Making use of (3.37) one can find

$$l\boldsymbol{\Theta}^{-1} \cdot \mathbf{m} = \frac{1}{\sin\alpha}\left(\frac{l \sin\vartheta}{\theta_2}\mathbf{d}_2 + \sin(\alpha - \vartheta)\boldsymbol{\omega}_0\right), \quad \boldsymbol{\Theta}^{-1} \cdot \mathbf{L} = \boldsymbol{\omega}_0 \equiv \boldsymbol{\omega}(0).$$

Now the second equation of (3.38) takes the form

$$\dot{\vartheta} = \frac{\sin(\alpha - \vartheta)}{\sin\alpha}\mathbf{e} \cdot \boldsymbol{\omega}_0 \equiv -\frac{1}{2}\gamma \sin(\alpha - \vartheta), \quad \gamma \equiv -2\frac{\mathbf{e} \cdot \boldsymbol{\omega}_0}{\sin\alpha}. \tag{3.39}$$

The solutions of equations (3.38) and (3.39) are obvious and have the form

$$\psi(t)=\frac{ht}{l}=\frac{lt}{\theta_2};\quad \cos(\alpha-\vartheta)=\frac{1+\cos\alpha-(1-\cos\alpha)e^{\gamma t}}{1+\cos\alpha+(1-\cos\alpha)e^{\gamma t}}, \tag{3.40}$$

where the initial conditions $\psi(0)=0$, $\vartheta(0)=0$ are used.

The solution (3.40) depends on the parameter γ,

$$\mathbf{e}\cdot\boldsymbol{\omega}_0=\left(\frac{1}{\theta_1}-\frac{1}{\theta_3}\right)\frac{l_1l_3}{\sin\alpha}\quad\Rightarrow\quad \gamma=-\left(\frac{1}{\theta_1}-\frac{1}{\theta_3}\right)\frac{2l_1l_3}{\sin\alpha}.$$

The sign of γ is determined by the initial conditions. If $l_1l_3>0$, then $\gamma<0$ and $\cos(\alpha-\vartheta)$ is approaching to 1 under $t\to\infty$. In this case the vector $\mathbf{L}$ belongs to the first plane from (3.35), as it is shown on Fig.3. The angle ϑ is tending to α. The rotation of the body in this case can be readily seen. To this end let us write down the vector $\mathbf{D}_2(t)$ corresponding to the middle moment of inertia θ_2 :

$$\mathbf{D}_2(t)=\frac{\sin\vartheta}{l\sin\alpha}\mathbf{L}+\frac{\sin(\alpha-\vartheta)}{\sin\alpha}\mathbf{Q}\left(\frac{lt}{\theta_2}\hat{\mathbf{L}}\right)\cdot\mathbf{d}_2\equiv\mathbf{P}(t)\cdot\mathbf{d}_2.$$

At the instant $t=0$ the vector $\mathbf{D}_2(0)$ coincides with the vector $\mathbf{d}_2$. When $t>0$ the vector $\mathbf{D}_2(t)$ rotating around the vector $\mathbf{L}$ is asymptotically approaching the vector $\mathbf{L}$. When t has a great magnitude the body rotates permanently around the axis with the moment of inertia θ_2.

It is not necessary to discuss this case in more details, because it can be readily proved, that the rotation of the body in the case $\sigma=0$ is unstable. This means that if the vector $\mathbf{L}$ does not belong to the plane (3.35) exactly (what is impossible in the reality) then the rotation of the body will differ from (3.40) very much. In fact the case $\sigma=0$ is a separatrix between the stable rotations, which will be constructed below.

3.2.4 Rotations of rigid bodies in the case of positive σ

If parameter σ, determined by (3.32), has a positive magnitude, then the coefficient at variable y_m^2 is plus one. Let us eliminate the coordinate $z_m^2=1-x_m^2-y_m^2$ from the equation (3.31), which then will take a form

$$\frac{x_m^2}{a^2}+\frac{y_m^2}{b^2}=1,\quad a^2=\frac{\theta_1(h\theta_3-l^2)}{l^2(\theta_3-\theta_1)},\quad b^2=\frac{\theta_2(h\theta_3-l^2)}{l^2(\theta_3-\theta_2)}. \tag{3.41}$$

The equation (3.41) is the equation of a cylindrical surface with an elliptic cross-section. The curve, described by the end of vector $\mathbf{m}(t)$, is the curve of intersection of the surface (3.41) and the unit sphere $x_m^2 + y_m^2 + z_m^2 = 1$. Let us show that the semi-axes of ellipse (3.41) are less than 1. To this end let us note that

$$h\theta_3 - l^2 = \frac{\theta_3 - \theta_1}{\theta_1} l_1^2 + \frac{\theta_3 - \theta_2}{\theta_2} l_2^2.$$

For the semi-axes a and b we have

$$a^2 = \frac{1}{l^2}\left(l_1^2 + \frac{\theta_1}{\theta_2}\frac{\theta_3 - \theta_2}{\theta_3 - \theta_1} l_2^2\right) < \frac{1}{l^2}\left(l_1^2 + l_2^2\right) < 1,$$

$$b^2 = \frac{1}{l^2}\left(\frac{\theta_2}{\theta_1}\frac{\theta_3 - \theta_1}{\theta_3 - \theta_2} l_1^2 + l_2^2\right) = \frac{1}{l^2}\left(l_1^2 + l_2^2 + \underline{\frac{\theta_3(\theta_2 - \theta_1)}{\theta_1(\theta_3 - \theta_2)} l_1^2}\right) < 1.$$

The underlined term here is less than l_3^2 because of (3.34) and of $\sigma > 0$. It may be readily seen that

$$\frac{a^2}{b^2} = \frac{\theta_1(\theta_3 - \theta_2)}{\theta_2(\theta_3 - \theta_1)} < 1. \tag{3.42}$$

This ratio is a characteristic of body and does not depend on initial conditions. Due to the equation (3.41) the coordinates x_m, y_m and z_m can be expressed in terms of the angle $\gamma(t)$

$$x_m = a\cos\gamma, \quad y_m = b\sin\gamma, \quad z_m = \sqrt{1 - a^2\cos^2\gamma - b^2\sin^2\gamma}. \tag{3.43}$$

For the coordinate z_m the sign "+" is chosen, but it makes no difference. It is clear that inequalities

$$\sqrt{1 - b^2} \equiv z_{min} \le z_m \le z_{max} \equiv \sqrt{1 - a^2}$$

are valid. In order to have a clear idea about the turns of a body for the case $\sigma > 0$, let us rewrite equation (3.31) in the form

$$z^2 = (1 - a^2)\frac{x^2}{a^2} + (1 - b^2)\frac{y^2}{b^2}. \tag{3.44}$$

This is an equation of a conical surface with an elliptic cross-section. Let us consider the cross-section of the surface (3.44) with the plane $z = z_{min} = \sqrt{1 - b^2}$, i.e., let us consider the ellipse

$$\frac{x^2}{a_1^2} + \frac{y^2}{b^2} = 1, \quad a_1^2 = \frac{1 - b^2}{1 - a^2} a^2 < a^2. \tag{3.45}$$

Here the coordinates x, y are not coordinates of the end of a vector $\mathbf{m}(t)$ any more. Let us show the cone (3.44) in Fig.4, in which the ellipse (3.45) is the ellipse

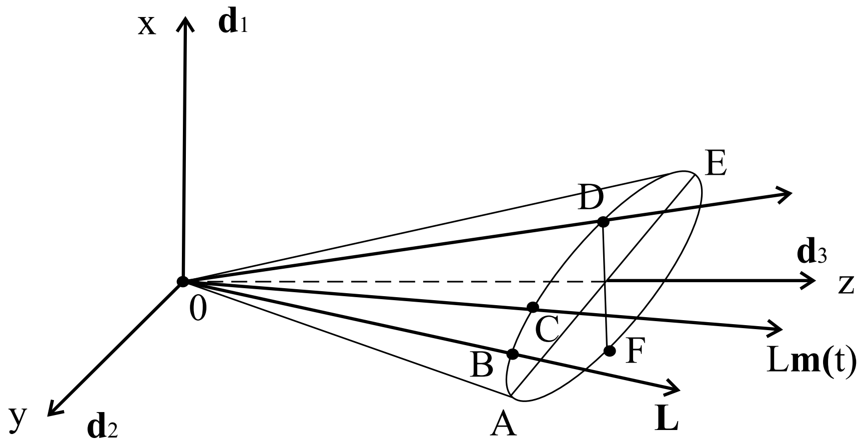

Figure 4: The case $\sigma > 0$

$ADEFA$ with semi-axes a_1 and b and distances $OO_1 = \sqrt{1-b^2}$, $OA = OE = 1$. All admitted vectors $\mathbf{m}(t)$ at any instant t belong to the conic surface (3.44). Two of them, namely $\mathbf{m} = \overrightarrow{OA}$ and $\mathbf{m} = \overrightarrow{OE}$ have its ends on the ellipse (3.45). The other vectors $\mathbf{m}(t)$ are tangent to the ellipse (3.45) — see Fig.4. The vector of angular momentum $\mathbf{L}$ is also tangent to this ellipse and belongs to the conic surface (3.44) since $\mathbf{m}(0) = \hat{\mathbf{L}}$. Now it is easy to understand the structure of turn-tensor (3.23). First of all the vector $\mathbf{p}$ in (3.23) must be chosen as $\hat{\mathbf{L}} = \mathbf{L}/l$. Really, if it is so, then

$$\mathbf{L}\cdot\mathbf{P} = \mathbf{L}\cdot\mathbf{Q}(\psi\hat{\mathbf{L}})\cdot\mathbf{Q}(\vartheta\mathbf{e})\cdot\mathbf{Q}(\varphi\mathbf{n}) = \mathbf{L}\cdot\mathbf{Q}(\vartheta\mathbf{e})\cdot\mathbf{Q}(\varphi\mathbf{n}) \tag{3.46}$$

and the angle of precession $\psi(t)$ goes out from an energy integral (3.25). Thus an energy will be conserved for any value of the angle ψ. Angles of notation ϑ and angle of own rotation φ are related by (3.25). The turn-tensor $\mathbf{Q}^T(\varphi\mathbf{n})\cdot\mathbf{Q}^T(\vartheta\mathbf{e})$ must transfer the vector $\mathbf{L}$ up to coincidence with a vector $\mathbf{m}(t)$ — see Fig.4. This may be done by two steps — see Fig.5, where three ellipses are shown.

The ellipse $ADEFA$ is the cross-section of the cone (3.44) with the plane $z = \sqrt{1-b^2}$, i.e., it is an ellipse (3.45). The ellipse $A_1D_1E_1F_1A_1$ is the cross-section of the cone (3.44) with the plane $z = z_{max} = \sqrt{1-a^2}$, i.e., it is an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b_1^2} = 1, \quad b_1^2 = \frac{1-a^2}{1-b^2}b^2 > b^2.$$

An ellipse AD_1EF_1A is ellipse (3.41). A vector $l\mathbf{m}(t)$ belongs to the cone (3.44) and intersects these ellipses. Let the point C_2 be a point of intersection of a vector $l\mathbf{m}(t)$ with the ellipse AD_1EF_1A. Let us construct the cone with an apex 0 and with the circular cross-section of the radius O_1C_2. The circle $A'C_2D'E'F'A'$ is the cross-section of this circular cone with the plane $z = z_*$, where $z_{min} < z_* < z_{max}$.

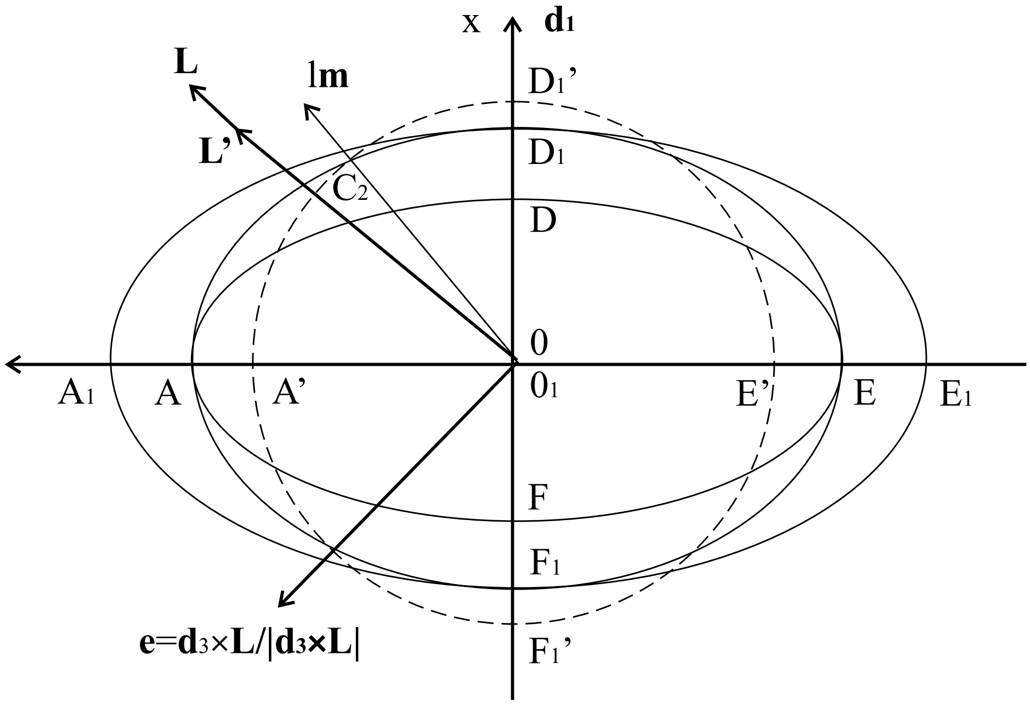

Figure 5: Determination of the turn

Now we are able to realize the two steps mentioned above. In the first step we rotate a vector $\mathbf{L}$ by the angle ϑ around the axis $\mathbf{e}$

$$\mathbf{e} = \mathbf{d}_3 \times \mathbf{L}/|\mathbf{d}_3 \times \mathbf{L}| = (\mathbf{d}_3 \times \mathbf{L})/\sqrt{l^2 - l_3^2}.$$

As a result the vector $\mathbf{L}$ will become a vector $\mathbf{L}'$,

$$\mathbf{L}' = \mathbf{Q}^{\mathsf{T}}(\vartheta\mathbf{e}) \cdot \mathbf{L}.$$

Vector $\mathbf{L}'$ belongs to the circular cone. In the second step we rotate vector $\mathbf{L}'$ around the axis $\mathbf{d}_3$. As a result of this rotation vector $\mathbf{L}'$ will be sliding along the circular cone up to coincidence with vector $l\mathbf{m}(t)$,

$$l\mathbf{m}(t) = \mathbf{Q}^{\mathsf{T}}(\varphi\mathbf{d}_3) \cdot \mathbf{L}' = \mathbf{Q}^{\mathsf{T}}(\varphi\mathbf{d}_3) \cdot \mathbf{Q}^{\mathsf{T}}(\vartheta\mathbf{e}) \cdot \mathbf{L}.$$

Thus the vector $\mathbf{n}$ in (3.46) is the vector $\mathbf{d}_3$. Now the form of a turn-tensor is determined,

$$\mathbf{P}(t) = \mathbf{Q}(\psi\hat{\mathbf{L}}) \cdot \mathbf{Q}(\vartheta\mathbf{e}) \cdot \mathbf{Q}(\varphi\mathbf{d}_3). \tag{3.47}$$

The angles ϑ and φ are in a relation, which can be found as follows:

$$\mathbf{m}(t) = \hat{\mathbf{L}} \cdot \mathbf{P}(t) =$$
$$\hat{\mathbf{L}} \cdot \mathbf{Q}(\vartheta\mathbf{e}) \cdot \mathbf{Q}(\varphi\mathbf{d}_3) = \frac{1}{\sin\alpha}\left(\sin\vartheta\,\mathbf{d}_3 + \sin(\alpha - \vartheta)\hat{\mathbf{L}} \cdot \mathbf{Q}(\varphi\mathbf{d}_3)\right), \tag{3.48}$$

where

$$\sin\alpha = \sqrt{l^2 - l_3^2}/l, \quad \cos\alpha = l_3/l, \quad (l_k \equiv \mathbf{L}\cdot\mathbf{d}_k). \tag{3.49}$$

The coordinates x_m, y_m, z_m of vector $\mathbf{m}$ can be defined from (3.48):

$$x_m = \sin(\alpha{-}\vartheta)\cos(\beta{-}\varphi),\ y_m = \sin(\alpha{-}\vartheta)\sin(\beta{-}\varphi),\ z_m = \cos(\alpha{-}\vartheta), \tag{3.50}$$

where

$$\sin\beta = l_2/\sqrt{l^2 - l_3^2}, \quad \cos\beta = l_1/\sqrt{l^2 - l_3^2}.$$

Making use of (3.43) and (3.50) we can write down

$$\tan(\beta - \varphi) = \frac{b}{a}\tan\gamma, \quad \cos(\alpha - \vartheta) = \sqrt{1 - a^2\cos^2\gamma - b^2\sin^2\gamma}. \tag{3.51}$$

A two-parameter representation of a turn-tensor conserving energy is known. Let us note that the conditions $\varphi(0) = 0$ and $\vartheta(0) = 0$ must be provided by the unique value of angle $\gamma_0 = \gamma(0)$. Thus for γ_0 there are two equations,

$$\tan\beta = \frac{b}{a}\tan\gamma_0, \quad \cos\alpha = \sqrt{1 - a^2\cos^2\gamma_0 - b^2\sin^2\gamma_0}. \tag{3.52}$$

It can be readily shown that these equations are compatible. It is only now we are ready to solve equation (3.22). For this purpose it is convenient to rewrite equation (3.22) in terms of the right vector of angular velocity and the vector $\mathbf{m}(t)$. Making use of the formula (3.4) and (3.47) we get

$$\boldsymbol{\Omega}(t) = \mathbf{P}^{\mathsf{T}}\cdot\boldsymbol{\omega}(t) = \dot\varphi\mathbf{d}_3 + \dot\vartheta\mathbf{Q}^{\mathsf{T}}(\varphi\mathbf{d}_3)\cdot\mathbf{e} + \dot\psi\mathbf{m}, \quad (\mathbf{m} = \mathbf{P}^{\mathsf{T}}\cdot\hat{\mathbf{L}}).$$

The equation (3.45) takes the form

$$\dot\varphi\mathbf{d}_3 + \dot\vartheta\mathbf{Q}^{\mathsf{T}}(\varphi\mathbf{d}_3)\cdot\mathbf{e} + \dot\psi\mathbf{m} = l\boldsymbol{\Theta}^{-1}\cdot\mathbf{m}. \tag{3.53}$$

Let us take into account the identity

$$\mathbf{Q}^{\mathsf{T}}(\varphi\mathbf{d}_3)\cdot\mathbf{e} = \frac{1}{\sqrt{l^2 - l_3^2}}\left(\mathbf{d}_3\times\mathbf{Q}^{\mathsf{T}}(\varphi\mathbf{d}_3)\cdot\mathbf{L}\right) = \frac{1}{\sin(\alpha - \vartheta)}\mathbf{d}_3\times\mathbf{m}.$$

Here the identity (3.10) was used. The equation (3.53) may be rewritten as

$$\dot\varphi\mathbf{d}_3 + \frac{\dot\vartheta}{\sin(\alpha - \vartheta)}\mathbf{d}_3\times\mathbf{m} + \dot\psi\mathbf{m} = l\boldsymbol{\Theta}^{-1}\cdot\mathbf{m}. \tag{3.54}$$

This is the basic equation that we have to solve. The first integral of (3.54) is known. It means that only two of the three scalar equations in (3.54) are independent. Projections of (3.54) on the vectors $\mathbf{d}_3$ and $\mathbf{m}(t)$ give

$$\dot\varphi - \left(\dot\psi - \frac{l}{\theta_3}\right)z_m = 0, \quad \dot\varphi z_m + \dot\psi = \frac{h}{l}; \quad z_m = \mathbf{m}\cdot\mathbf{d}_3 = \cos(\alpha - \vartheta).$$

It will be more useful to write down these equations in another form

$$\dot{\psi} = \frac{l}{\theta_3} + \frac{h\theta_3 - l^2}{l\theta_3}\frac{1}{1 - z_m^2} > 0, \quad \dot{\varphi} = -\frac{h\theta_3 - l^2}{l\theta_3}\frac{z_m}{1 - z_m^2} < 0. \tag{3.55}$$

From the first expression of (3.51) follows

$$(1 - z_m^2)\,\dot{\varphi} = -ab\dot{\gamma}.$$

Now the second equation from (3.55) takes the form

$$\dot{\gamma} = \omega_* \sqrt{1 - k^2 \sin^2\gamma} > 0, \tag{3.56}$$

where

$$\omega_* \equiv \frac{l}{\theta_3}\sqrt{(1 - a^2)\frac{(\theta_3 - \theta_1)(\theta_3 - \theta_2)}{\theta_1\theta_2}}, \quad k^2 = \frac{b^2 - a^2}{1 - a^2} < 1.$$

The initial condition for the angle γ follows from (3.52). The solution of equation (3.56) may be readily constructed in terms of elliptical functions. However, the qualitative analysis does not require any integration. Let us consider Fig.4. The form of the ellipse $ADEFA$ is determined by a body only — see (3.42). Initial conditions determine a dilatation of the ellipse. If $h\theta_3 - l^2 = 0$, then the ellipse is a point ($a = b = 0$) and we have the permanent rotation around the axis $\mathbf{d}_3 = \mathbf{D}_3(t)$: $\varphi = \vartheta = 0$, $\dot{\psi} = l/\theta_3$. When the quantity $h\theta_3 - l^2$ is increasing, then the semi-axis a and b grow in the same rate. The total turn of a body is the composition of two turns, the first of which is defined by a turn-tensor $\mathbf{Q}(\vartheta\mathbf{e}) \cdot \mathbf{Q}(\varphi\mathbf{d}_3)$ and the second one is determined by a turn-tensor $\mathbf{Q}(\psi\hat{\mathbf{L}})$. The meaning of the first turn is: the body is rotating in such a way, that the ellipse $ADEFA$, which is fixed with respect to the body, is turning in its plane touching to the vector $\mathbf{L}$, which is fixed in the space. The turn goes in the direction of a clock-wise movement since $\dot{\varphi} < 0$ and at the same time the body is turning around a fixed vector $\mathbf{d}_3 \times \mathbf{L}$ in order to provide the point of contact with the vector $\mathbf{L}$. The angle between vectors $\mathbf{L}$ and $\mathbf{D}_3(t)$ is changing in time. The speed of this rotation is changing too: it is maximal, when the point of contact coincides with the end of the small diameter of the ellipse and it is minimal, when the point of a contact coincides with the end of the big diameter of the ellipse. In the position shown on Fig.4 the angular velocity of the first turn is decreasing and takes the minimal value, when the point A of the ellipse will touch the vector $\mathbf{L}$. After that the angular velocity will be increasing and will take the maximal value when the point F of the ellipse will touch the vector $\mathbf{L}$ and so on. From (3.56) it follows that

$$\omega_* \sqrt{1 - k^2} \le \dot{\gamma}(t) \le \omega_*.$$

The second turn, called precession, is superposed on the turn described above. It is a rotation around the vector **L**. The angular velocity of rotation, i.e. $\dot{\psi}$, is positive and changes: it is maximal, when $z_m = z_{max}$; it is minimal, when $z_m = z_{min}$. After the determination of the function $\gamma(t)$ all characteristics of the rotation of a body can be expressed through this function. As an example let us show the formula for an absolute value of angular velocity

$$|\boldsymbol{\omega}(t)|^2 = |\boldsymbol{\Omega}(t)|^2 = \frac{l^2}{\theta_3^2} + \frac{h\theta_3 - l^2}{\theta_3^2}\left(\frac{\theta_1 + \theta_3}{\theta_1} - \frac{\theta_3(\theta_2 - \theta_1)}{\theta_1\theta_2}\sin^2\gamma\right).$$

As a conclusion of this section let us remind that the solution of equation (3.56) can be found in the form of the series

$$\gamma(t) = \gamma_0 + \omega_a t + \sum_{n=1}^{\infty} \frac{\sin(2n\omega_a t)}{n\cosh(n\pi K'/K)},$$

where

$$\omega_a = \frac{\pi\omega_*}{2K}, \quad K = \int_0^{\frac{\pi}{2}} \frac{d\gamma}{\sqrt{1 - k^2\sin^2\gamma}}, \quad K' = \int_0^{\frac{\pi}{2}} \frac{d\gamma}{\sqrt{1 - (1 - k^2)\sin^2\gamma}}.$$

The convergence of the series is sufficiently good if $0 \le k^2 \le 0.98$, since even for $k^2 = 0.98$ we have $K'/K = 0.4707$.

3.2.5 Rotation of the body in the case $\sigma < 0$

The only important difference of this case to that described above is that the first rotation of the body is going around the axis $\mathbf{d}_1$ instead of $\mathbf{d}_3$. If $\sigma = l^2 - h\theta_2 < 0$, then the coefficient of y_m^2 in equation (3.31) is negative. Eliminating the coordinate $x_m^2 = 1 - y_m^2 - z_m^2$ from (3.31) we will get

$$\frac{y_m^2}{c^2} + \frac{z_m^2}{d^2} = 1, \quad c^2 = \frac{\theta_2(l^2 - h\theta_1)}{l^2(\theta_2 - \theta_1)}, \quad d^2 = \frac{\theta_3(l^2 - h\theta_1)}{l^2(\theta_3 - \theta_1)}, \quad \frac{d^2}{c^2} < 1.$$

Let us introduce the angle $\gamma(t)$ such that

$$x_m = \sqrt{1 - d^2\cos^2\gamma - c^2\sin^2\gamma}, \quad y_m = c\sin\gamma, \quad z_m = d\cos\gamma. \qquad (3.57)$$

The turn-tensor $\mathbf{P}(t)$ can be presented in the form

$$\mathbf{P}(t) = \mathbf{Q}(\psi\hat{\mathbf{L}}) \cdot \mathbf{Q}(\vartheta\mathbf{e}) \cdot \mathbf{Q}(\varphi\mathbf{d}_1), \quad \mathbf{e} = (\mathbf{d}_1 \times \mathbf{L})/\sqrt{l^2 - l_1^2}. \qquad (3.58)$$

The angles ϑ and φ here are related. In order to find out this relation it is necessary to express the vector $\mathbf{m}(t) = \mathbf{P}^{\mathsf{T}} \cdot \hat{\mathbf{L}}$ in terms of the angles ϑ and φ,

$$\mathbf{m}(t) = \frac{1}{\sin\alpha}\left(\sin\vartheta\,\mathbf{d}_1 + \sin(\alpha-\vartheta)\hat{\mathbf{L}}\cdot\mathbf{Q}(\varphi\mathbf{d}_1)\right),$$

where

$$\sin\alpha = \sqrt{l^2 - l_1^2}\,/\,l, \quad \cos\alpha = l_1/l.$$

The projections of $\mathbf{m}(t)$ on the unit vectors $\mathbf{d}_k$ are

$$x_m = \cos(\alpha-\vartheta), \quad y_m = \sin(\alpha-\vartheta)\sin(\beta+\varphi),$$
$$z_m = \sin(\alpha-\vartheta)\cos(\beta+\varphi), \quad (3.59)$$

where

$$\sin\beta = l_2/\sqrt{l^2-l_1^2}, \quad \cos\beta = l_3/\sqrt{l^2-l_1^2}.$$

From (3.57) and (3.59) follows

$$\tan(\beta+\varphi) = \frac{c}{d}\tan\gamma, \quad \cos(\alpha-\vartheta) = \sqrt{1 - d^2\cos^2\gamma - c^2\sin^2\gamma}. \qquad (3.60)$$

The right vector of an angular velocity corresponding to the turn-tensor (3.58) is calculated from the formula

$$\mathbf{\Omega}(t) = \dot\varphi\mathbf{d}_1 + \frac{\dot\vartheta}{\sin(\alpha-\vartheta)}\mathbf{d}_1\times\mathbf{m} + \dot\psi\mathbf{m}.$$

The equation (3.22) takes the form

$$\dot\varphi\mathbf{d}_1 + \frac{\dot\vartheta}{\sin(\alpha-\vartheta)}\,\mathbf{d}_1\times\mathbf{m} + \dot\psi\mathbf{m} = l\mathbf{\Theta}^{-1}\cdot\mathbf{m}(t).$$

From this equation it follows that

$$\dot\varphi + \left(\dot\psi - \frac{l}{\theta_1}\right)x_m = 0, \quad \dot\varphi x_m + \dot\psi = \frac{h}{l}$$

or, in another form,

$$\dot\psi = \frac{l}{\theta_1} - \frac{l^2 - h\theta_1}{l\theta_1}\frac{1}{1-x_m^2} > 0, \quad \dot\varphi = \frac{l^2-h\theta_1}{l\theta_1}\frac{x_m}{1-x_m^2} > 0. \qquad (3.61)$$

The positiveness of $\dot\varphi$ is obvious, and the positiveness of $\dot\psi$ follows from the inequalities

$$\dot\psi \geq \frac{l}{\theta_1} - \frac{l^2-h\theta_1}{l\theta_1}\frac{1}{1-(x_m^2)_{max}}, \quad 1 - x_m^2 = d^2\cos^2\gamma + c^2\sin^2\gamma =$$
$$d^2 + (c^2-d^2)\sin^2\gamma \quad \Rightarrow \quad \dot\psi \geq \frac{l}{\theta_1} - \frac{l^2-h\theta_1}{l\theta_1}\frac{1}{d^2} = \frac{l}{\theta_3} > 0.$$

The first equation from (3.60) gives

$$(1 - x_m^2)\,\dot{\varphi} = c d\,\dot{\gamma}.$$

Substituting this expression in the second equation of (3.61) we get

$$\dot{\gamma}(t) = \omega_* \sqrt{1 - k^2 \sin^2 \gamma},$$

where

$$\omega_* \equiv \frac{l}{\theta_1} \sqrt{(1 - d^2)\frac{(\theta_2 - \theta_1)(\theta_3 - \theta_1)}{\theta_2 \theta_3}}, \quad k^2 = \frac{c^2 - d^2}{1 - d^2} < 1.$$

The initial condition for the function $\gamma(t)$ follows from (3.60) and conditions $\vartheta(0) = 0,\ \varphi(0) = 0$:

$$\tan \gamma_0 = \frac{d}{c} \tan \beta, \quad \gamma_0 \equiv \gamma(0). \tag{3.62}$$

If the function $\gamma(t)$ is known then all characteristics of the rotation of a body can be found. For example,

$$|\boldsymbol{\omega}(t)|^2 = \frac{l^2}{\theta_1^2} - \frac{l^2 - h\theta_1}{\theta_1^2} \left(\frac{\theta_1 + \theta_3}{\theta_3} + \frac{\theta_1(\theta_3 - \theta_2)}{\theta_2 \theta_3} \sin^2 \gamma \right).$$

From a formal point of view the cases $\sigma > 0$ and $\sigma < 0$ are almost the same. However, the rotation of the body in the case $\sigma > 0$ differ from that one in the case $\sigma < 0$ significantly.

3.2.6 Discussion

In order to understand and predict the behavior of a body it is not necessary to solve the problem as a whole. It is sufficient to know the form of the turn-tensor of the body

$$\mathbf{P}(t) = \mathbf{Q}(\psi \hat{\mathbf{L}}) \cdot \mathbf{Q}(\vartheta \mathbf{e}) \cdot \mathbf{Q}(\varphi \mathbf{d}_1), \quad \mathbf{e} = (\mathbf{d}_1 \times \mathbf{L})/\sqrt{l^2 - l_1^2}, \quad \sigma < 0; \tag{3.63}$$

$$\mathbf{P}(t) = \mathbf{Q}(\psi \hat{\mathbf{L}}) \cdot \mathbf{Q}(\vartheta \mathbf{e}), \quad \mathbf{e} = (\mathbf{d}_2 \times \mathbf{L})/\sqrt{l^2 - l_2^2}, \quad \sigma = 0; \tag{3.64}$$

$$\mathbf{P}(t) = \mathbf{Q}(\psi \hat{\mathbf{L}}) \cdot \mathbf{Q}(\vartheta \mathbf{e}) \cdot \mathbf{Q}(\varphi \mathbf{d}_3), \quad \mathbf{e} = (\mathbf{d}_3 \times \mathbf{L})/\sqrt{l^2 - l_3^2},\ \sigma > 0. \tag{3.65}$$

The angles of a precession, ψ, and own rotation, φ, are varying monotonically in all cases ($\dot{\varphi} = 0$ if $\sigma = 0$). The variation of the angle of nutation, ϑ, for $\sigma > 0$ or $\sigma < 0$ has an oscillating nature. The rotations of a body in the case $\sigma = 0$ are unstable. The rotations of a body in the cases $\sigma > 0$ and $\sigma < 0$ are stable. However, such a conclusion is correct only in linear approximation. If the value of σ is very small ($0 < |\sigma| \leq \varepsilon \ll 1$), then the analysis of stability in a linear approximation becomes useless from the practical point of view. Let us consider the following initial conditions ($\varepsilon > 0$)

$$l_1 = \frac{l}{\sqrt{1 + (\sqrt{2} + \varepsilon)^2}}, \quad l_2 = 0, \quad l_3 = \frac{(\sqrt{2} + \varepsilon)\, l}{\sqrt{1 + (\sqrt{2} + \varepsilon)^2}}, \quad \sigma > 0; \tag{3.66}$$

$$l_1 = \frac{l}{\sqrt{1 + (\sqrt{2} - \varepsilon)^2}}, \quad l_2 = 0, \quad l_3 = \frac{(\sqrt{2} - \varepsilon)\, l}{\sqrt{1 + (\sqrt{2} - \varepsilon)^2}}, \quad \sigma < 0. \tag{3.67}$$

If the value of $\varepsilon > 0$ is very small, then the conditions (3.66) and (3.67) are very close. Nevertheless the rotations of a body in these cases are essentially different. Moreover, it quite well is possible for the body to jump from the "stable" solution (3.63) (corresponding to the conditions (3.66)) to another "stable" solution (3.65) which corresponds to conditions (3.67). In the first case the body was rotating around the axis with minimal moment of inertia, but after such a jump the body will be rotating around the axis with maximal moment of inertia. For this only very small perturbations acting on a body are needed. Thus if we want the body to have a stable rotation in reality, we have to avoid the case of a small value of σ.

3.3 Classical solution

The classical solution is constructed in two steps. In the first step the right angular velocity is determined. Then, in a second step, the Euler angles can be found from the solution of the right Darboux problem.

The angular momentum vector $\mathbf{L}$ had been expressed through the right angular velocity $\mathbf{\Omega}(t)$

$$\mathbf{L} = \mathbf{P} \cdot \mathbf{\Theta} \cdot \mathbf{P}^T \cdot \omega = \mathbf{P} \cdot \mathbf{\Theta} \cdot \mathbf{\Omega} = \text{const.} \tag{3.68}$$

Let us write down the two identities

$$\begin{aligned} &(\mathbf{A} \cdot \mathbf{a}) \times (\mathbf{A} \cdot \mathbf{b}) = (\det \mathbf{A})\, (\mathbf{A}^T)^{-1} \cdot (\mathbf{a} \times \mathbf{b}), \\ &\qquad \mathbf{P} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{P} \cdot \mathbf{a}) \times (\mathbf{P} \cdot \mathbf{b}), \end{aligned} \tag{3.69}$$

where $\mathbf{a}$, $\mathbf{b}$ are arbitrary vectors, $\mathbf{A}$ is a nonsingular second rank tensor, $\mathbf{P}$ is a turn-tensor. If we differentiate (3.68) with respect to time, then we get — see, for example, Arnold V.I. [1].

$$\dot{\mathbf{L}} = \mathbf{P} \cdot (\mathbf{\Theta} \cdot \dot{\mathbf{\Omega}} + \mathbf{\Omega} \times \mathbf{\Theta} \cdot \mathbf{\Omega}) = 0 \quad \Rightarrow \quad \mathbf{\Theta} \cdot \dot{\mathbf{\Omega}} + \mathbf{\Omega} \times \mathbf{\Theta} \cdot \mathbf{\Omega} = 0. \tag{3.70}$$

These are the classical Euler equations. The tensor of inertia, $\boldsymbol{\Theta}$, in (3.70) can be represented in any form (any basis). Making use of (3.69) equation (3.70) can be written in the form

$$\dot{\boldsymbol{\Omega}} + \frac{1}{\theta_1\theta_2\theta_3}(\boldsymbol{\Theta}\cdot\boldsymbol{\Omega})\times(\boldsymbol{\Theta}^2\cdot\boldsymbol{\Omega}) = 0. \tag{3.71}$$

This equation was derived by L. Silberstein [7]. The merit of equations (3.70) and (3.71) is that they do not contain the turns. If all eigenvalues of $\boldsymbol{\Theta}$ are different, and the vector $\boldsymbol{\Omega}(t)$ is not an eigenvector of $\boldsymbol{\Theta}$, then the vectors $\boldsymbol{\Omega}(t)$, $\boldsymbol{\Theta}\cdot\boldsymbol{\Omega}(t)$, $\boldsymbol{\Theta}^2\cdot\boldsymbol{\Omega}(t)$ are linearly independent. The equation (3.71), multiplied scalarly at first by $\boldsymbol{\Theta}\cdot\boldsymbol{\Omega}$ and than by $\boldsymbol{\Theta}^2\cdot\boldsymbol{\Omega}$, gives two first integrals

$$h = 2K = \boldsymbol{\Omega}\cdot\boldsymbol{\Theta}\cdot\boldsymbol{\Omega} = \text{const}, \quad l^2 = \mathbf{L}\cdot\mathbf{L} = \boldsymbol{\Omega}\cdot\boldsymbol{\Theta}^2\cdot\boldsymbol{\Omega} = \text{const}. \tag{3.72}$$

Besides, equation (3.71), multiplied scalarly by the vector $\boldsymbol{\Omega}$, gives the equation for $\Omega^2 = \boldsymbol{\Omega}\cdot\boldsymbol{\Omega}$

$$(\Omega^2)^{\bullet} = 2A\Omega_1\Omega_2\Omega_3, \quad A = \frac{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_2-\theta_1)}{\theta_1\theta_2\theta_3} > 0, \tag{3.73}$$

where $\mathbf{d}_k$ are eigenvectors of $\boldsymbol{\Theta}$ and $\Omega_k = \boldsymbol{\Omega}\cdot\mathbf{d}_k$.

Equation (3.73) shows, that a rotation of a body with constant absolute value of angular velocity, $\Omega = |\boldsymbol{\Omega}(t)|$, is possible only in two cases: a) for permanent rotations, when only one Ω_k is not zero, i.e., $\boldsymbol{\Omega}$ is eigenvector of $\boldsymbol{\Theta}$; and b) if two or three eigenvalue of $\boldsymbol{\Theta}$ are the same. Integrals (3.72) hold, if we accept

$$\Omega_1^2 = \Omega_{10}^2 + \alpha(\Omega^2 - \Omega_0^2), \quad \Omega_2^2 = \Omega_{20}^2 - \beta(\Omega^2 - \Omega_0^2),$$
$$\Omega_3^2 = \Omega_{30}^2 + \gamma(\Omega^2 - \Omega_0^2), \tag{3.74}$$

where

$$\alpha = \frac{\theta_2-\theta_1}{A\theta_3}, \quad \beta = \frac{\theta_3-\theta_1}{A\theta_2}, \quad \gamma = \frac{\theta_3-\theta_2}{A\theta_1}, \quad \Omega_0 \equiv \Omega\Big|_{t=0}.$$

Substituting (3.74) into (3.73) we get the equation for the quantity Ω. The solution of this equation can be found in terms of elliptic functions — see [8].

Up to here the investigation was a standard one. Now we have to consider the initial conditions for the right angular velocity $\boldsymbol{\Omega}(t)$. Often it is supposed that the initial condition for $\boldsymbol{\Omega}(t)$ does not depend on the initial position of the body. However, it is not so. At the instant $t = 0$ we know the value of the left angular velocity $\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0$, i.e., we know true angular velocity. The relation (3.4) gives

$$\boldsymbol{\Omega}(0) \equiv \boldsymbol{\Omega}_0 = \mathbf{P}_0^T\cdot\boldsymbol{\omega}_0, \quad \mathbf{P}_0 \equiv \mathbf{P}(0).$$

If we choose the initial position of the body as a reference position, then we will get $\mathbf{P}(0) = \mathbf{E}$. In such a case the vector $\mathbf{\Omega}(t)$ will contain three arbitrary parameters (vector $\boldsymbol{\omega}_0$). If the reference position is chosen a priory and does not coincide with the initial position of the body, then $\mathbf{P}_0 \neq \mathbf{E}$. In this case the vector $\mathbf{\Omega}(t)$ depends on six arbitrary parameters (vector $\boldsymbol{\omega}_0$ and turn-tensor $\mathbf{P}_0$). In order to find out the turn-tensor $\mathbf{P}(t)$ we have to solve the right Darboux problem

$$\dot{\mathbf{P}}(t) = \mathbf{P}(t) \times \mathbf{\Omega}(t), \quad \mathbf{P}\Big|_{t=0} = \mathbf{P}_0. \tag{3.75}$$

The solution of this problem does not add the new arbitrary parameters. Thus the general solution, for example the left (true) vector of angular velocity, $\boldsymbol{\omega}(t)$, will contain only six arbitrary parameters and all of them are contained in the vector $\mathbf{\Omega}(t)$.

Let us consider the solution of problem (3.75). We shall give two different approaches to the solution. The first approach is based on the representation of the turn-tensor in terms of Eulerian angles and contains the classical solution as a particular case. The second approach is new.

Let us represent the turn-tensor $\mathbf{P}(t)$ in terms of Eulerian angles — see the introduction.

$$\mathbf{P}(t) = \mathbf{Q}(\psi \mathbf{e}_3) \cdot \mathbf{Q}(\vartheta \mathbf{e}_1) \cdot \mathbf{Q}(\varphi \mathbf{e}_3),$$

where

$$\mathbf{e}_3 = \hat{\mathbf{L}} \equiv \mathbf{L}/l, \quad \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1, \quad \mathbf{e}_1 \qquad (\mathbf{e}_1 \cdot \mathbf{L} = 0, \quad \mathbf{e}_2 \cdot \boldsymbol{\omega}_0 = 0)$$

are the unit vectors, the vector $\mathbf{e}_1$ is placed in the plane spanned from the vectors $\mathbf{L}$ and $\boldsymbol{\omega}_0$.

In fact, we do not need to solve the Darboux problem (3.75) because it had been partly solved. Indeed, multiplying (3.75) by the vector $\mathbf{L}$ from the left we get

$$(\mathbf{L} \cdot \mathbf{P})^{\bullet} = (\mathbf{L} \cdot \mathbf{P}) \times \mathbf{\Omega} \quad \Rightarrow \quad \mathbf{\Theta} \cdot \dot{\mathbf{\Omega}} = (\mathbf{\Theta} \cdot \mathbf{\Omega}) \times \mathbf{\Omega},$$

where the expression (3.68) was used. It means that we know the vector $\mathbf{L} \cdot \mathbf{P} = \mathbf{\Theta} \cdot \mathbf{\Omega}$,

$$\mathbf{\Theta} \cdot \mathbf{\Omega} = \mathbf{P}^T \cdot \mathbf{L} = l \sin\vartheta\,(\sin\varphi\, \mathbf{e}_1 + \cos\varphi\, \mathbf{e}_2) + l\cos\vartheta\, \mathbf{e}_3. \tag{3.76}$$

From (3.76) follows

$$l\sin\vartheta\sin\varphi = \mathbf{e}_1 \cdot \mathbf{\Theta} \cdot \mathbf{\Omega}, \quad l\sin\vartheta\cos\varphi = \mathbf{e}_2 \cdot \mathbf{\Theta} \cdot \mathbf{\Omega}, \quad l\cos\vartheta = \mathbf{e}_3 \cdot \mathbf{\Theta} \cdot \mathbf{\Omega}, \tag{3.77}$$

or

$$\tan\varphi = (\mathbf{e}_1 \cdot \mathbf{\Theta} \cdot \mathbf{\Omega})/(\mathbf{e}_2 \cdot \mathbf{\Theta} \cdot \mathbf{\Omega}), \quad \cos\vartheta = (\mathbf{e}_3 \cdot \mathbf{\Theta} \cdot \mathbf{\Omega})/l. \tag{3.78}$$

In order to find the angle of precession, ψ, it is necessary to use equation (3.75), which can be rewritten in the form

$$\mathbf{\Omega}(t) = \dot{\varphi}\mathbf{e}_3 + \dot{\vartheta}(\cos\varphi\,\mathbf{e}_1 - \sin\varphi\,\mathbf{e}_2) + \dot{\psi}(\mathbf{\Theta}\cdot\mathbf{\Omega})/l. \tag{3.79}$$

From this equation it follows

$$\dot{\psi} = \frac{h - l\cos\vartheta\Omega_3}{l^2\sin^2\vartheta}, \quad \psi\Big|_{t=0} = \psi_0, \quad (\Omega_k = \mathbf{\Omega}\cdot\mathbf{e}_k). \tag{3.80}$$

Up to here we do not fix the reference position of the body. Let us consider two cases of a choice of a reference position.

A. The reference position is an initial position of the body. In this case we have

$$\mathbf{P}_0 = \mathbf{E} \quad \Rightarrow \quad \vartheta\Big|_{t=0} = 0, \quad \varphi_0 + \psi_0 = 0, \quad \boldsymbol{\omega}_0 = \mathbf{\Omega}_0, \quad \mathbf{\Theta}\cdot\mathbf{\Omega}_0 = \mathbf{L}. \tag{3.81}$$

When $t \to 0$ the quantities $\mathbf{e}_1\cdot\mathbf{\Theta}\cdot\mathbf{\Omega}$ and $\mathbf{e}_2\cdot\mathbf{\Theta}\cdot\mathbf{\Omega}$ tends to zero, but $\mathbf{e}_3\cdot\mathbf{\Theta}\cdot\mathbf{\Omega}/l$ tends to 1. We see that equations (3.77) come to identities when $t \to 0$. From (3.78) it follows that $\vartheta \to 0$ when $t \to 0$. However, it is very difficult to find out the value φ_0 from equations (3.78). Let us consider equation (3.79) at the instant $t = 0$,

$$\boldsymbol{\omega}_0 = \mathbf{\Omega}_0 = (\dot{\varphi}_0 + \dot{\psi}_0)\mathbf{e}_3 + \dot{\vartheta}_0(\cos\varphi_0\,\mathbf{e}_1 - \sin\varphi_0\,\mathbf{e}_2).$$

From this equation we see that $\varphi_0 = 0$, since $\mathbf{e}_2\cdot\boldsymbol{\omega}_0 = 0$ and $\dot{\vartheta}_0 \neq 0$ in the general case. From (3.81) it follows that $\psi_0 = 0$. This means that the numerator in (3.78) tends to zero more quickly than the denominator. It is easy to see that the numerator in (3.80) has a zero of order 2 when $t \to 0$. Really, we have

$$h = \boldsymbol{\omega}\cdot\mathbf{L} = \mathbf{L}\cdot\mathbf{P}\cdot\mathbf{\Omega} = l\,(\cos\vartheta\,\Omega_3 + \sin\vartheta\,(\cos\varphi\,\Omega_2 + \sin\varphi\,\Omega_1))$$

and

$$\Omega_{20} = \mathbf{\Omega}_0\cdot\mathbf{e}_2 = \boldsymbol{\omega}_0\cdot\mathbf{e}_2 = 0.$$

Thus we have the expression (3.78) and (3.80) in order to find Eulerian angles ψ, ϑ, and φ. They are not good for practical calculation, but they can be used.

B. Let a reference position of the body be a position, when the eigenvector $\mathbf{d}_3$ of the tensor $\mathbf{\Theta}$ coincides with the vector $\mathbf{L}$. This case is presented in the most of books — see, for example, Macmillan [6] and Suslov [8]. In this case we have $\mathbf{e}_k = \mathbf{d}_k$. Equations (3.77), (3.78) take the form

$$l\sin\vartheta\sin\varphi = \theta_1\Omega_1, \quad l\sin\vartheta\cos\varphi = \theta_2\Omega_2, \quad l\cos\vartheta = \theta_3\Omega_3; \tag{3.82}$$

$$\tan\varphi = (\theta_1\Omega_1)/(\theta_2\Omega_2), \quad \cos\vartheta = (\theta_3\Omega_3)/l. \tag{3.83}$$

Here a lot of questions arises. First of all, it is difficult to find out the angle φ. Really, from equation (3.73) it follows that the quantities Ω_1 and Ω_2 must be zero at the some instants. It means that the functions $\psi(t)$ and $\varphi(t)$ will be discontinuous functions of time. The same fact follows from equations (3.82). Indeed, if we write down (3.82) at the instant $t = 0$, then we must get the identities with respect to six parameters $\psi_0,\ \varphi_0,\ \vartheta_0,\ \boldsymbol{\omega}_0$. Is it really so? Nobody shows it. Anyway one must be very careful to use expression (3.83) and equation (3.80).

Let us consider another way of constructing the turn-tensor $\mathbf{P}(t)$. In this approach we do not use Eulerian angles. The vector $\mathbf{m}(t) = \boldsymbol{\Theta} \cdot \boldsymbol{\Omega}/l$ is supposed to be known. The turn-tensor $\mathbf{P}(t)$ is represented in the form

$$\mathbf{P}(t) = \mathbf{Q}(\psi\hat{\mathbf{L}}) \cdot \mathbf{Q}(\alpha\mathbf{b}), \quad \mathbf{b} = \mathbf{m} \times \hat{\mathbf{L}}/|\mathbf{m} \times \hat{\mathbf{L}}| \equiv (\mathbf{m} \times \hat{\mathbf{L}})/|\sin\alpha|, \tag{3.84}$$

where

$$\cos\alpha = \mathbf{m}(t) \cdot \hat{\mathbf{L}} = \boldsymbol{\Omega}(t) \cdot \boldsymbol{\Theta}^2 \cdot \boldsymbol{\Omega}_0/l^2, \quad \alpha(0) = 0. \tag{3.85}$$

The angle $\alpha(t)$ is known. It can be readily shown that the inequality

$$0 \leq \alpha(t) < \pi$$

holds good. Let the vector $\boldsymbol{\omega}_*(t)$ be the left angular velocity of the turn-tensor $\mathbf{Q}(\alpha\mathbf{b})$,

$$\boldsymbol{\omega}_*(t) = \dot{\alpha}(t)\mathbf{b}(t) + \sin\alpha\,\dot{\mathbf{b}} + (1 - \cos\alpha)\mathbf{b} \times \dot{\mathbf{b}}.$$

The left angular velocity $\boldsymbol{\omega}(t)$ of the turn-tensor (3.84) is equal to

$$\boldsymbol{\omega}(t) = \dot{\psi}\hat{\mathbf{L}} + \mathbf{Q}(\psi\hat{\mathbf{L}}) \cdot \boldsymbol{\omega}_*. \tag{3.86}$$

Let us remind that $\boldsymbol{\omega} \cdot \mathbf{L} = h$. From equation (3.86) it follows

$$\dot{\psi} = \frac{h}{l} - \boldsymbol{\omega}_* \cdot \hat{\mathbf{L}} = \frac{h + \boldsymbol{\Omega}(t) \cdot \mathbf{L}}{l(1 + \cos\alpha)}, \quad \psi\Big|_{t=0} = \psi_0. \tag{3.87}$$

The next chain of equalities,

$$\begin{aligned}
\boldsymbol{\omega}_* \cdot \mathbf{L} &= (1 - \cos\alpha)(\mathbf{b} \times \dot{\mathbf{b}}) \cdot \mathbf{L} = \frac{1 - \cos\alpha}{\sin\alpha}\left(\hat{\mathbf{L}} \times (\mathbf{m} \times \hat{\mathbf{L}})\right) \cdot \dot{\mathbf{b}} = \\
&= \frac{1 - \cos\alpha}{\sin\alpha}(\mathbf{m} - \cos\alpha\hat{\mathbf{L}}) \cdot \dot{\mathbf{b}} = \frac{1 - \cos\alpha}{\sin\alpha}\mathbf{m} \cdot \dot{\mathbf{b}} = -\frac{1 - \cos\alpha}{\sin\alpha}\dot{\mathbf{m}} \cdot \mathbf{b} = \\
&= \frac{1 - \cos\alpha}{\sin\alpha}(\boldsymbol{\Omega} \times \mathbf{m}) \cdot \mathbf{b} = \frac{1 - \cos\alpha}{\sin^2\alpha}(\boldsymbol{\Omega} \times \mathbf{m}) \cdot (\mathbf{m} \times \hat{\mathbf{L}}) = \frac{h\cos\alpha - \boldsymbol{\Omega} \cdot \mathbf{L}}{l(1 + \cos\alpha)}
\end{aligned}$$

must be used in order to get equation (3.87). The solution (3.84), (3.85) and (3.87) contains six arbitrary parameters. The angles $\alpha(t)$ and $\psi(t)$ are continuous function of time.

3.4 Final remarks

The success of the solution of many problems in dynamics depends on the appropriate choice of the turn-tensor representation. There exist many different ways. The most popular way is the representation of the turn-tensor in terms of Eulerian angles. Our own experience shows that in most cases Eulerian angles are not advantageous tools. The only exception is the case when the angle of nutation is small. In this case the application of Eulerian angles leads to very simple solutions. Of course, we are able to use another sets of parameters instead of Eulerian angles. The main point is in which way we will do it. If we make our choice a priory we must be lucky to find a successful solution. What we wanted to show is the fact, that the set of basic parameters must be chosen in the process of the solving the task. Especially for this aim we need the theorems like the representation theorem (3.12). The turn-tensor is a necessary tool in order to keep the freedom of choice of the basic set of parameters.

The Euler – Poinsot problem is the simplest case in dynamics of rigid bodies. In the most cases we have no chances to find something like the first integrals. Does it mean that the turn-tensor is useless in such cases? We are quite sure that it is not so. As an example we can point out one important (but not most important) case. In many problems of continuum mechanics of multipolar media, dynamics of gyroscopic system, dynamics of centrifuges and ultracentrifuges and so on, we deal with the case of rotation under small angles of nutation. If we use conventional methods then we will get nonlinear equations for arbitrary small angles of nutation if the other Eulerian angles are not small. Application of turn-tensor and identities like (3.9) and (3.10) allows to simplify the task significantly. Let us briefly show the way how to do it. Let us accept representation (3.15) of the turn-tensor in terms of Eulerian angles,

$$\mathbf{P}(t) = \mathbf{Q}\left(\psi(t)\mathbf{m}\right) \cdot \mathbf{Q}\left(\vartheta(t)\mathbf{e}\right) \cdot \mathbf{Q}\left(\varphi(t)\mathbf{m}\right), \quad \mathbf{e} \cdot \mathbf{m} = 0, \quad |\vartheta(t)| << 1. \tag{3.88}$$

If $|\vartheta(t)|$ is small then we can use the expression

$$\mathbf{Q}(\vartheta\mathbf{e}) = \mathbf{E} + \vartheta\mathbf{e} \times \mathbf{E} + O(\vartheta^3)$$

and rewrite (3.88) in the form

$$\mathbf{P}(t) = (\mathbf{E} + \boldsymbol{\gamma}(t) \times \mathbf{E}) \cdot \mathbf{Q}(\beta(t)\mathbf{m}), \tag{3.89}$$

where

$$\beta(t) = \varphi(t) + \psi(t), \quad \boldsymbol{\gamma}(t) = \vartheta(t)\mathbf{Q}(\psi(t)\mathbf{m}) \cdot \mathbf{e}, \quad \mathbf{m} \cdot \boldsymbol{\gamma} = 0.$$

Usually the quantities ψ, φ, β are not small but the vector $\boldsymbol{\gamma}(t)$ is small since $|\boldsymbol{\gamma}(t)| = |\vartheta(t)| << 1$. From (3.88) and (3.89) follows

$$\boldsymbol{\omega} = \dot{\beta}\mathbf{m} + \dot{\boldsymbol{\gamma}} + \dot{\beta}\boldsymbol{\gamma} \times \mathbf{m}.$$

After that it is possible to linearize the basic equations with respect to the small vector $\boldsymbol{\gamma}$ and to construct a solution. Let us underline that linear equation can be derived only for the vector $\boldsymbol{\gamma}(t)$ rather than for a small angle $\vartheta(t)$. The equation for $\vartheta(t)$ will be always nonlinear. If we know the vector $\boldsymbol{\gamma}(t)$ it is easy to find the angle of precession, $\psi(t)$,

$$\dot{\psi}(t) = \mathbf{m} \cdot (\boldsymbol{\gamma} \times \dot{\boldsymbol{\gamma}}) / (\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}).$$

The angle $\varphi(t)$ can be found if ψ and β are known. Usually $\beta(t)$ can be found without any difficulty.

References

1. **Arnold V.I.** The mathematical methods of the classical mechanics. Moscow: Nauka, 1979. (In Russian).

2. **Goldstein H.** Classical Mechanics/ Addision-Wesley Press. Cambridge, 1950.

3. **Golubev U.F.** Foundations of Theoretical Mechanics/ Moscow State Univ. Press. Moscow, 1992. (In Russian).

4. **Lagally M.** Vorlesungen über Vektor-Rechnung/ Akad. Verlagsgesellschaft M.B.H. Leipzig, 1928.

5. **Lurie A.I.** Theory of Elasticity. Moscow: Nauka, 1970. (In Russian).

6. **Macmillan A.M.** Theoretical Mechanics. Vol. 3: Dynamics of rigid bodies. New York – London, 1936.

7. **Silberstein L.** Vectorial Mechanics. London: Macmillan and Co, 1913.

8. **Suslov G.K.** Theoretical mechanics. Moscow, 1944. (In Russian).

9. **Zhilin P.A.** Tensor of a turn in kinematics of rigid bodies// Mechanics and Control process. Proc. of St. Petersburg's State Techn. Univ., 443. St. Petersburg, 1992. P. 100 – 121. (In Russian).