Symmetries and Orthogonal Invariants in Oriented Space*

Abstract

The theory of the tensor symmetry is modified in order to include into consideration the Non-Euclidean tensors. The polar (Euclidean) and axial (Non-Euclidean) tensors are discussed. A new definition of the tensor invariants is given. This definition coincides with the conventional one only for the Euclidean tensors. It is shown that any invariant is the solution of some partial differential equation of the first order. The number of the independent solutions of this equation determines the minimal number of the invariants which are necessary in order to fix a system of tensors as a rigid whole. This result was not found in the known publications. The examples of some systems of tensors are discussed in order to give the comparison with known results.

1 Introduction

Symmetries and orthogonal invariants are important theoretical tools for many fields of mechanics. Therefore these tools must be applicable to all objects widely used in mechanics. Unfortunately this is not so. The application of the classical theory of symmetry leads to the meaningless results in shell theory and not only in shell theory. The main reason of this is that in many cases we are forced to work in multi-oriented spaces. The classical theory of symmetry and invariants is well defined in non-oriented vector space only. In order to define the cross product of vectors we have to introduce the oriented vector space. There are two different types of tensors acting in oriented space. These tensors are known as polar and axial ones. In oriented space the classical theory of symmetry is well defined for polar tensors. There exist many formally equivalent ways for introduction of the space orientation. In this paper we introduce a definition of the space orientation in such a way that the physical sense of this concept is quiet clear. Besides we restrict ourselves by consideration of the oriented space. In general case this is not enough. For example, in shell theory it is necessary to use multi-oriented space. Briefly speaking in shell theory 3D-space E_3 must be represented as a direct sum of 2D-space E_2 and 1D-space E_1: $\mathsf{E}_3 = \mathsf{E}_2 \oplus \mathsf{E}_1$. If we orientate each of these spaces, then we obtain three oriented spaces $\mathsf{E}_3^{\mathsf{O}}$, $\mathsf{E}_2^{\mathsf{O}}$ and $\mathsf{E}_1^{\mathsf{O}}$. Suppose that there is a relation $\mathsf{E}_3^{\mathsf{O}} = \mathsf{E}_2^{\mathsf{O}} \oplus \mathsf{E}_1^{\mathsf{O}}$. In such a case only

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two orientations are independent and we have 2-oriented space. In this 2-oriented space the four different types of tensors may be defined. The classical theory of symmetry is correct for one of these types of tensors called polar tensors. The modified theory of symmetry for other types of tensors may be found in the paper [1].

At the moment oriented space is the most important and popular vector space in mechanics. By this reason the paper deals with the symmetries and orthogonal invariants in oriented space. Necessary generalizations may be obtained without any problems.

1.1 Classical theory

The classical theory of the tensor symmetry and the tensor invariants was born due to O. Cauchy and is extensively developing up to now. The books [2, 3, 4] contain the conventional statements of the problem. The modern applications of symmetries and invariants to mechanics may be found in the book [5]. Recall that almost all modern results of the invariant theory are obtained for the polynomial invariants [6]. The sufficiently complete list of the modern papers on the subject may be found in the paper [7].

In this subsection we reproduce the basic definitions of the classical theory in order to avoid the possible misunderstandings. In the sequel the direct tensor notation [8] is used. In some works [6] the term "direct notation" has another meaning: a notation $\mathbf{a}$ is assigned to the triple of vector coordinates.

In the sequel the next notation will be used

$$\underbrace{f,\ g,\ldots,h}_{\text{scalars}};\quad \underbrace{\mathbf{a}\equiv a^i\mathbf{g}_i,\ \mathbf{b},\ldots,\mathbf{c}}_{\text{vectors}};\quad \underbrace{\mathbf{A}\equiv A^{ij}\mathbf{g}_i\otimes\mathbf{g}_j,\ \mathbf{B},\ldots,\mathbf{C}}_{\text{2-rank tensors}};\quad \ldots,$$

where vectors $\mathbf{g}_i$ consist arbitrary basis in the reference system.

The set of second-rank tensors $\mathbf{Q}$ such that

$$\mathbf{Q}\cdot\mathbf{Q}^T=\mathbf{E},\quad \det\mathbf{Q}=\pm 1$$

is called orthogonal group, which contains infinitely many different elements but any of them may be generated by two orthogonal tensors. First of them is the tensor of a mirror reflection from the plane with unit normal $\mathbf{n}$. This tensor is determined by the expression

$$\mathbf{Q}=\mathbf{E}-2\mathbf{n}\otimes\mathbf{n},\quad \mathbf{Q}\cdot\mathbf{Q}^T=\mathbf{E},\quad \det\mathbf{Q}=-1. \tag{1}$$

The second tensor is the tensor of turn (rotation). With the help of Euler's theorem this tensor may be represented in the following form

$$\mathbf{Q}(\varphi\mathbf{m})\equiv(1-\cos\varphi)\mathbf{m}\otimes\mathbf{m}+\cos\varphi\mathbf{E}+\sin\varphi\,\mathbf{m}\times\mathbf{E},\quad \det\mathbf{Q}=+1, \tag{2}$$

where the unit vector $\mathbf{m}$ determines a strait line called the turn axis, an angle φ is called the turn angle. The action of the tensor (2) on a vector $\mathbf{a}$ is the turn of $\mathbf{a}$ around the vector $\mathbf{m}$ by the angle φ. Any orthogonal tensor may be represented as the composition of the tensors (1) and (2).

In classical theory the orthogonal transformation of the n-rank tensor $\mathbf{D}$ is defined by the formula

$$\mathbf{D}'\equiv\mathbf{Q}^n\odot\mathbf{D}\equiv\mathbf{Q}^n\odot\left(D^{i_1\ldots i_n}\mathbf{g}_{i_1}\otimes\ldots\otimes\mathbf{g}_{i_n}\right)\equiv D^{i_1\ldots i_n}\mathbf{Q}\cdot\mathbf{g}_{i_1}\otimes\ldots\otimes\mathbf{Q}\cdot\mathbf{g}_{i_n} \tag{3}$$

For example, for scalars, vectors and 2-rank tensors we have

$$f' \equiv f, \quad \mathbf{a}' \equiv \mathbf{Q} \cdot \mathbf{a}, \quad \mathbf{A}' \equiv \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^T.$$

Classical definition of the symmetry group*: the sets of orthogonal solutions of the equations*

$$\mathbf{a}' \equiv \mathbf{Q} \cdot \mathbf{a} = \mathbf{a}, \quad \mathbf{A}' \equiv \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^T = \mathbf{A}, \quad \mathbf{D}' \equiv \mathbf{Q}^n \odot \mathbf{D} = \mathbf{D} \tag{4}$$

are called the groups of symmetry (SG) of the vector $\mathbf{a}$*, 2-rank tensor* $\mathbf{A}$ *and n-rank tensor* $\mathbf{D}$ *correspondingly, where the vector* $\mathbf{a}$*, 2-rank tensor* $\mathbf{A}$ *and n-rank tensor* $\mathbf{D}$ *are given and orthogonal tensors* $\mathbf{Q}$ *must be found.*

Definition. *The n-rank tensor* $\mathbf{D}$ *is called isotropic if its group of symmetry contains all orthogonal tensors.*

There are two basic problems in the theory of symmetry:

Direct problem*: to find SG for the given system of tensors*.

Inverse problem*: to find the structure of a tensor of some order with given elements of symmetry*.

In non-oriented vector space the definition (4) leads to the correct results both from the mathematical and from physical points of view. However in oriented vector space with cross product of vectors this definition generates some paradoxical results from physical point of view.

Physical paradoxes.

1. Let $\mathbf{V}$ be a vector of translation velocity and $\boldsymbol{\omega}$ be a vector of the angular velocity such that $\mathbf{V} \times \boldsymbol{\omega} = 0$. Accordingly to the definition (4) these vectors have the same groups of symmetry. This is nonsense from physical point of view.

2. Tensor $\mathbf{E} \times \mathbf{E}$ is not isotropic. From physical point of view this result seems to be doubtful.

Definition*. A scalar-valued tensor function* $\psi(f, \mathbf{a}, \mathbf{A})$ *is said to be the orthogonal invariant if the equation*

$$\psi(f', \mathbf{a}', \mathbf{A}') = \psi(f, \mathbf{a}, \mathbf{A}) \tag{5}$$

holds for all orthogonal tensors $\mathbf{Q}$.

Theorem *(Gilbert). For any finite system of tensors there exist a finite basis of invariants, that means the finite system of the functionally independent scalar invariants such that all other invariants can be expressed in terms of basis invariants.*

The central problem in classical invariant theory: *for a given set of tensors and a given transformation group, determine a set of invariants from which all other invariants can be generated.*

Physical paradox: *a mixed product of three vectors* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ *is not invariant with respect to orthogonal group.*

It is easy to give a lot of other examples in which the classical theory leads to the physical mistakes. All these examples are connected with axial objects in oriented system of reference [9]. A lot of important discussions on the subject may be found in [10] – [22]. Some of them will be considered below.

1.2 A modified statement of the problem

The main purpose of the paper is to modify the classical theory in order to avoid the contradictions between mathematics and physics.

For this end we have to slightly change the definitions of the invariants and groups of symmetry. For example, the problem of invariants may be reformulated by the next manner. Let there be given two sets of tensors

$$\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_m, \quad \mathbf{A}_1, \mathbf{A}_2, \ldots \mathbf{A}_n \tag{6}$$

and

$$\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_m, \quad \mathbf{B}_1, \mathbf{B}_2, \ldots \mathbf{B}_n. \tag{7}$$

The invariant problem. *To find the minimal collections of the invariants for the system (6) and (7) whose coinciding is the guarantee of existing of the proper orthogonal tensor* $\mathbf{P}$: det $\mathbf{P} = 1$ *such that equalities*

$$\mathbf{b}_1 = \mathbf{P} \cdot \mathbf{a}_1, \quad \mathbf{b}_2 = \mathbf{P} \cdot \mathbf{a}_2, \ldots, \mathbf{b}_m = \mathbf{P} \cdot \mathbf{a}_m,$$
$$\mathbf{B}_1 = \mathbf{P} \cdot \mathbf{A}_1 \cdot \mathbf{P}^T, \quad \ldots, \quad \mathbf{B}_n = \mathbf{P} \cdot \mathbf{A}_n \cdot \mathbf{P}^T \tag{8}$$

holds. This statement will be called ***I-problem*** in what follows.

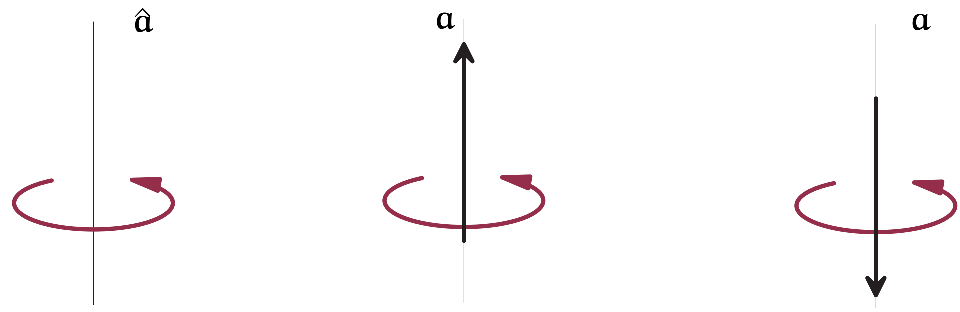
In other words, if the basis invariants for the system (6) and (7) coincide, then the system (7) may be obtained from the system (6) by the rigid rotation.

In the classical statement of the problem the tensor $\mathbf{P}$ in (8) may be orthogonal one rather then proper orthogonal tensor. For physical applications the tensor $\mathbf{P}$ must be the proper orthogonal tensor. This fact will be shown in what follows.

2 Orientation of Reference System. Polar and Axial Objects

The necessity of orientation of reference system arises due to our desire to take into account the moment interaction in mechanics. In the nature there are two principally different kinds of motion: the translation motion and the spinor (rotational) motion. Under translational motion a body is changing the position in the space. Under spinor motion a body is changing an orientation in the space without changing of position. The changing of translational motion is determined by forces. The changing of spinor motion is determined by moments. Note that in general moments can not be reduced to the concept of the force moment. In order to describe the spinor movements and the moment interactions we must orient the reference system and to introduce some new objects called axial objects in addition to the conventional objects called polar. There are many different but mathematically almost equivalent ways to introduce the space orientation. We prefer a way with clear physical sense. The physical (and mathematical) image of the spinor movement is given by so-called spin-vector whose introduction does not require the space orientation. Let there be given some system of reference (SR). Polar vector is represented in SR as an arrow. In addition to polar vector let us introduce a new object called spin-vector. For this it is necessary to take a strait line in SR called axis of a spin-vector. After that a circular arrow around the axis of a spin-vector must be drown in the plain orthogonal to the axis. Now we have a visual image of the spin-vector — see Figure 1a.

The length of the circular arrow is called a modulus of the spin-vector. A direction of a circular arrow shows the direction of a rotation. Spin-vectors describe characteristics of spinor movements. They are convenient for an intuition. However for the formal calculations it is much better to use so-called axial vectors. An axial vector can be obtained from a spin-vector with the



a). Object $\hat{\mathbf{a}}$ is called a spin vector

b). Strait arrow is the axial vector $\mathbf{a}$ corresponding to $\hat{\mathbf{a}}$ in the right-oriented SR

c). Strait arrow is the axial vector $\mathbf{a}$ corresponding to $\hat{\mathbf{a}}$ in the left-oriented SR

Figure 1: Oriented System of Reference

help of special rule called an orientation of the reference system. An axial vector $\mathbf{a}$ is associated with the spin-vector $\hat{\mathbf{a}}$ by means of the following rule:

1) $\mathbf{a}$ is placed on the axis of spin-vector $\hat{\mathbf{a}}$, 2) modulus of $\mathbf{a}$ is equal to the modulus of $\hat{\mathbf{a}}$, 3) the vector $\mathbf{a}$ is directed as shown at the Fig.1b (in such a case we have the right-oriented SR) ***or*** *as shown at the Fig.1c (in such a case we have the left-oriented SR).*

The concept of an axial vector introduced above is the exact expression of a physical idea about angular velocity, moment and so on. The introduction of axial vector does not require any system of coordinates. For example, the coordinate free introduction of the cross product of vectors with the using of spin-vector may be found in [9]. Let us consider two different coordinate systems in SR. Let $\mathbf{g}_i$ and $\mathbf{g}_{i'}$ be the local bases of these coordinate systems. If

$$\mathbf{g}_{i'} = h^m_{i'}\mathbf{g}_m, \quad \mathbf{g}^{i'} = h^{i'}_m \mathbf{g}^m, \quad h^{i'}_m h^m_{k'} = \delta^i_k, \quad h^{i'}_m h^n_{i'} = \delta^n_m,$$

then we have

$$\mathbf{a} = a^i\mathbf{g}_i = a^{i'}\mathbf{g}_{i'}, \quad a^{i'} = h^{i'}_m a^m. \tag{9}$$

This transformation of the vector components is valid both for polar vectors and for axial vectors. Here there is a contradiction with the conventional determinations for axial vectors whose coordinates are transformed accordingly to the rule (see, for example, [6])

$$a^{i'} = \det(h^{k'}_n) h^{i'}_m a^m,$$

where matrix $h^{k'}_n$ is supposed to be orthogonal.

From the pure mathematical point of view this definition is possible. However from physical point of view only the definition (9) must be used. Indeed the simplest example of axial vector is given by the cross product of two polar vectors

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} \quad \Rightarrow \quad c_p = a^m b^n (\mathbf{g}_m \times \mathbf{g}_n) \cdot \mathbf{g}_p.$$

In other system of coordinates we have

$$c_{p'} = a^{m'} b^{n'} (\mathbf{g}_{m'} \times \mathbf{g}_{n'}) \cdot \mathbf{g}_{p'} = a^r h^{m'}_r b^s h^{n'}_s (\mathbf{g}_{m'} \times \mathbf{g}_{n'}) \cdot h^t_{p'} \mathbf{g}_t =$$

$$= h^t_{p'} a^r b^s (\mathbf{g}_r \times \mathbf{g}_s) \cdot \mathbf{g}_t = h^t_{p'} c_t.$$

For axial vector we obtain a standard law of transformation for covariant coordinates of vectors.

Thus in the oriented SR there are two kinds of objects: polar objects and axial objects.

Definition. *Objects, which are independent of the choice of orientation of SR, are called the polar objects; objects, which depend on the choice of orientation of SR and are multiplied by (-1) under changing of orientation of SR, are called the axial objects.*

In according with the definition axial objects may be represented by scalars, vectors and tensors of any rank. The well known examples of axial objects are: the mixed product of three polar vectors $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, vector of angular velocity, the Levi - Civita tensor $\mathbf{E} \times \mathbf{E}$ and so on.

3 Modified Definition of Orthogonal Transformation

In oriented space the definition of orthogonal transformation (3) must be slightly changed.

Definition. *The orthogonal transformation of the scalar* g*, of the vector* $\mathbf{a}$*, of the second-rank tensor* $\mathbf{A}$ *and of the n-rank tensor* $\mathbf{D}$ *are defined by formulae*

$$g' \equiv (\det\mathbf{Q})^\alpha g, \quad \mathbf{a}' \equiv (\det\mathbf{Q})^\alpha \mathbf{Q} \cdot \mathbf{a}, \quad \mathbf{A}' \equiv (\det\mathbf{Q})^\alpha \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^T,$$
$$\mathbf{D}' \equiv (\det\mathbf{Q})^\alpha \mathbf{Q}^n \odot \mathbf{D}, \quad (10)$$

where $\alpha = 0$ *for polar objects,* $\alpha = 1$ *for axial objects.*

The definition (10) coincides with the classical definition (3) for the polar tensors. Correctness of the definition (10) can be shown by means of simple examples. Let us consider the axial scalar $\psi = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, where all vectors are polar ones. Then the orthogonal transformation of the scalar ψ may me found directly

$$\psi' = \mathbf{a}' \cdot \left(\mathbf{b}' \times \mathbf{c}'\right) = (\mathbf{Q} \cdot \mathbf{a}) \cdot [(\mathbf{Q} \cdot \mathbf{b}) \times (\mathbf{Q} \cdot \mathbf{c})] =$$
$$= (\mathbf{a} \cdot \mathbf{Q}^T) \cdot [(\det\mathbf{Q})\mathbf{Q} \cdot (\mathbf{b} \times \mathbf{c})] = (\det\mathbf{Q})\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\det\mathbf{Q})\psi,$$

where the identity

$$(\mathbf{Q} \cdot \mathbf{b}) \times (\mathbf{Q} \cdot \mathbf{c}) = (\det\mathbf{Q})\, \mathbf{Q} \cdot (\mathbf{b} \times \mathbf{c})$$

was used.

As a result we obtain the definition (10). The vector product of two polar vectors is a typical example of an axial vector. In this case the direct definition is possible as well

$$\mathbf{c}' = \mathbf{a}' \times \mathbf{b}' = (\mathbf{Q} \cdot \mathbf{a}) \times (\mathbf{Q} \cdot \mathbf{b}) = (\det\mathbf{Q})\, \mathbf{Q} \cdot (\mathbf{a} \times \mathbf{b}) = (\det\mathbf{Q})\, \mathbf{Q} \cdot \mathbf{c}.$$

By this way the definition (10) may be derived for a tensor of any rank.

4 Symmetry Groups of Tensors

Definition. *The sets of the orthogonal solutions of the equations*

$$(\det\mathbf{Q})^\alpha \psi = \psi, \;\; (\det\mathbf{Q})^\alpha \mathbf{Q} \cdot \mathbf{a} = \mathbf{a}, \;\; (\det\mathbf{Q})^\alpha \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^T = \mathbf{A}, \;\; (\det\mathbf{Q})^\alpha \mathbf{Q}^n \odot \mathbf{D} = \mathbf{D} \quad (11)$$

are called the symmetry groups (SG) of the scalar ψ, *the vector* $\mathbf{a}$, *the second rank tensor* $\mathbf{A}$ *and the n-rank tensor* $\mathbf{D}$ *correspondingly, where* ψ, $\mathbf{a}$, $\mathbf{A}$ *and* $\mathbf{D}$ *are given, orthogonal tensors* $\mathbf{Q}$ *must be found.*

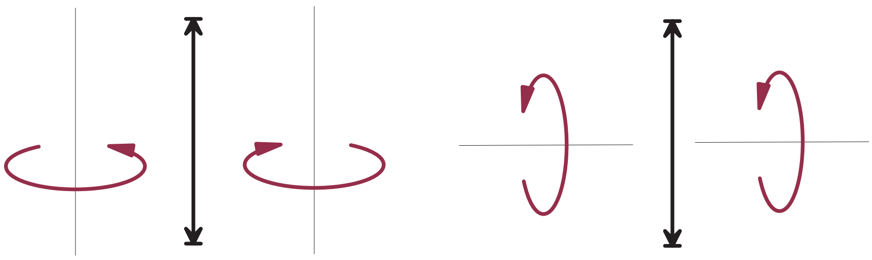
Let us consider some examples.

Symmetry group of vector. For a polar vector $\mathbf{a}$ the SG contain tensors (2), where $\mathbf{m} = \mathbf{a}/|\mathbf{a}|$, and tensors (1), where $\mathbf{n}$ is any unit vector such that $\mathbf{n} \cdot \mathbf{a} = 0$.

In order to establish SG of an axial vector $\mathbf{a}$ the orthogonal solutions of the equation

$$(\det\mathbf{Q})\mathbf{Q} \cdot \mathbf{a} = \mathbf{a}$$

must be found. It is easy to see that the SG of the axial vector $\mathbf{a}$ contains tensors (2), where $\mathbf{m} \times \mathbf{a} = 0$, and tensors (1), where $\mathbf{n} \times \mathbf{a} = 0$. The correctness of this result may be easy seen at the Fig. 2.



a). The mirror reflection from the plane parallel to the axis of spin-vector

b). The mirror reflection from the plane orthogonal to the axis of spin-vector

Figure 2: On a symmetry of an axial vector

Thus the symmetry elements of polar and axial vectors are different under the mirror reflections.

Usually it is not difficult to find the SG of any given tensor. However in the theory of constitutive equations the solution of the inverse problem is much more important. For this it is necessary to find the structure of a tensor with given elements of symmetry.

Second rank tensor with one plane of the mirror symmetry. Polar and axial tensors with the same elements of symmetry have different structures. Let, for example, the mirror reflection $\mathbf{E} - 2\mathbf{m} \otimes \mathbf{m}$ belongs to the SG of a polar tensor $\mathbf{A}$ and an axial tensor $\mathbf{B}$. It is possible if and only if these tensors have the form

$$\mathbf{A} = A_{11}\mathbf{m} \otimes \mathbf{m} + A_{22}\mathbf{n} \otimes \mathbf{n} + A_{23}\mathbf{n} \otimes \mathbf{p} + A_{32}\mathbf{p} \otimes \mathbf{n} + A_{33}\mathbf{p} \otimes \mathbf{p},$$

$$\mathbf{B} = B_{12}\mathbf{m} \otimes \mathbf{n} + B_{13}\mathbf{m} \otimes \mathbf{p} + B_{21}\mathbf{n} \otimes \mathbf{m} + B_{31}\mathbf{p} \otimes \mathbf{m},$$

where A_{ik} are absolute scalars and B_{ik} are axial scalars, $\mathbf{m}$, $\mathbf{n}$, $\mathbf{p}$ is an orthogonal basis. If there are two planes of mirror symmetry with unit normals $\mathbf{m}$ and $\mathbf{n}$, then

$$\mathbf{A} = A_{11}\mathbf{m} \otimes \mathbf{m} + A_{22}\mathbf{n} \otimes \mathbf{n} + A_{33}\mathbf{p} \otimes \mathbf{p}, \quad \mathbf{B} = B_{12}\mathbf{m} \otimes \mathbf{n} + B_{21}\mathbf{n} \otimes \mathbf{m}. \qquad (12)$$

Example: naturally twisted rods. *The specific energy of thin elatic rods is determined by quadratic form*

$$\mathcal{U} = \frac{1}{2}\mathbf{e} \cdot \mathbf{A} \cdot \mathbf{e} + \mathbf{e} \cdot \mathbf{B} \cdot \boldsymbol{\kappa} + \frac{1}{2}\boldsymbol{\kappa} \cdot \mathbf{C} \cdot \boldsymbol{\kappa}, \tag{13}$$

where the vectors of deformation are defined by the expressions

$$\mathbf{e} = \mathbf{u}' + \mathbf{p} \times \boldsymbol{\varphi}, \quad \boldsymbol{\kappa} = \boldsymbol{\varphi}'.$$

Let vectors $\mathbf{m}$ *and* $\mathbf{n}$ *be principle directions of the rod cross section. Let tensors* $\mathbf{E} - 2\mathbf{m} \otimes \mathbf{m}$ *and* $\mathbf{E} - 2\mathbf{n} \otimes \mathbf{n}$ *be elements of symmetry for the cross section. If we apply the classical theory of symmetry, then we obtain that in such a case the tensors of elasticity* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$ *in (13) have the form of the tensor* $\mathbf{A}$ *in (12). This is nonsense from physical point of view. If we use the modified theory of symmetry, then only the polar tensors of elasticity* $\mathbf{A}$ *and* $\mathbf{C}$ *in (13) have the form of the tensor* $\mathbf{A}$ *in (12) but the axial tensor* $\mathbf{B}$ *in (13) has the form of the tensor* $\mathbf{B}$ *in (12). Let* $\mathbf{p}$ *be the unit normal vector to the cross section. As a rule tensor* $\mathbf{E} - 2\mathbf{p} \otimes \mathbf{p}$ *belongs to the SG of the rod. In such a case the axial tensor of elasticity* $\mathbf{B}$ *in (13) is equal to zero. However for naturally twisted rods (for example, drills) the tensor* $\mathbf{E} - 2\mathbf{p} \otimes \mathbf{p}$ *does not belong to the SG of the rod. Because of this the axial tensor of elasticity* $\mathbf{B}$ *in (13) has the form of the tensor* $\mathbf{B}$ *in (12). This result can not be obtained with the help of the classical theory of symmetry.*

Definition. *The n-rank tensor is called isotropic one if its group of symmetry contains all orthogonal tensors.*

There is one polar isotropic 2nd-rank tensor $f\mathbf{E}$, where f is an absolute scalar. There are no axial isotropic tensors of the 2nd-rank. There are no polar isotropic tensors of the 3d-rank. However there is one axial isotropic 3d-rank tensor $f\mathbf{E} \times \mathbf{E}$, where f is an absolute scalar. Indeed, according to the definition (10) we have

$$(\mathbf{E} \times \mathbf{E})' \equiv (\mathbf{g}^m \otimes \mathbf{g}_m \times \mathbf{g}^n \otimes \mathbf{g}_n)' \equiv (\det\mathbf{Q})\mathbf{Q} \cdot \mathbf{g}^m \otimes \mathbf{Q} \cdot (\mathbf{g}_m \times \mathbf{g}^n) \otimes \mathbf{Q} \cdot \mathbf{g}_n =$$

$$= [\mathbf{Q} \cdot (\mathbf{g}^m \otimes \mathbf{g}_m) \cdot \mathbf{Q}^T] \times [\mathbf{Q} \cdot (\mathbf{g}^n \otimes \mathbf{g}_n) \cdot \mathbf{Q}^T] = [\mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^T] \times [\mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^T] = \mathbf{E} \times \mathbf{E}.$$

The tensor $f\mathbf{E} \times \mathbf{E}$ is supposed to be non-isotropic under the conventional approach. This fact is important in the theory of piezoelectricity.

5 Orthogonal invariants and theorem on basis

Let there be given the finite collection of the tensors (6).

Definition. *A scalar-valued tensor function*

$$F = F(\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_m, \mathbf{A}_1, \mathbf{A}_2, \ldots \mathbf{A}_n)$$

is called an orthogonal invariant of the collection (6) if the equality

$$F(\mathbf{a}_1', \ldots, \mathbf{a}_m', \mathbf{A}_1', \ldots, \mathbf{A}_n') = (\det\mathbf{Q})^\alpha F(\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{A}_1, \ldots \mathbf{A}_n), \tag{14}$$

holds for all orthogonal tensors $\mathbf{Q}$*; the quantities with primes are defined by (10);* $\alpha = 0$*, if values of* F *are absolute scalars, and* $\alpha = 1$*, if values of* F *are axial scalars.*

Let us consider the function

$$\psi(\mathbf{a}, \mathbf{b}, \mathbf{c}) \equiv (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}, \tag{15}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are polar vectors. Accordingly to classical definition (5) the function ψ is not orthogonal invariant. In accordance with the definition (14) the function ψ is orthogonal invariant since

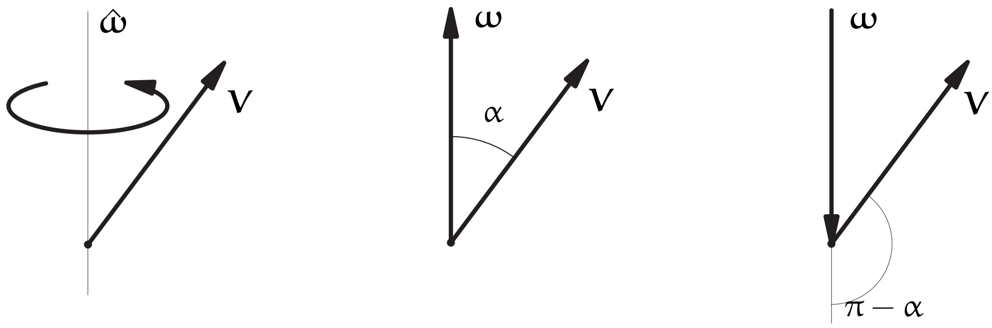
$$\psi(\mathbf{a}', \mathbf{b}', \mathbf{c}') = (\mathbf{a}' \times \mathbf{b}') \cdot \mathbf{c}' = (\det \mathbf{Q})(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\det \mathbf{Q})\psi(\mathbf{a}, \mathbf{b}, \mathbf{c}).$$

Another example is the scalar product of polar $\mathbf{V}$ and axial $\boldsymbol{\omega}$ vectors

$$\psi(\mathbf{V}, \boldsymbol{\omega}) \equiv \mathbf{V} \cdot \boldsymbol{\omega}. \tag{16}$$

With respect to definition (5) this function is not an orthogonal invariant. In accordance with the definition (14) the function (16) is orthogonal invariant. From the pure mathematical point of view this is a question of the definitions and there is no subject for discussions. However the situation is quiet different if we consider the problem from physical point of view.

Let us consider an one-spin particle — see Figure 3. An one-spin particle, shown with using



a). One-spin particle: a physical object

b). One-spin particle: the mathematical image in the right-oriented SR

c). One-spin particle: the mathematical image in the left-oriented SR

Figure 3: One spin-particle: physical and mathematical images

of spin-vector, is presented at the Figure 3a. This image does not depend on the SR orientation. The mathematical image, obtained with using of an axial vector, is shown on the Figure 3b in the right-oriented SR and on the Figure 3c in the left-oriented SR. It is seen that

$$(\mathbf{V} \cdot \boldsymbol{\omega})_R \equiv V\omega \cos\alpha = -(\mathbf{V} \cdot \boldsymbol{\omega})_L \equiv -V\omega \cos(\pi - \alpha).$$

Let us note that the nature and physical objects, for example spin-vectors, know nothing about orientation of SR. Axial vectors are some mathematical inventions and they feel the change of the SR orientation. The scalar product $(\mathbf{V} \cdot \boldsymbol{\omega})_R$ in the right-oriented SR is not equal to the scalar product $(\mathbf{V} \cdot \boldsymbol{\omega})_L$ in the left-oriented SR. However both of them correspond to the single physical object. By this reason the scalar product (16) must be called an invariant. This fact is taken into account by the definition (14).

Now we are able to discuss the problem of invariants. It is obvious that Hilbert's theorem is valid both for the definition (14) of orthogonal invariants of the collection (6) and for the classical definition (5). However Hilbert's theorem says nothing about the number of functionally independent invariants consisting the basis. From the pure physical point of view the basis dimension may be found by simple calculation. The result of this calculation leads to the following statement.

Theorem. *The dimension* N_* *of the invariant basis of the collection (6) is related with the number* N *of independent coordinates of objects in (6) by the next formulae*

$$N_* = 1 \ \textit{when} \ m = 1,\, n = 0; \quad N_* = N - 3 \tag{17}$$

in all other cases.

The proof of the theorem will be given in what follows. Let us note that well-known Rivlin's theorem states that $N_* = N - 2$.

6 Generic Equation for Invariants

The definition (14) of an invariant of the collection (6) contain an arbitrary orthogonal tensor $\mathbf{Q}$. In what follows without loss of generality we may suppose that tensor $\mathbf{Q}$ is a proper orthogonal tensor since an arbitrary orthogonal tensor $\mathbf{Q}$ can be represented as composition $\mathbf{Q} = (-\mathbf{E}) \cdot \mathbf{P}$, where $\mathbf{P}$ is a proper orthogonal tensor. The inversion transformation may be taken into account later. In such a case let us consider a continuous set of orthogonal tensors $\mathbf{Q}(\tau)$, depending on real parameter τ. It is easy to prove that there exist an axial vector $\boldsymbol{\omega}(\tau)$ satisfying the equation

$$\frac{d}{d\tau}\mathbf{Q}(\tau) = \boldsymbol{\omega}(\tau) \times \mathbf{Q}(\tau), \quad \mathbf{Q}(0) = \mathbf{E}, \qquad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 \neq \mathbf{0}. \tag{18}$$

If the orthogonal tensor $\mathbf{Q}(\tau)$ in (14) depend on the parameter τ, then we may differentiate both sides of (14) with respect to τ. In such a case we obtain the following equation

$$\sum_{i=1}^{m} \frac{\partial F}{\partial \mathbf{a}'_i} \cdot \frac{d\mathbf{a}'_i}{d\tau} + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial \mathbf{A}'_i}\right)^T \cdot\cdot \frac{d\mathbf{A}'_i}{d\tau} = 0. \tag{19}$$

The derivatives of a scalar function with respect to vector and tensor arguments is defined by the rule

$$dF = \sum_{i=1}^{m} \frac{\partial F}{\partial \mathbf{a}_i} \cdot d\mathbf{a}_i + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial \mathbf{A}_i}\right)^T \cdot\cdot d\mathbf{A}_i. \tag{20}$$

An example. *Let there be given a scalar function*

$$F(\mathbf{a}, \mathbf{b}, \mathbf{A}) = \mathbf{a} \cdot \mathbf{A} \cdot \mathbf{b} \quad \Rightarrow \quad dF = (\mathbf{A} \cdot \mathbf{b}) \cdot d\mathbf{a} + (\mathbf{a} \cdot \mathbf{A}) \cdot d\mathbf{b} + (\mathbf{a} \otimes \mathbf{b})^T \cdot\cdot d\mathbf{A}.$$

According to (20) one has

$$\frac{\partial F}{\partial \mathbf{a}} = \mathbf{A} \cdot \mathbf{b}, \quad \frac{\partial F}{\partial \mathbf{b}} = \mathbf{a} \cdot \mathbf{A}, \quad \frac{\partial F}{\partial \mathbf{A}} = \mathbf{a} \otimes \mathbf{b}.$$

Making use of equality (18), the derivatives may be calculated

$$\frac{d\mathbf{a}'_i}{d\tau} = \boldsymbol{\omega}(\tau) \times \mathbf{a}'_i, \quad \frac{d\mathbf{A}'_i}{d\tau} = \boldsymbol{\omega}(\tau) \times \mathbf{A}'_i - \mathbf{A}'_i \times \boldsymbol{\omega}(\tau).$$

Let us take into account the equalities

$$\mathbf{a}'_i(0) = \mathbf{a}_i, \quad \mathbf{A}'_i(0) = \mathbf{A}_i.$$

Let τ in the equality (19) be equal to zero. Using the equalities written above we obtain

$$\sum_{i=1}^{n} \left(\frac{\partial F}{\partial \mathbf{A}_i}\right)^T \cdot\cdot (\boldsymbol{\omega}_0 \times \mathbf{A}_i - \mathbf{A}_i \times \boldsymbol{\omega}_0) + \sum_{i=1}^{m} \frac{\partial F}{\partial \mathbf{a}_i} \cdot (\boldsymbol{\omega}_0 \times \mathbf{a}_i) = 0. \tag{21}$$

The equation (21) is the linear homogenous equation in the partial derivatives of the first order. This equation must be valid for any vector $\boldsymbol{\omega}_0$. Because of this the equation (21) is equivalent to the three scalar equations. Any scalar invariant of the collection (6) must be solution of the equation (21). Instead of the eq. (21) it is more convenient to use an equation obtained from the eq. (21) by dividing on the modulus of vector $\boldsymbol{\omega}_0$

$$\sum_{i=1}^{n} \left(\frac{\partial F}{\partial \mathbf{A}_i}\right)^T \cdot\cdot (\mathbf{m} \times \mathbf{A}_i - \mathbf{A}_i \times \mathbf{m}) + \sum_{i=1}^{m} \frac{\partial F}{\partial \mathbf{a}_i} \cdot (\mathbf{m} \times \mathbf{a}_i) = 0, \tag{22}$$

which must be valid for any unit vector $\mathbf{m}$.

In what follows the equation (22) will be called the generic equation for the invariants. The unit vector $\mathbf{m}$ may be excluded from the equation (22). In such a case instead of the scalar equation (22) we obtain the vector equation

$$\sum_{i=1}^{n} \left[\left(\frac{\partial F}{\partial \mathbf{A}_i}\right)^T \cdot \mathbf{A}_i + \mathbf{A}_i \cdot \left(\frac{\partial F}{\partial \mathbf{A}_i}\right)^T\right]_\times + \sum_{i=1}^{m} \frac{\partial F}{\partial \mathbf{a}_i} \times \mathbf{a}_i = \mathbf{0}, \quad (\mathbf{a} \otimes \mathbf{b})_\times \equiv \mathbf{a} \times \mathbf{b}, \tag{23}$$

where the vector $\mathbf{A}_\times$ is called a vector invariant of the 2nd-rank tensor $\mathbf{A}$. The vector equation (23) is equivalent to the three scalar equations. Any orthogonal invariant must satisfy this three equations. However in general case not all of these equations are independent. If the collection (6) contains more than one vector, then all three equations are independent.

The equation (23) is a system of three linear equation in partial derivatives. Any scalar invariant of the collection (6) must be solution of the equation (21). And conversely, any solution of equation (23) is the invariant of the collection (6). Coordinates of vectors and tensors of the set (6) are independent variables. Therefore this equation is defined in the space of dimension $N = 3m + 6n$. The function $F(\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_m, \quad \mathbf{A}_1, \mathbf{A}_2, \ldots \mathbf{A}_n)$ depends on N arguments. The theory of the equations in the partial derivatives of the first order is well developed. It may be said that each of the scalar equation of the system (23) decreases the number of independent variables on one. The number of the rest independent variables is the number of functionally independent invariants. Thus the dimension of the invariant basis is equal to $N - q$, where $2 \le q \le 3$ is the number of independent equation of the system (23). If $N = 3$, then $q = 2$. If $N > 3$, then $q = 3$.

Below the application of this theorem will be considered.

7 The basis invariant of vector

In this case the equation (23) takes a simple form

$$\frac{\partial F}{\partial \mathbf{a}} \times \mathbf{a} = 0 \quad \Leftrightarrow$$

$$a_1 \frac{\partial F}{\partial a_2} - a_2 \frac{\partial F}{\partial a_1} = 0, \; a_2 \frac{\partial F}{\partial a_3} - a_3 \frac{\partial F}{\partial a_2} = 0, \; a_1 \frac{\partial F}{\partial a_3} - a_3 \frac{\partial F}{\partial a_1} = 0,$$

where a_i are coordinates of vector $\mathbf{a}$ with respect to some orthogonal basis.

The last equation is a consequence of two previous equations. Because of this there are only two independent equations

$$a_1 \frac{\partial F}{\partial a_2} - a_2 \frac{\partial F}{\partial a_1} = 0, \quad a_2 \frac{\partial F}{\partial a_3} - a_3 \frac{\partial F}{\partial a_2} = 0. \tag{24}$$

The function F must satisfy two independent equation (24) in partial derivatives of the first order. We have to find a general solution of the first equation. After that this general solution must be obeyed to the second equation.

In order to find a general solution of the first equation we have to write down the characteristic system for this equation

$$\frac{da_1}{ds} = -a_2, \; \frac{da_2}{ds} = a_1 \;\Rightarrow\; \frac{d}{ds}\left(a_1^2 + a_2^2\right) = 0, \tag{25}$$

where $a_i(s)$ is the parametric form of a curve in 3D-space of coordinates a_i, s is the parameter. The linear system (25) of the second order has not more then one independent integral. Any function of this integral is an integral of (25) as well. The integral shown by the last equality in (25) may be taken as such independent integral. Thus a general solution of the first equation from (25) is given by expression

$$F(\mathbf{a}) = F(a_1, \, a_2, \, a_3) = f(a_1^2 + a_2^2, \, a_3).$$

Substituting this solution into the second equation from system (24), we obtain

$$\frac{\partial f}{\partial a_3} - 2a_3 \frac{\partial f}{\partial q} = 0, \quad q \equiv a_1^2 + a_2^2.$$

The characteristic system for this equation has a form

$$\frac{dq}{ds} = -2\,a_3, \quad \frac{da_3}{ds} = 1 \quad \Rightarrow \quad \frac{d}{ds}\left(q + a_3^2\right) = 0.$$

This system of the second order has the only independent integral. One may choose the integral $q + a_3^2$. Therefore any orthogonal invariant may be expressed as a function of the modulus of vector $\mathbf{a}$

$$F(\mathbf{a}) = f(q, \, a_3) = g(\mathbf{a} \cdot \mathbf{a}).$$

In this case the basic theorem has been almost proved. Of course this result is well-known and was obtained by O. Cauchy.

Let us obtain this result by direct coordinate-free approach. The equation (22) and corresponding to it the characteristic system take a form

$$\frac{\partial F}{\partial \mathbf{a}} \cdot (\mathbf{m} \times \mathbf{a}) = 0, \quad \frac{d\mathbf{a}}{ds} = \mathbf{m} \times \mathbf{a}. \tag{26}$$

The vector characteristic equation (26) has the third order and two independent integrals

$$\mathbf{a} \cdot \mathbf{a} = \mathrm{const}, \quad \mathbf{m} \cdot \mathbf{a} = \mathrm{const}.$$

The second integral depends on arbitrary vector $\mathbf{m}$ and must be taken off. The coordinate-free approach is much shorter and will be used below.

In order to finish the proof of theorem we have to show that two vectors with the same moduli may be transformed from one to other by means of pure turn. Let there be given two vectors $\mathbf{a}$ and $\mathbf{b}$, whose moduli and kinds are the same. A general transformation of a vector, conserving its modulus, is given by

$$\mathbf{a} = (\det\mathbf{Q})^{\alpha}\mathbf{Q} \cdot \mathbf{b}, \quad \mathbf{Q} \cdot \mathbf{Q}^{T} = \mathbf{E}, \quad \det\mathbf{Q} = \pm 1,$$

where $\alpha = 0$ for polar vectors and $\alpha = 1$ for axial vectors. It is not what we wont. In ***I-problem*** the tensor $\mathbf{Q}$ must be proper orthogonal tensor. Let us note that the tensor $\mathbf{Q}$ in the last equality is not completely defined. Indeed this equality may be rewritten in the equivalent form

$$\mathbf{a} = (\det\mathbf{Q})^{\alpha}\mathbf{Q} \cdot (\det\mathbf{S})^{\alpha}\mathbf{S} \cdot \mathbf{b}, \quad (\det\mathbf{S})^{\alpha}\mathbf{S} \cdot \mathbf{b} = \mathbf{b},$$

where tensor $\mathbf{S}$ belongs to SG of vector $\mathbf{b}$. Thus we have

$$\mathbf{a} = \mathbf{Q}_* \cdot \mathbf{b}, \quad \mathbf{Q}_* \equiv \det(\mathbf{Q} \cdot \mathbf{S})^{\alpha}\, \mathbf{Q} \cdot \mathbf{S}, \quad \det\mathbf{Q}_* = [\det(\mathbf{Q} \cdot \mathbf{S})]^{1+\alpha} = 1.$$

If $\alpha = 1$, then the last condition is valid identically. If $\alpha = 0$, then it may be satisfied by appropriate choice of the tensor $\mathbf{S}$. For example, if $\det \mathbf{Q} = -1$, then we may take $\mathbf{S} = \mathbf{E} - 2\mathbf{m} \otimes \mathbf{m}$, where $\mathbf{m} \cdot \mathbf{b} = 0, \quad |\mathbf{m}| = 1$.

8 Basic invariants for a set of three vectors

Let us find the minimally complete set of basic invariants for a collection of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The equation (22) for invariant $F(\mathbf{a}, \mathbf{b}, \mathbf{c})$ takes the form

$$\frac{\partial F}{\partial \mathbf{a}} \cdot (\mathbf{m} \times \mathbf{a}) + \frac{\partial F}{\partial \mathbf{b}} \cdot (\mathbf{m} \times \mathbf{b}) + \frac{\partial F}{\partial \mathbf{c}} \cdot (\mathbf{m} \times \mathbf{c}) = 0.$$

The characteristic system for this equation is

$$\frac{d\mathbf{a}}{ds} = \mathbf{m} \times \mathbf{a}, \quad \frac{d\mathbf{b}}{ds} = \mathbf{m} \times \mathbf{b}, \quad \frac{d\mathbf{c}}{ds} = \mathbf{m} \times \mathbf{c}. \tag{27}$$

General solution of (27) is given by

$$\mathbf{a}(s) = \mathbf{Q}(s\mathbf{m}) \cdot \mathbf{a}_0, \; \mathbf{b}(s) = \mathbf{Q}(s\mathbf{m}) \cdot \mathbf{b}_0, \; \mathbf{c}(s) = \mathbf{Q}(s\mathbf{m}) \cdot \mathbf{c}_0, \tag{28}$$

where vectors $\mathbf{a}_0$, $\mathbf{b}_0$, $\mathbf{c}_0$ are arbitrary constant vectors. The system (27) of ninth order has not more than eight integrals. In order to find these integrals, it is necessary to exclude the variable s and the vector $\mathbf{m}$ from (28). It is easy to build up ten integrals. Three of them depend on arbitrary vector $\mathbf{m}$

$$\mathbf{m}\cdot\mathbf{a} = \mathbf{m}\cdot\mathbf{a}_0, \quad \mathbf{m}\cdot\mathbf{b} = \mathbf{m}\cdot\mathbf{b}_0, \quad \mathbf{m}\cdot\mathbf{c} = \mathbf{m}\cdot\mathbf{c}_0.$$

These integrals must be ignored.

Seven invariants are determined by the next integrals

$$I_1 = \mathbf{a}\cdot\mathbf{a}, \quad I_2 = \mathbf{b}\cdot\mathbf{b}, \quad I_3 = \mathbf{c}\cdot\mathbf{c}, \quad I_4 = \mathbf{a}\cdot\mathbf{b}, \quad I_5 = \mathbf{a}\cdot\mathbf{c}, \quad I_6 = \mathbf{b}\cdot\mathbf{c}, \tag{29}$$

$$I_7 = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}). \tag{30}$$

Invariants $I_1 - I_7$ are not independent since there is an obvious relation

$$I_7^2 \equiv \begin{vmatrix} I_1 & I_4 & I_5 \\ I_4 & I_2 & I_6 \\ I_5 & I_6 & I_3 \end{vmatrix}.$$

Therefore seven integrals (29) and (30) may be expressed in terms of six functionally independent integrals.

This example clearly shows the distinction between classical approach and the approach under consideration. Accordingly to classical definitions the axial scalar is not invariant. Because of this some scalars must be excluded from the list (29). For example, let vectors $\mathbf{a}$ and $\mathbf{b}$ be polar ones but $\mathbf{c}$ is an axial vector. In such a case the invariants I_5 and I_6 are the axial scalars and must be excluded from the list (29). That means that the classical problem of invariants has no solution. If all three vectors are polar ones, then from the classical point of view the invariant I_7 must be excluded from the list (29)–(30). However the fixation of the invariants $I_1 - I_6$ does not fix the triple of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as a rigid whole. In such a case the ***I-problem*** has no solution. Indeed let us consider the two triples of vectors: $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and

$$\mathbf{a}, \quad \mathbf{b}, \quad (\mathbf{E} - 2\mathbf{k}\otimes\mathbf{k})\cdot\mathbf{c}, \quad \mathbf{k} = \mathbf{a}\times\mathbf{b}/|\mathbf{a}\times\mathbf{b}|.$$

This triple of vectors is formed from polar vectors. Invariants $I_1 - I_6$ are the same for both triples of vectors. Nevertheless the second triple can not be obtained from the first triple by rigid rotation. Invariants (29)–(30) for these triples are different.

In order to solve the ***I-problem*** we must prove the existing of the proper orthogonal tensor $\mathbf{P}$ such that the relations (8) holds good. Let there be given two triples of vectors

$$\mathbf{a}, \mathbf{b}, \mathbf{c} \qquad \text{and} \qquad \mathbf{a}_* = a\mathbf{m}, \quad \mathbf{b}_* = b\mathbf{n}, \quad \mathbf{c}_* = c\mathbf{p}.$$

Invariants $I_1 - I_3$ for these triples are supposed to be the same. In such a case we have

$$\mathbf{a} = a\mathbf{Q}_a\cdot\mathbf{m}, \quad \mathbf{b} = b\mathbf{Q}_b\cdot\mathbf{n}, \quad \mathbf{c} = c\mathbf{Q}_c\cdot\mathbf{p}, \tag{31}$$

where $\mathbf{Q}_a$, $\mathbf{Q}_b$, $\mathbf{Q}_c$ are orthogonal tensors. The coincidence of invariants $I_4 - I_6$ gives equations for determining of tensors $\mathbf{Q}_a$, $\mathbf{Q}_b$, $\mathbf{Q}_c$

$$\mathbf{m}\cdot\mathbf{Q}_a^T\cdot\mathbf{Q}_b\cdot\mathbf{n} = \mathbf{m}\cdot\mathbf{n}, \quad \mathbf{m}\cdot\mathbf{Q}_a^T\cdot\mathbf{Q}_c\cdot\mathbf{p} = \mathbf{m}\cdot\mathbf{p}, \quad \mathbf{n}\cdot\mathbf{Q}_b^T\cdot\mathbf{Q}_c\cdot\mathbf{p} = \mathbf{n}\cdot\mathbf{p}. \tag{32}$$

From (32) the next equalities may be obtained

$$\mathbf{Q}_a^T \cdot \mathbf{Q}_b = \mathbf{S}_{(m,1)} \cdot \mathbf{S}_{(n,1)}, \quad \mathbf{Q}_a^T \cdot \mathbf{Q}_c = \mathbf{S}_{(m,2)} \cdot \mathbf{S}_{(p,1)}, \quad \mathbf{Q}_b^T \cdot \mathbf{Q}_c = \mathbf{S}_{(n,2)} \cdot \mathbf{S}_{(p,2)}, \quad (33)$$

where orthogonal tensors $\mathbf{S}_{(m,k)}$, $\mathbf{S}_{(n,k)}$, $\mathbf{S}_{(p,k)}$ are some elements of symmetry of the vectors $\mathbf{m}$, $\mathbf{n}$ and $\mathbf{p}$ correspondingly. The equalities (33) lead to the restrictions

$$\mathbf{Q}_b^T \cdot \mathbf{Q}_c = \mathbf{Q}_b^T \cdot \mathbf{Q}_a \cdot \mathbf{Q}_a^T \cdot \mathbf{Q}_c = \mathbf{S}_{(n,1)}^T \cdot \mathbf{S}_{(m,1)}^T \cdot \mathbf{S}_{(m,2)} \cdot \mathbf{S}_{(p,1)} = \mathbf{S}_{(n,2)} \cdot \mathbf{S}_{(p,2)}. \quad (34)$$

Now the equalities (31) takes a form

$$\mathbf{a} = a\mathbf{Q}_a \cdot \mathbf{m}, \ \mathbf{b} = b\mathbf{Q}_a \cdot \mathbf{S}_{(m,1)} \cdot \mathbf{n}, \ \mathbf{c} = c\mathbf{Q}_a \cdot \mathbf{S}_{(m,2)} \cdot \mathbf{p},$$

where tensors $\mathbf{S}_{(m,1)}$, $\mathbf{S}_{(m,2)}$ are two arbitrary elements of symmetry of $\mathbf{m}$. Invariants $I_1 - I_6$ of two triples under consideration for any orthogonal tensor $\mathbf{Q}_a$ coincide. However the distinction between these triples of vectors can not be reduced to the rigid rotation. If the invariants I_7 for these triple of vectors coincide, then the equation

$$\det\mathbf{Q}_a \det\mathbf{S}_{m,1} (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{S}_{(m,1)}^T \cdot \mathbf{S}_{(m,2)} \cdot \mathbf{p} = (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{p}.$$

must be valid. It will be so if and only if

$$\det\mathbf{Q}_a \det\mathbf{S}_{(m,1)} = 1, \quad \mathbf{S}_{(m,1)} = \mathbf{S}_{(m,2)}.$$

Finally we obtain

$$\mathbf{a} = \mathbf{P} \cdot \mathbf{a}_*, \quad \mathbf{b} = \mathbf{P} \cdot \mathbf{b}_*, \quad \mathbf{c} = \mathbf{P} \cdot \mathbf{c}_*, \quad \mathbf{P} = \mathbf{Q}_a \cdot \mathbf{S}_{(m,1)},$$

where $\mathbf{Q}_a$ is an arbitrary orthogonal tensor, the tensor $\mathbf{S}_{(m,1)}$ is a such element of symmetry of vector $\mathbf{a}$, that tensor $\mathbf{P}$ is a proper orthogonal tensor. Thus we obtain the solution of the ***I-problem***.

9 Basic invariants of a symmetric second-rank tensor

It is known that any orthogonal invariant of the symmetric second-rank tensor may be represented as a function of its principle invariants. Let us obtain this result by means of our approach. Let $F(\mathbf{A})$ be some orthogonal invariant of $\mathbf{A}$. Then it must satisfy the equation (22) for $m = 0$, $n = 1$, which takes a form

$$\left(\frac{\partial F}{\partial \mathbf{A}}\right)^T \cdot\cdot (\mathbf{m} \times \mathbf{A} - \mathbf{A} \times \mathbf{m}) = 0.$$

Let us write down the characteristic system for (9)

$$\frac{d\mathbf{A}(s)}{ds} = \mathbf{m} \times \mathbf{A}(s) - \mathbf{A}(s) \times \mathbf{m}. \quad (35)$$

We obtain the sixth-order system which has exactly five independent integrals. However two of them depends on arbitrary unit vector $\mathbf{m}$ and may be ignored. A general solution of (35) is given by expression

$$\mathbf{A}(s) = \mathbf{Q}(s\mathbf{m}) \cdot \mathbf{A}_0 \cdot \mathbf{Q}^T(s\mathbf{m}), \quad (36)$$

where $\mathbf{A}_0$ is the tensor $\mathbf{A}$ in some fixed position, $\mathbf{Q}$ is a turn-tensor.

Excluding tensor $\mathbf{Q}$ from the solution (36) we obtain five independent integrals of (35)

$$\mathrm{I}_1 = \operatorname{tr}\mathbf{A} = \operatorname{tr}\mathbf{A}_0, \quad \mathrm{I}_2 = \operatorname{tr}\mathbf{A}^2 = \operatorname{tr}\mathbf{A}_0^2, \quad \mathrm{I}_3 = \operatorname{tr}\mathbf{A}^3 = \operatorname{tr}\mathbf{A}_0^3,$$

$$\mathrm{I}_4 = \mathbf{m}\cdot\mathbf{A}\cdot\mathbf{m} = \mathbf{m}\cdot\mathbf{A}_0\cdot\mathbf{m}, \quad \mathrm{I}_5 = \mathbf{m}\cdot\mathbf{A}^2\cdot\mathbf{m} = \mathbf{m}\cdot\mathbf{A}_0^2\cdot\mathbf{m}.$$

Any integral of (35) may be represented as a function of these integrals: $f(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_4, \mathrm{I}_5)$. It is easy to see that the function $f(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_4, \mathrm{I}_5)$ is an orthogonal invariant $F(\mathbf{A})$ of the tensor $\mathbf{A}$, if and only if it is independent of $\mathrm{I}_4, \mathrm{I}_5$. In other words any orthogonal invariant $F(\mathbf{A})$ of the tensor $\mathbf{A}$ is a function of the form $F(\mathbf{A}) = f(\operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^2, \operatorname{tr}\mathbf{A}^3)$. Now we have to show that two symmetric tensors with the same eigenvalues are connected by means of transformation of pure rotation. Let us write down the theorem on spectral decomposition

$$\mathbf{A} = \sum A_i\, \mathbf{d}_i \otimes \mathbf{d}_i, \quad \mathbf{A}^* = \sum A_k\, \mathbf{d}_k^* \otimes \mathbf{d}_k^*,$$

where the triples of vectors $\mathbf{d}_i$ and $\mathbf{d}_k^*$ are orthonormal but they may have different orientation. Thus we may write down

$$\mathbf{d}_m^* = \mathbf{Q}\cdot\mathbf{d}_m = \mathbf{d}_m\cdot\mathbf{Q}^T \quad \Rightarrow \quad \mathbf{A}^* = \mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T,$$

where $\mathbf{Q}$ is an orthogonal tensor. If det $\mathbf{Q} = 1$, then the ***I-problem*** has been solved. If det $\mathbf{Q} = -1$, then $\mathbf{Q}$ may be represented as decomposition $\mathbf{Q} = \mathbf{P}\cdot(-\mathbf{E})$. Using this decomposition we have

$$\mathbf{A}^* = \mathbf{P}\cdot(-\mathbf{E})\cdot\mathbf{A}\cdot(-\mathbf{E})^T\cdot\mathbf{P}^T = \mathbf{P}\cdot\mathbf{A}\cdot\mathbf{P}^T, \quad \det\mathbf{P} = 1,$$

and the ***I-problem*** has been solved as well.

10 Basic invariants of a collection of vector and of tensor

Let us consider a collection of vector $\mathbf{a}$ and of symmetric second-rank tensor $\mathbf{A}$. In literature [6] it is supposed that the set of six invariants

$$\mathrm{I}_1 = \operatorname{tr}\mathbf{A}, \quad \mathrm{I}_2 = \operatorname{tr}\mathbf{A}^2, \quad \mathrm{I}_3 = \operatorname{tr}\mathbf{A}^3, \quad \mathrm{I}_4 = \mathbf{a}\cdot\mathbf{a}, \quad \mathrm{I}_5 = \mathbf{a}\cdot\mathbf{A}\cdot\mathbf{a}, \quad \mathrm{I}_6 = \mathbf{a}\cdot\mathbf{A}^2\cdot\mathbf{a} \tag{37}$$

fixes this collection.

Almost the same case we have for a collection of symmetric second-rank tensor $\mathbf{A}$ and of skew-symmetric second-rank tensor $\mathbf{W}$. It is supposed [6] that for this case it is necessary to set seven integrals

$$\mathrm{I}_1 = \operatorname{tr}\mathbf{A}, \quad \mathrm{I}_2 = \operatorname{tr}\mathbf{A}^2, \quad \mathrm{I}_3 = \operatorname{tr}\mathbf{A}^3,$$

$$\mathrm{I}_4' = \operatorname{tr}\mathbf{W}^2, \quad \mathrm{I}_5' = \operatorname{tr}\left(\mathbf{W}^2\cdot\mathbf{A}\right), \quad \mathrm{I}_6' = \operatorname{tr}\left(\mathbf{W}^2\cdot\mathbf{A}^2\right), \quad \mathrm{I}_7' = \operatorname{tr}\left(\mathbf{W}^2\cdot\mathbf{A}\cdot\mathbf{W}\cdot\mathbf{A}^2\right). \tag{38}$$

From the other hand it is well known that for any skew-symmetric tensor $\mathbf{W}$ there exists uniquely defined vector $\mathbf{a}$ such that

$$\mathbf{W} = \mathbf{a}\times\mathbf{E}.$$

Making use of this representation one may obtain

$$I_4' = -2\mathbf{a}\cdot\mathbf{a}, \quad I_5' = \mathbf{a}\cdot\mathbf{A}\cdot\mathbf{a} - a^2\mathrm{tr}\mathbf{A}, \quad I_6' = \mathbf{a}\cdot\mathbf{A}^2\cdot\mathbf{a} - a^2\mathrm{tr}\mathbf{A}^2, \quad I_7' = -\mathbf{a}\cdot\mathbf{A}^2\cdot(\mathbf{a}\times\mathbf{A}\cdot\mathbf{a})\,. \tag{39}$$

By this reason it seems to be clear that the lists of invariants (37) and (38) must be the same. But it is not so. One may think that the difference is arising due to following fact. The vector $\mathbf{a}$ in (37) is a polar one, but the vector $\mathbf{a}$ in (39) is an axial one. However it can not be the reason since the dimension of invariant basis is independent of the tensor kind.

Let us apply our approach to this case. For visual perception it is useful to keep in mind that a collection of vector $\mathbf{a}$ and symmetric second-rank tensor $\mathbf{A}$ is equivalent to the collection

$$\mathbf{A}, \quad \mathbf{a}, \quad \mathbf{A}\cdot\mathbf{a}, \quad \mathbf{A}^2\cdot\mathbf{a}.$$

Therefore it is necessary to find the list of invariants, whose fixation determines this collection as a rigid whole. For a collection of vector $\mathbf{a}$ and symmetric second-rank tensor $\mathbf{A}$ the basic equation (22) takes the form

$$\frac{\partial F}{\partial \mathbf{a}}\cdot(\mathbf{m}\times\mathbf{a}) + \left(\frac{\partial F}{\partial \mathbf{A}}\right)^T \cdot\cdot\,(\mathbf{m}\times\mathbf{A} - \mathbf{A}\times\mathbf{m}) = 0. \tag{40}$$

The characteristic system for (40)

$$\frac{d\mathbf{a}}{ds} = \mathbf{m}\times\mathbf{a}, \quad \frac{d\mathbf{A}}{ds} = \mathbf{m}\times\mathbf{A} - \mathbf{A}\times\mathbf{m}.$$

This system of ninth order has exactly eight independent integrals. However only six from them are independent of arbitrary vector $\mathbf{m}$. These integrals are given by expressions

$$I_1 = \mathrm{tr}\,\mathbf{A}, \quad I_2 = \mathrm{tr}\,\mathbf{A}^2, \quad I_3 = \mathrm{tr}\,\mathbf{A}^3, \quad I_4 = \mathbf{a}\cdot\mathbf{a}, \quad I_5 = \mathbf{a}\cdot\mathbf{A}\cdot\mathbf{a},$$
$$I_6 = \mathbf{a}\cdot\mathbf{A}^2\cdot\mathbf{a}, \quad I_7 = \mathbf{a}\cdot\mathbf{A}^2\cdot(\mathbf{a}\times\mathbf{A}\cdot\mathbf{a})\,. \tag{41}$$

The list (41) contain seven integrals but between them there is a relation

$$I_7^2 = \begin{vmatrix} I_4 & I_5 & I_6 \\ I_5 & I_6 & \mathbf{a}\cdot\mathbf{A}^3\cdot\mathbf{a} \\ I_6 & \mathbf{a}\cdot\mathbf{A}^3\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{A}^4\cdot\mathbf{a} \end{vmatrix}, \tag{42}$$

which was not mentioned in literature. The invariants $\mathbf{a}\cdot\mathbf{A}^3\cdot\mathbf{a}$ and $\mathbf{a}\cdot\mathbf{A}^4\cdot\mathbf{a}$ in the determinant (42) must be expressed in terms of invariants I_1 – I_6 with the help of the Cayley–Hamilton identity.

The invariant I_7 must be taken into account and can not be ignored. In order to verify this fact it is enough to consider two collections

$$\mathbf{A}, \quad \mathbf{a} \quad \text{and} \quad \mathbf{B} = \mathbf{S}_n\cdot\mathbf{A}\cdot\mathbf{S}_n^T, \quad \mathbf{a},$$

where $\mathbf{S}_n = \mathbf{E} - 2\mathbf{n}\otimes\mathbf{n}$, $\mathbf{n}\cdot\mathbf{n} = 1$, $\mathbf{n}\cdot\mathbf{a} = 0$. It is easy to check that invariants I_1 – I_6 for these two sets are the same. However we have

$$\mathbf{a}\cdot\mathbf{B}^2\cdot(\mathbf{a}\times\mathbf{B}\cdot\mathbf{a}) = -\mathbf{a}\cdot\mathbf{A}^2\cdot(\mathbf{a}\times\mathbf{A}\cdot\mathbf{a})\,,$$

i.e. the triples of vectors $\mathbf{a}$, $\mathbf{A}\cdot\mathbf{a}$, $\mathbf{A}^2\cdot\mathbf{a}$ and $\mathbf{a}$, $\mathbf{B}\cdot\mathbf{a}$, $\mathbf{B}^2\cdot\mathbf{a}$ have different orientations and can not be combined by a rotation.

Finally we obtain that the collection of vector $\mathbf{a}$ and symmetric second-rank tensor $\mathbf{A}$ has exactly six functionally independent invariants (41)–(42). Now we have to show that the fixation of the invariants (41)–(42) determines the collection of vector $\mathbf{a}$ and symmetric second-rank tensor $\mathbf{A}$ as a rigid whole. Let us consider two sets $\mathbf{a}$, $\mathbf{A}$ and $\mathbf{b}$, $\mathbf{B}$. If invariants $I_1 - I_4$ for these sets are the same, then we have

$$\mathbf{a} = \mathbf{Q}_a\cdot\mathbf{b}, \quad \mathbf{A} = \mathbf{Q}_A\cdot\mathbf{B}\cdot\mathbf{Q}_A^T, \tag{43}$$

where $\mathbf{Q}_a$ and $\mathbf{Q}_A$ are any orthogonal tensors. Tensors $\mathbf{Q}_a$ and $\mathbf{Q}_A$ must ensure the coincidence of invariants $I_5 - I_7$ for these two sets. Thus we have

$$\mathbf{a}\cdot\mathbf{A}\cdot\mathbf{a} \equiv \mathbf{b}\cdot\mathbf{Q}\cdot\mathbf{B}\cdot\mathbf{Q}^T\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{B}\cdot\mathbf{b}, \quad \mathbf{a}\cdot\mathbf{A}^2\cdot\mathbf{a} \equiv \mathbf{b}\cdot\mathbf{Q}\cdot\mathbf{B}^2\cdot\mathbf{Q}^T\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{B}^2\cdot\mathbf{b},$$

where

$$\mathbf{Q} \equiv \mathbf{Q}_a^T\cdot\mathbf{Q}_A.$$

From these equation it follows

$$\mathbf{Q} = (\det\mathbf{S}_b)^\alpha(\det\mathbf{S}_B)^\beta\,\mathbf{S}_b\cdot\mathbf{S}_B,$$

where $\mathbf{S}_b$ and $\mathbf{S}_B$ are some elements of symmetry of vector $\mathbf{b}$ and of tensor $\mathbf{B}$ correspondingly, $\alpha = 0$ for polar $\mathbf{b}$, $\alpha = 1$ for axial $\mathbf{b}$, $\beta = 0$ for polar $\mathbf{B}$, $\beta = 1$ for axial $\mathbf{B}$. Now we have

$$\mathbf{Q}_A = (\det\mathbf{S}_b)^\alpha(\det\mathbf{S}_B)^\beta\,\mathbf{Q}_a\cdot\mathbf{S}_b\cdot\mathbf{S}_B,$$

The relation (43) takes the form

$$\mathbf{a} = \mathbf{Q}_a\cdot\mathbf{b}, \quad \mathbf{A} = \mathbf{Q}_a\cdot\mathbf{S}_b\cdot\mathbf{B}\cdot\mathbf{S}_b^T\cdot\mathbf{Q}_a^T,$$

or

$$\mathbf{a} = \mathbf{P}\cdot\mathbf{b}, \quad \mathbf{A} = \mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^T, \quad \mathbf{P} \equiv (\det\mathbf{S}_b)^\alpha\mathbf{Q}_a\cdot\mathbf{S}_b \tag{44}$$

Taking into account (44) and the expression for I_7 one may write down

$$\mathbf{a}\cdot\mathbf{A}^2\cdot(\mathbf{a}\times\mathbf{A}\cdot\mathbf{a}) = (\det\mathbf{P})\,\mathbf{b}\cdot\mathbf{B}^2\cdot(\mathbf{b}\times\mathbf{B}\cdot\mathbf{b}) \tag{45}$$

Since the $I_7(\mathbf{a}, \mathbf{A})$ must be equal to $I_7(\mathbf{b}, \mathbf{B})$ then we obtain $\det\mathbf{P} = 1$, i.e. the tensor $\mathbf{P}$ must be the proper orthogonal tensor.

11 Basic invariants for a set of two tensors

Let us consider a collection of two symmetric tensors

$$\mathbf{A} = \sum_{k=1}^{3} A_k\,\mathbf{a}_k\otimes\mathbf{a}_k, \quad \mathbf{B} = \sum_{k=1}^{3} B_k\,\mathbf{b}_k\otimes\mathbf{b}_k. \tag{46}$$

It is claimed that in order to fix this system it is necessary to fix the next ten invariants

$$I_k^A = \operatorname{tr}\mathbf{A}^k, \quad I_k^B = \operatorname{tr}\mathbf{B}^k, \quad x = \operatorname{tr}(\mathbf{A}\cdot\mathbf{B}),$$

$$y = \mathrm{tr}\,(\mathbf{A}^2 \cdot \mathbf{B}), \quad z = \mathrm{tr}\,(\mathbf{A} \cdot \mathbf{B}^2), \quad u = \mathrm{tr}\,(\mathbf{A}^2 \cdot \mathbf{B}^2). \tag{47}$$

In the literature the tensors $\mathbf{A}$, $\mathbf{B}$ are supposed to be polar. Let us note that for application the case, when $\mathbf{A}$ is a polar tensor and $\mathbf{B}$ is an axial tensor, is much more important. Accordingly to the theorem on the dimension of the invariant basis, the number of functionally independent invariants is equal to nine. Therefore the invariants shown in the list (47) can not be functionally independent. That means that there exists some relation superposed on the invariants (47).

Let us show that the invariants x, y, z, u in (47) can be expressed through the invariants A_k, B_n and four parameters on which one obvious relation is superposed. For sake of simplicity the eigenvalues A_k and B_n are supposed to be different. In such a case we may write

$$\mathbf{a}_k \otimes \mathbf{a}_k = \alpha_k \mathbf{\Gamma}_k(\mathbf{A}), \ \alpha_k^{-1} = (A_k - A_i)(A_k - A_j), \tag{48}$$

$$\mathbf{\Gamma}_k(\mathbf{A}) \equiv \mathbf{A}^2 - (I_1^A - A_k)\mathbf{A} + A_k^{-1} I_3^A \mathbf{E}, \quad i \neq j \neq k \neq i;$$

$$\mathbf{b}_m \otimes \mathbf{b}_m = \beta_m \mathbf{\Gamma}_m(\mathbf{B}), \quad \beta_m^{-1} = (B_m - B_n)(B_m - B_p),$$

$$\mathbf{\Gamma}_m(\mathbf{B}) \equiv \mathbf{B}^2 - (I_1^B - B_m)\mathbf{B} + B_m^{-1} I_3^B \mathbf{E}, \ m \neq n \neq p \neq m.$$

Making use of (48) and (11) one may obtain

$$\begin{aligned}(\mathbf{a}_k \cdot \mathbf{b}_m)^2 = \alpha_k \beta_m \big[&u + (A_k - I_1^A)(B_m - I_1^B)x - (I_1^B - B_m)y - (I_1^A - A_k)z + \\ &+ \left(B_m I_1^B - 2J_B\right) A_k^{-1} I_3^A + \left(A_k I_1^A - 2J_A\right) B_m^{-1} I_3^B + 3\,(A_k B_m)^{-1} I_3^A I_3^B\big],\end{aligned} \tag{49}$$

$$2J_A = (\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2), \quad 2J_B = (\mathrm{tr}\mathbf{B})^2 - \mathrm{tr}(\mathbf{B}^2).$$

The system (49) of nine equations is the linear system for four invariants x, y, z and u. It may be verified that the rank of the system (49) is equal to four. Therefore the invariants x, y, z and u may be expressed as function of eigenvalues A_k, B_n and and numbers $\mathbf{a}_k \cdot \mathbf{b}_m$, which are connected by six constraints

$$\sum_{m=1}^{3} (\mathbf{a}_k \cdot \mathbf{b}_m)^2 = 1, \ \sum_{k=1}^{3} (\mathbf{a}_k \cdot \mathbf{b}_m)^2 = 1.$$

Let the triples $\mathbf{a}_k$ and $\mathbf{a}_k$ have the same orientation. This assumption does not restrict the analysis. In such a case we may write down

$$\mathbf{b}_k = \mathbf{Q} \cdot \mathbf{a}_k, \quad det\mathbf{Q} = 1. \tag{50}$$

The turn-tensor $\mathbf{Q}$ may be expressed in terms of turn-vector $\boldsymbol{\theta}$

$$\mathbf{Q}(\boldsymbol{\theta}) = \cos\theta\,\mathbf{E} + \frac{(1 - \cos\theta)}{\theta^2}\boldsymbol{\theta} \otimes \boldsymbol{\theta} + \frac{\sin\theta}{\theta}\,\boldsymbol{\theta} \times \mathbf{E}, \tag{51}$$

where $\theta = |\boldsymbol{\theta}|$.

Making use of (51) and (50), one may obtain

$$\theta^2 \mathbf{a}_p \cdot \mathbf{b}_p = \theta_p^2 + \cos\theta\,(\theta^2 - \theta_p^2), \quad \theta^2 = \theta_1^2 + \theta_2^2 + \theta_3^2, \tag{52}$$

$$(\mathbf{a}_m \cdot \mathbf{b}_p)^2 = \frac{(1-\cos\theta)^2}{\theta^4}\theta_m^2\theta_p^2 + \frac{\sin^2\theta}{\theta^2}\theta_s^2 + \\ + e_{mps}\frac{2\sin\theta(1-\cos\theta)}{\theta^3}\theta_1\theta_2\theta_3, \quad m \neq p \neq s \neq m, \tag{53}$$

where e_{mks} is the permutation symbol, $\theta_k = \boldsymbol{\theta} \cdot \mathbf{a}_k$.

Let us show that invariants x, y, z, u can be expressed in term of the turn-vector components and eigenvalues of the tensor $\mathbf{A}$ and $\mathbf{B}$. For this end in the right side of (49) we may ignore the terms depending on the eigenvalues of the tensor $\mathbf{A}$ and $\mathbf{B}$ only and rewrite (49) in the form of three independent systems

$$u_1 - (B_2 + B_3)y_1 = a_{11}, \quad u_1 - (B_1 + B_3)y_1 = a_{12}, \quad u_1 - (B_1 + B_2)y_1 = a_{13}; \tag{54}$$

$$u_2 - (B_2 + B_3)y_2 = a_{21}, \quad u_2 - (B_1 + B_3)y_2 = a_{22}, \quad u_2 - (B_1 + B_2)y_2 = a_{23}; \tag{55}$$

$$u_3 - (B_2 + B_3)y_3 = a_{31}, \quad u_3 - (B_1 + B_3)y_3 = a_{32}, \quad u_3 - (B_1 + B_2)y_3 = a_{33}. \tag{56}$$

In (54)–(56) the next new unknown variables are introduced

$$u_m = u - (A_n + A_p)z, \quad y_m = y - (A_n + A_p)x, \tag{57}$$

where $m \neq n \neq p \neq m$. Besides the notation are used

$$a_{ik} = \mathbf{a}_i \cdot \mathbf{b}_k / \alpha_i \beta_k.$$

Solutions of (54)–(56) are given by expressions

$$u_m = a_{mm} + \frac{B_n + B_p}{B_n - B_p}(a_{mn} - a_{mp}), \quad y_m = \frac{a_{mn} - a_{mp}}{B_n - B_p}, \tag{58}$$

where the numbers m, n, p consist the even permutation of 1, 2, 3. It is easy to find the solution of (57)

$$x = \frac{y_1 - y_2}{A_1 - A_2}, \quad y = y_3 + \frac{A_1 + A_2}{A_1 - A_2}(y_1 - y_2),$$
$$z = \frac{u_1 - u_2}{A_1 - A_2}, \quad u = u_3 + \frac{A_1 + A_2}{A_1 - A_2}(u_1 - u_2).$$

Making use of (58) we finally obtain

$$x = (A_1 - A_3)\left[(B_2 - B_1)(\mathbf{a}_1 \cdot \mathbf{b}_2)^2 + (B_3 - B_1)(\mathbf{a}_1 \cdot \mathbf{b}_3)^2\right].$$

The analogous formulae may be obtained for invariants y, z and u. It may be noted that invariants x, y, z, u are represented in terms of four numbers θ_1^2, θ_2^2, θ_3^2, $\theta_1\theta_2\theta_3$. However there is one obvious constraint $\theta_1^2\theta_2^2\theta_3^2 = (\theta_1\theta_2\theta_3)^2$. Thus we see that the number of functionally independent invariants is equal to nine. This is the statement of the theorem on the dimension of the invariant basis.

The using of the turn-vector is possible but it is not convenient. So it would be better to use more simple invariants with clear physical sense. To this end let us introduce the tensors

$$\mathbf{C}(\mathbf{A}, \mathbf{B}) = \sum_{k=1}^{3} B_k \mathbf{b}_k \times \mathbf{A} \times \mathbf{b}_k = \sum_{k=1}^{3} A_k \mathbf{a}_k \times \mathbf{B} \times \mathbf{a}_k = \mathbf{C}(\mathbf{B}, \mathbf{A})$$

and

$$\mathbf{D}(\mathbf{A},\, \mathbf{B}) = \mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A} = -\mathbf{D}(\mathbf{B},\, \mathbf{A}).$$

The skew-symmetric tensor $\mathbf{D}(\mathbf{A}, \mathbf{B})$ may be represented as

$$\mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A} = -\mathbf{r}\times\mathbf{E} \quad \Rightarrow \quad \mathbf{r} = (\mathbf{A}\cdot\mathbf{B})_{\times}. \tag{59}$$

The vector $\mathbf{r}$ characterizes a non-coaxiality of the tensors $\mathbf{A}$ and $\mathbf{B}$. The tensor $\mathbf{C}$ is the isotropic function of the tensors $\mathbf{A}$ and $\mathbf{B}$. It may be represented a combination of the form

$$\mathbf{C} = [\mathrm{tr}(\mathbf{A}\cdot\mathbf{B}) - \mathrm{tr}\mathbf{A}\,\mathrm{tr}\mathbf{B}]\,\mathbf{E} + (\mathrm{tr}\mathbf{B})\,\mathbf{A} + (\mathrm{tr}\mathbf{A})\,\mathbf{B} - \mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A}.$$

The tensor $\mathbf{C}$ is convenient for applications since its eigenvalues characterize the reciprocal position of the tensors $\mathbf{A}$ and $\mathbf{B}$.

In order to fix the collection (46) it is possible to set the next nine invariants. Firstly, they contain the principal invariants of the tensors $\mathbf{A}$ and $\mathbf{B}$

$$I_1^A = \mathrm{tr}\,\mathbf{A}, \quad I_2^A = \frac{1}{2}\left[(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}\,\mathbf{A}^2)\right], \quad I_3^A = \det\mathbf{A},$$

$$I_1^B = \mathrm{tr}\,\mathbf{B}, \quad I_2^B = \frac{1}{2}\left[(\mathrm{tr}\mathbf{B})^2 - \mathrm{tr}\,\mathbf{B}^2)\right], \quad I_3^B = \det\mathbf{B}. \tag{60}$$

Invariants (60) determine the eigenvalues A_k and B_k of the tensors $\mathbf{A}$ and $\mathbf{B}$. Secondly, in order to fix the reciprocal position of the tensors $\mathbf{A}$ and $\mathbf{B}$ it is possible to use the next three invariants

$$\mathrm{tr}\mathbf{C}, \quad \mathrm{tr}\mathbf{C}^2, \quad \mathbf{r}\cdot\mathbf{r}.$$

These invariants may be expressed in terms of invariants (47)

$$\mathrm{tr}\mathbf{C} = \mathrm{tr}(\mathbf{A}\cdot\mathbf{B}) - \mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{B}, \quad \mathbf{r}\cdot\mathbf{r} = \mathrm{tr}(\mathbf{A}^2\cdot\mathbf{B}^2) - \mathrm{tr}\left[(\mathbf{A}\cdot\mathbf{B})^2\right],$$

$$\mathrm{tr}\mathbf{C}^2 = [\mathrm{tr}(\mathbf{A}\cdot\mathbf{B})]^2 + \mathrm{tr}\mathbf{A}^2\mathrm{tr}\mathbf{B}^2 - 2\mathrm{tr}(\mathbf{A}^2\cdot\mathbf{B}^2).$$

Thus we may fix the next three invariants

$$\mathrm{tr}(\mathbf{A}\cdot\mathbf{B}), \quad \mathrm{tr}(\mathbf{A}^2\cdot\mathbf{B}^2), \quad \mathrm{tr}(\mathbf{A}^2\cdot\mathbf{B}^2) - \mathrm{tr}\left[(\mathbf{A}\cdot\mathbf{B})^2\right]. \tag{61}$$

Now we have to show that fixation of the nine invariants (60), (61) determines the set of the tensors $\mathbf{A}$ and $\mathbf{B}$ as a rigid whole. Let there be given two sets of the tensors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{A}_*$, $\mathbf{B}_*$. A fixation of the invariants (60) leads to the relations

$$\mathbf{A}_* = \mathbf{Q}_A\cdot\mathbf{A}\cdot\mathbf{Q}_A^T, \quad \mathbf{B}_* = \mathbf{Q}_B\cdot\mathbf{B}\cdot\mathbf{Q}_B^T.$$

where $\mathbf{Q}_A$ and $\mathbf{Q}_B$ are orthogonal tensors.

Fixation of the invariants (61) gives the next system of equation

$$\mathbf{A}\cdot\cdot\left(\mathbf{Q}\cdot\mathbf{B}\cdot\mathbf{Q}^T - \mathbf{B}\right) = 0, \quad \mathbf{A}^2\cdot\cdot\left(\mathbf{Q}\cdot\mathbf{B}^2\cdot\mathbf{Q}^T - \mathbf{B}^2\right) = 0,$$

$$\mathrm{tr}\left[\left(\mathbf{A}\cdot\mathbf{Q}\cdot\mathbf{B}\cdot\mathbf{Q}^T\right)^2\right] = \mathrm{tr}\left[(\mathbf{A}\cdot\mathbf{B})^2\right], \tag{62}$$

where

$$\mathbf{Q} \equiv \mathbf{Q}_A^T \cdot \mathbf{Q}_B.$$

From (62) it follows

$$\mathbf{Q} = \mathbf{S}_A \cdot \mathbf{S}_B \quad \Rightarrow \quad \mathbf{Q}_B = \mathbf{Q}_A \cdot \mathbf{S}_A \cdot \mathbf{S}_B,$$

where $\mathbf{S}_A$ and $\mathbf{S}_B$ are some elements of the tensors $\mathbf{A}$ and $\mathbf{B}$ correspondingly. Thus we have

$$\mathbf{A}_* = \mathbf{Q}_A \cdot \mathbf{S}_A \cdot \mathbf{A} \cdot \mathbf{S}_A^T \cdot \mathbf{Q}_A^T, \quad \mathbf{B}_* = \mathbf{Q}_A \cdot \mathbf{S}_A \cdot \mathbf{B} \cdot \mathbf{S}_A^T \cdot \mathbf{Q}_A^T.$$

For any $\mathbf{Q}_A$ we may chose such element of symmetry $\mathbf{S}_A$ that

$$\det(\mathbf{Q}_A \cdot \mathbf{S}_A) = 1.$$

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