# A General Model of Rigid Body Oscillator*

**Abstract**

The present discourse develops a new model named by a rigid body oscillator. In Eulerian mechanics this model plays the same role as the model of nonlinear oscillator in Newtonian mechanics. The importance of the introduction of the rigid body oscillator, i.e. a rigid body oscillator on an elastic foundation of general kind, into consideration was pointed out by many scientists. However the problem is not formalized up to now. In the paper all necessary for a mathematical description concepts are introduced. Some of them are new. The equations of motion are represented in unusual for dynamics of rigid body form, which has a clearly expressed simple structure but contain the nonlinearity of a complex kind. These equations give the very interesting object for the theory of nonlinear oscillations. The solutions of some problems are given. For the simplest case the exact solution was found by an essentially new method of an integration of basic equations.

## 1 Introduction

The nonlinear (linear) oscillator is the most important model of classical physics. An investigation of many physical phenomenons and a development of many methods of nonlinear mechanics had arisen in the science due to this model. At the same time it was recognized the necessity of construction of models with new properties. Especially it was important in quantum mechanics, where many authors pointed out that a new model must be something like a rigid body on an elastic foundation. However, such model was not created up to now. Why? The full answer on this question will be found by historians later.

A rigid body on an elastic foundation will be called the rigid body oscillator in what follows. A general model of such object can be used in many cases, for example, in mechanics of continuum multipolar media. For the construction of model the three new elements are needed: the vector of turn, the integrating tensor, and the potential torque. Let us briefly discuss these concepts.

An unusual situation takes place with the **vector of turn**. From the one side, the well-known theorem of Euler proves that any turn of the body can be realized as the

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turn around some unit vector $\mathbf{m}$ by a certain angle $\theta$. Thus the turn can be described by the vector $\boldsymbol{\theta} = \theta\mathbf{m}$. This fact can be found in many books on mechanics. From the other side the same books claim that the vector $\theta\mathbf{m}$ is not a vector, and a description of a turn in terms of vector is impossible. May be by this reason a vector of turn has no applications in conventional dynamics of rigid body. However namely the vector of turn plays the main role in dynamics of rigid body on an elastic foundation.

**Integrating tensor.** In classical mechanics the linear differential form $\mathbf{v}dt$ is the total differential of the vector of position $\mathbf{v}dt = d\mathbf{R}$. It is not true for spinor movements. If the vector $\boldsymbol{\omega}$ is a vector of angular velocity, then the linear differential form $\boldsymbol{\omega}dt$ is not a total differential of the vector of turn. However, it can be proved that there exists a tensor $\mathbf{Z}$ that transforms the linear differential form $\boldsymbol{\omega}dt$ into the total differential $d\boldsymbol{\theta}$ of the vector of turn $\boldsymbol{\theta}$. This fact was established in the work [2]. The integrating tensor $\mathbf{Z}$ plays the decisive role for an introduction of **a potential torque**. The latter expresses an action of the elastic foundation on the rigid body. Thus it is an essential element of a general model of rigid body oscillator.

The basic equations of dynamics of rigid body oscillator contain a strong nonlinearity but their form is rather simple. These equations give the very interesting object for methods of nonlinear mechanics. In the paper some simple examples are considering. In particular a new method of integration of the basic equations is given in the case of simplest model.

Author hopes that the clarity of the mathematical formulas in the paper will be able to compensate a helplessness of its language of words.

# 2 Mathematical preliminaries

In the section certain aspects of the tensor of turn and the vector of turn will be briefly presented. Some initial definitions can be found in the paper [1].

## 2.1 Vector of turn

A vector of turn is the very old concept. It is difficult to find another concept, for which there exist so many inconsistent propositions as for the vector of turn. The latter plays the main role in the present work. Because of this it seems to be necessary to give the strict introduction of the vector of turn and to describe its basic properties. The introduction of the vector of turn is determined by the well-known statement of Euler: any turn can be represented as the turn around some axis $\mathbf{n}$ by the certain angle $\theta$. The vector $\theta\mathbf{n}$, $|\mathbf{n}|=1$, is called the vector of turn. Note that two different mathematical concepts correspond to one physical (or geometrical) idea of turn. One of them is described by a tensor of turn and another is described by a vector of turn. Of course both of them are connected by a unique manner. For the turn-tensor we shall use the notation [1]

$$\mathbf{Q}\left(\theta\mathbf{n}\right) = \left(1 - \cos\theta\right)\mathbf{n}\otimes\mathbf{n} + \cos\theta\mathbf{E} + \sin\theta\mathbf{n}\times\mathbf{E}. \tag{1}$$

An action of the tensor $\mathbf{Q}\left(\theta\mathbf{n}\right)$ on the vector $\mathbf{a}$ can be expressed in the form

$$\mathbf{a}' = \mathbf{Q}\left(\theta\mathbf{n}\right)\cdot\mathbf{a} = \left(\mathbf{a}\cdot\mathbf{n}\right)\mathbf{n} + \cos\theta\left(\mathbf{a} - \mathbf{a}\cdot\mathbf{n}\mathbf{n}\right) + \sin\theta\mathbf{n}\times\mathbf{a}. \tag{2}$$

If $\mathbf{n} \times \mathbf{a} = \mathbf{0}$, then $\mathbf{a}' = \mathbf{a}$. If $\mathbf{a} \cdot \mathbf{n} = 0$, then we have

$$\mathbf{a}' = \cos\theta\mathbf{a} + \sin\theta\mathbf{n} \times \mathbf{a}.$$

This means that vector $\mathbf{a}'$ is the vector $\mathbf{a}$ turned around the vector $\mathbf{n}$ by the angle $\theta$.

Representation (1) can be rewritten in another form

$$\mathbf{Q}(\boldsymbol{\theta}) = \mathbf{E} + \frac{\sin\theta}{\theta}\mathbf{R} + \frac{1-\cos\theta}{\theta^2}\mathbf{R}^2 = \exp\mathbf{R}, \tag{3}$$

where

$$\mathbf{R} = \boldsymbol{\theta} \times \mathbf{E}, \qquad \theta = |\boldsymbol{\theta}|. \tag{4}$$

The vector $\boldsymbol{\theta}$ in (3), (4) is called the vector of turn. Note that there exists a little difference between representations (1) and (3). In (1) the quantity $\theta$ is the angle of turn and can be both positive and negative. In (3) the quantity $\theta$ is the modulus of the vector of turn, i.e. the modulus of the angle of turn. such interpretation is possible since, for example, $\sin\theta/\theta = \sin|\theta|\,/\,|\theta|$. As a rule, representation (3) is more convenient for applications then expression (1). Let us consider a superposition of two turns

$$\mathbf{Q}(\boldsymbol{\theta}) = \mathbf{Q}(\boldsymbol{\varphi}) \cdot \mathbf{Q}(\boldsymbol{\psi}). \tag{5}$$

The vector of total turn $\boldsymbol{\theta}$ is connected with the vectors of turn $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ by the formulas

$$\begin{gathered} 1 + 2\cos\theta = \cos\varphi + \cos\psi + \cos\varphi\cos\psi - \\ -2\frac{\sin\varphi}{\varphi}\frac{\sin\psi}{\psi}\boldsymbol{\varphi}\cdot\boldsymbol{\psi} + \frac{(1-\cos\varphi)}{\varphi^2}\frac{(1-\cos\psi)}{\psi^2}(\boldsymbol{\varphi}\cdot\boldsymbol{\psi})^2, \end{gathered} \tag{6}$$

$$\begin{aligned} 2\frac{\sin\theta}{\theta}\boldsymbol{\theta} = &\left[\frac{\sin\varphi}{\varphi}(1+\cos\psi) - \frac{(1-\cos\varphi)}{\varphi^2}\frac{\sin\psi}{\psi}\boldsymbol{\varphi}\cdot\boldsymbol{\psi}\right]\boldsymbol{\varphi} + \\ &+\left[\frac{\sin\psi}{\psi}(1+\cos\varphi) - \frac{(1-\cos\psi)}{\psi^2}\frac{\sin\varphi}{\varphi}\boldsymbol{\varphi}\cdot\boldsymbol{\psi}\right]\boldsymbol{\psi} + \\ &+\left[\frac{\sin\varphi\sin\psi}{\varphi\psi} - \frac{(1-\cos\varphi)}{\varphi^2}\frac{(1-\cos\psi)}{\psi^2}\boldsymbol{\varphi}\cdot\boldsymbol{\psi}\right]\boldsymbol{\varphi}\times\boldsymbol{\psi}. \end{aligned} \tag{7}$$

Note that from expressions (3), (4) it follows

$$\mathbf{R} \cdot \boldsymbol{\theta} = \mathbf{0}, \qquad \mathbf{Q}(\boldsymbol{\theta}) \cdot \boldsymbol{\theta} = \boldsymbol{\theta}. \tag{8}$$

## 2.2 Integrating tensor

The vector of turn $\boldsymbol{\theta}(t)$ plays for spinor movements the same role as the vector of position $\mathbf{R}(t)$ for translation movements. In the latter case the translation velocity $\mathbf{v}$ can be found by means of simplest formula $\mathbf{v} = \dot{\mathbf{R}}(t)$. This means that the linear form $\mathbf{v}dt$ is the total differential of the vector of position. For spinor movements the situation is more complicated, since the linear form $\boldsymbol{\omega}dt$, where $\boldsymbol{\omega}$ is the vector of angular velocity, is not the total differential of the vector of turn $\boldsymbol{\theta}$. Thus it is necessary to find an integrating factor that transforms the linear form $\boldsymbol{\omega} dt$ into the total differential of vector of turn $d\boldsymbol{\theta}$. For this end let us consider the left Poisson equation [1]

$$\dot{\mathbf{Q}}(\boldsymbol{\theta}) = \boldsymbol{\omega} \times \mathbf{Q}(\boldsymbol{\theta}), \qquad \dot{f} \equiv df/dt. \tag{9}$$

This equation for the tensor of turn $\mathbf{Q}(\boldsymbol{\theta})$ is equivalent to a system of nine scalar equations but only three of them are independent. In order to find these independent equations it is possible to substitute expression (3) into equation (9). After some transformations the next equation can be derived

$$\dot{\boldsymbol{\theta}}(t) = \mathbf{Z}(\boldsymbol{\theta}) \cdot \boldsymbol{\omega}(t), \tag{10}$$

where

$$\mathbf{Z}(\boldsymbol{\theta}) = \mathbf{E} - \frac{1}{2}\mathbf{R} + \frac{1-g}{\theta^2}\mathbf{R}^2, \qquad g = \frac{\theta \sin\theta}{2(1-\cos\theta)}. \tag{11}$$

The tensor $\mathbf{Z}(\boldsymbol{\theta})$ will be called the integrating tensor in what follows. The nonsingular tensor $\mathbf{Z}$ has the determinant

$$\det \mathbf{Z}(\boldsymbol{\theta}) = \theta^2/2(1-\cos\theta) \neq 0.$$

The integrating tensor has a number useful properties. Let us describe some of them. First of all, the tensor $\mathbf{Z}(\boldsymbol{\theta})$ is an isotropic function of the vector of turn $\boldsymbol{\theta}$. This means that

$$\mathbf{Z}(\mathbf{S} \cdot \boldsymbol{\theta}) = \mathbf{S} \cdot \mathbf{Z}(\boldsymbol{\theta}) \cdot \mathbf{S}^T, \qquad \forall \mathbf{S} : \mathbf{S} \cdot \mathbf{S}^T = \mathbf{E}, \qquad \det \mathbf{S} = 1. \tag{12}$$

If $\mathbf{S} = \mathbf{Q}(\boldsymbol{\theta})$, then from (12) and (8) it follows

$$\mathbf{Z}(\boldsymbol{\theta}) \cdot \mathbf{Q}(\boldsymbol{\theta}) = \mathbf{Q}(\boldsymbol{\theta}) \cdot \mathbf{Z}(\boldsymbol{\theta}).$$

Besides, it can be checked the identity

$$\mathbf{Z}^T(\boldsymbol{\theta}) = \mathbf{Q}(\boldsymbol{\theta}) \cdot \mathbf{Z}(\boldsymbol{\theta}) = \mathbf{Z}(\boldsymbol{\theta}) \cdot \mathbf{Q}(\boldsymbol{\theta}). \tag{13}$$

For the right angular velocity $\boldsymbol{\Omega} = \mathbf{Q}^T(\boldsymbol{\theta}) \cdot \boldsymbol{\omega}$ — see [1] — from expressions (10) and (13) it follows

$$\dot{\boldsymbol{\theta}}(t) = \mathbf{Z}^T(\boldsymbol{\theta}) \cdot \boldsymbol{\Omega}(t). \tag{14}$$

This equation is equivalent to the right Poisson equation [1]. In the explicit form equations (10) and (14) can be rewritten by such manner

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}), \qquad \boldsymbol{\theta}|_{t=0} = \boldsymbol{\theta}_0, \tag{15}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\Omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\Omega}), \qquad \boldsymbol{\theta}|_{t=0} = \boldsymbol{\theta}_0. \tag{16}$$

Problem (15) is the left Darboux problem[1]. If the left angular velocity is known, then the vector of turn (and therefore the turn-tensor) can be found as the solution of problem (15). It is much more simple task (at least for numerical analysis) then a solution of the conventional Riccati equation. The same can be said with respect to the right Darboux problem (16). Expressions (15) and (16) can be rewritten in the equivalent form

$$\dot{\boldsymbol{\theta}} = g\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1-g}{\theta}\dot{\theta}\boldsymbol{\theta}, \tag{17}$$

$$\dot{\boldsymbol{\theta}} = g\boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\Omega} + \frac{1-g}{\theta}\dot{\theta}\boldsymbol{\theta}. \tag{18}$$

Here we take into account the identity

$$\boldsymbol{\theta} \cdot \boldsymbol{\omega} = \boldsymbol{\theta} \cdot \boldsymbol{\Omega} = \boldsymbol{\theta} \cdot \dot{\boldsymbol{\theta}} = \theta\dot{\theta}$$

Sometimes it is more convenient to use an inverse form of equations (10) and (14)

$$\boldsymbol{\omega}(t) = \mathbf{Z}^{-1}(\boldsymbol{\theta}) \cdot \dot{\boldsymbol{\theta}}(t), \qquad \boldsymbol{\Omega}(t) = \mathbf{Z}^{-T}(\boldsymbol{\theta}) \cdot \dot{\boldsymbol{\theta}}(t), \tag{19}$$

where

$$\mathbf{Z}^{-1}(\boldsymbol{\theta}) = \mathbf{E} + \frac{1-\cos\theta}{\theta^2}\mathbf{R} + \frac{\theta - \sin\theta}{\theta^3}\mathbf{R}^2. \tag{20}$$

## 2.3 Potential torque

Let us introduce a concept of potential torque. This concept is necessary for a statement and an analysis of many problems. Nevertheless a general definition of potential torque is absent in literature.

**Definition:** Torque $\mathbf{M}(t)$ is called potential if there exists scalar function $U(\boldsymbol{\theta})$ depending on a vector of turn such that the next equality is valid

$$\mathbf{M} \cdot \boldsymbol{\omega} = -\dot{U}(\boldsymbol{\theta}) = -\frac{dU}{d\boldsymbol{\theta}} \cdot \dot{\boldsymbol{\theta}}. \tag{21}$$

Making use of equation (10) this equality can be rewritten in the form

$$\left(\mathbf{M} + \frac{dU}{d\boldsymbol{\theta}} \cdot \mathbf{Z}\right) \cdot \boldsymbol{\omega} = \mathbf{0}.$$

This equality must be satisfied for any vector $\boldsymbol{\omega}$. It is possible if and only if

$$\mathbf{M} = -\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{dU}{d\boldsymbol{\theta}} + \mathbf{f}(\boldsymbol{\theta}, \boldsymbol{\omega}) \times \boldsymbol{\omega}, \tag{22}$$

where $\mathbf{f}(\boldsymbol{\theta}, \boldsymbol{\omega})$ is some functional of vectors $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$.

**Definition:** a torque $\mathbf{M}$ is called positional if $\mathbf{M}$ depends on the vector of turn $\boldsymbol{\theta}$ only. For the positional torque $\mathbf{M}(\boldsymbol{\theta})$ we have

$$\mathbf{M}(\boldsymbol{\theta}) = -\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{dU(\boldsymbol{\theta})}{d\boldsymbol{\theta}}. \tag{23}$$

Let us show two simple examples.

If the potential function has a form of an isotropic function of a vector of turn

$$U(\boldsymbol{\theta}) = F(\theta^2),$$

then from expression (23) it follows

$$\mathbf{M}(\boldsymbol{\theta}) = -2\frac{dF(\theta^2)}{d(\theta^2)}\boldsymbol{\theta}.$$

Let the potential function has the simplest form

$$U(\boldsymbol{\theta}) = C\mathbf{k}\cdot\boldsymbol{\theta}, \qquad C = const, \qquad \mathbf{k} = const.$$

However, for the torque we have rather complex expression

$$\mathbf{M} = -C\mathbf{Z}^T\cdot\mathbf{k} = -C\left[\mathbf{k} + \frac{1}{2}\boldsymbol{\theta}\times\mathbf{k} + \frac{1-g}{\theta^2}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\mathbf{k})\right].$$

Let there be given a unit vector $\mathbf{k}$.

**Definition:** the potential $U(\boldsymbol{\theta})$ is called transversally isotropic with the axis of symmetry $\mathbf{k}$ if the equality

$$U(\boldsymbol{\theta}) = U[\mathbf{Q}(\alpha\mathbf{k})\cdot\boldsymbol{\theta}]$$

holds good for any tensor of turn $\mathbf{Q}(\alpha\mathbf{k})$.

It can be proved that a general form of a transversally isotropic potential can be expressed as a function of two arguments

$$U(\boldsymbol{\theta}) = F\left(\mathbf{k}\cdot\boldsymbol{\theta}, \theta^2\right). \tag{24}$$

For this potential one can derive the expression

$$\mathbf{M}(\boldsymbol{\theta}) = -2\frac{\partial F}{\partial(\theta^2)}\boldsymbol{\theta} - \frac{\partial F}{\partial(\mathbf{k}\cdot\boldsymbol{\theta})}\mathbf{Z}^T\cdot\mathbf{k}. \tag{25}$$

There exists the obvious identity

$$(\mathbf{E} - \mathbf{Q}(\boldsymbol{\theta}))\cdot\boldsymbol{\theta} = \left(\mathbf{E} - \mathbf{Q}^T\right)\cdot\boldsymbol{\theta} = \mathbf{0} \Longrightarrow (\mathbf{a} - \mathbf{a}')\cdot\boldsymbol{\theta} = 0,$$
$$\mathbf{a}' = \mathbf{Q}\cdot\mathbf{a}.$$

Taking into account this identity and expression (25) one can get

$$(\mathbf{E} - \mathbf{Q}(\boldsymbol{\theta}))\cdot\mathbf{M} = -\frac{\partial F}{\partial(\mathbf{k}\cdot\boldsymbol{\theta})}\mathbf{k}\times\boldsymbol{\theta}.$$

Multiplying this equality by the vector $\mathbf{k}$ we shall obtain

$$(\mathbf{k} - \mathbf{k}')\cdot\mathbf{M} = \mathbf{0}. \tag{26}$$

For the isotropic potential equality (26) holds good for any vector $\mathbf{a}$. Sometimes equality (26) is very important — see, for example, section 4.

## 2.4 The perturbation method on the set of properly orthogonal tensors

Any turn-tensors must be subjected to restrictions

$$\mathbf{Q}\cdot\mathbf{Q}^{\mathsf{T}}=\mathbf{Q}^{\mathsf{T}}\cdot\mathbf{Q}=\mathbf{E},\qquad \det\mathbf{Q}=+1. \tag{27}$$

This means that the perturbed tensor of turn $\mathbf{Q}_\varepsilon$ must be subjected to conditions (27) as well. In contrast with this the vector of turn has no restrictions like (27). Because of this the perturbed vector of turn can be defined in the simplest form

$$\boldsymbol{\theta}_\varepsilon=\boldsymbol{\theta}+\varepsilon\boldsymbol{\varphi},\qquad |\varepsilon|\ll 1, \tag{28}$$

where the vector $\boldsymbol{\varphi}$ is called the first variation of the vector of turn. The perturbed tensor of turn can be found by a usual way

$$\mathbf{Q}_\varepsilon=\exp\mathbf{R}_\varepsilon=\exp\left(\boldsymbol{\theta}_\varepsilon\times\mathbf{E}\right). \tag{29}$$

Equations (27) are satisfied by the tensor $\mathbf{Q}_\varepsilon$ for arbitrary vector $\boldsymbol{\theta}_\varepsilon$. We shall consider the parameter $\varepsilon$ as an independent variable. In such case it is possible to introduce the left $\boldsymbol{\eta}_\varepsilon$ and the right $\boldsymbol{\zeta}_\varepsilon$ velocities of perturbation

$$\frac{\partial}{\partial\varepsilon}\mathbf{Q}_\varepsilon=\boldsymbol{\eta}_\varepsilon\times\mathbf{Q}_\varepsilon,\qquad \frac{\partial}{\partial\varepsilon}\mathbf{Q}_\varepsilon=\mathbf{Q}_\varepsilon\times\boldsymbol{\zeta}_\varepsilon,\qquad \boldsymbol{\eta}_\varepsilon=\mathbf{Q}_\varepsilon\cdot\boldsymbol{\zeta}_\varepsilon. \tag{30}$$

The perturbed angular velocities can be found from the Poisson equations

$$\dot{\mathbf{Q}}_\varepsilon=\boldsymbol{\omega}_\varepsilon\times\mathbf{Q}_\varepsilon,\qquad \dot{\mathbf{Q}}_\varepsilon=\mathbf{Q}_\varepsilon\times\boldsymbol{\Omega}_\varepsilon,\qquad \boldsymbol{\omega}_\varepsilon=\mathbf{Q}_\varepsilon\cdot\boldsymbol{\Omega}_\varepsilon. \tag{31}$$

The conditions of integrability for system (30), (31) can be written in the form

$$\frac{\partial}{\partial\varepsilon}\boldsymbol{\omega}_\varepsilon=\dot{\boldsymbol{\eta}}_\varepsilon+\boldsymbol{\eta}_\varepsilon\times\boldsymbol{\omega}_\varepsilon,\qquad \frac{\partial}{\partial\varepsilon}\boldsymbol{\Omega}_\varepsilon=\dot{\boldsymbol{\zeta}}_\varepsilon-\boldsymbol{\zeta}_\varepsilon\times\boldsymbol{\Omega}_\varepsilon. \tag{32}$$

For the velocities of perturbation we have the expressions that are analogous to equations (19)

$$\boldsymbol{\eta}_\varepsilon=\mathbf{Z}^{-1}\left(\boldsymbol{\theta}_\varepsilon\right)\cdot\frac{\partial}{\partial\varepsilon}\boldsymbol{\theta}_\varepsilon=\mathbf{Z}_\varepsilon^{-1}\cdot\boldsymbol{\varphi},\qquad \boldsymbol{\zeta}_\varepsilon=\mathbf{Z}_\varepsilon^{-\mathsf{T}}\cdot\boldsymbol{\varphi}. \tag{33}$$

The perturbed angular velocities can be found by means of expressions

$$\boldsymbol{\omega}_\varepsilon=\mathbf{Z}_\varepsilon^{-1}\cdot\dot{\boldsymbol{\theta}}_\varepsilon,\qquad \boldsymbol{\Omega}_\varepsilon=\mathbf{Z}_\varepsilon^{-\mathsf{T}}\cdot\dot{\boldsymbol{\theta}}_\varepsilon.$$

If an unperturbed vector $\boldsymbol{\theta}$ does not depend on time (a state of equilibrium), then

$$\boldsymbol{\omega}_\varepsilon=\varepsilon\mathbf{Z}_\varepsilon^{-1}\cdot\dot{\boldsymbol{\varphi}},\qquad \boldsymbol{\Omega}_\varepsilon=\varepsilon\mathbf{Z}_\varepsilon^{-\mathsf{T}}\cdot\dot{\boldsymbol{\varphi}}. \tag{34}$$

Let there be given the function $f\left(\varepsilon,t\right)$. The quantity

$$f^*\left(t\right)=\left[\partial f\left(\varepsilon,t\right)/\partial\varepsilon\right]_{\varepsilon=0} \tag{35}$$

is called the first variation of the function $f(\varepsilon, t)$. For the first variation of the turn-tensor and of the velocities of perturbation we have

$$\mathbf{Q}^* = \boldsymbol{\eta}_0 \times \mathbf{Q}_0, \qquad \boldsymbol{\eta}_0 = \mathbf{Z}_0^{-1} \cdot \boldsymbol{\varphi}, \qquad \boldsymbol{\omega}^* = \dot{\boldsymbol{\eta}}_0 + \boldsymbol{\eta}_0 \times \boldsymbol{\omega}_0, \tag{36}$$

where the subscripts $0$ marks the unperturbed state, $\boldsymbol{\eta}_0 = \boldsymbol{\eta}_\varepsilon \,|_{\varepsilon=0}$.

For the right quantities the next expressions are valid

$$\mathbf{Q}^* = \mathbf{Q}_0 \times \boldsymbol{\zeta}_0, \qquad \boldsymbol{\zeta}_0 = \mathbf{Z}_0^{-T} \cdot \boldsymbol{\varphi}, \qquad \boldsymbol{\Omega}^* = \dot{\boldsymbol{\zeta}}_0 - \boldsymbol{\zeta}_0 \times \boldsymbol{\Omega}_0. \tag{37}$$

If the perturbations are superposed on a state of equilibrium, then $\boldsymbol{\omega}_0 = \boldsymbol{\Omega}_0 = \mathbf{0}$.

Let us write down the formulas for the first variation of modulus of the vector of turn

$$\theta^* = \frac{1}{\theta_0}\boldsymbol{\theta}_0 \cdot \boldsymbol{\varphi} = \frac{1}{\theta_0}\boldsymbol{\theta}_0 \cdot \boldsymbol{\eta}_0 = \frac{1}{\theta_0}\boldsymbol{\theta}_0 \cdot \boldsymbol{\zeta}_0. \tag{38}$$

# 3 The equations of motion of the rigid body oscillator

Let us consider a rigid body with a fixed point $O$. The body is supposed to be clamped in an elastic foundation, which is resisting to any turn of the body. The position of the body, in which the elastic foundation is undeformed, we shall choose as the reference position. The tensor of inertia with respect to the fixed point $O$ of the body will be denoted as

$$\mathbf{A} = A_1 \mathbf{d}_1 \otimes \mathbf{d}_1 + A_2 \mathbf{d}_2 \otimes \mathbf{d}_2 + A_3 \mathbf{d}_3 \otimes \mathbf{d}_3, \tag{39}$$

where $A_i > 0$ are the principal moments of inertia and the vectors $\mathbf{d}_i$ are the principal axes of the inertia tensor. Of course the tensor $\mathbf{A}$ can be represented in terms of arbitrary basis $\mathbf{e}_i$

$$\mathbf{d}_i = \alpha_i^m \mathbf{e}_m, \qquad \mathbf{A} = A^{mn} \mathbf{e}_m \otimes \mathbf{e}_n, \qquad A^{mn} = \sum_{i=1}^{3} \alpha_i^m \alpha_i^n A_i.$$

If the body has the axis of symmetry $\mathbf{k}$, then the inertia tensor will be transversally isotropic

$$\mathbf{A} = A_1 \left(\mathbf{E} - \mathbf{k} \otimes \mathbf{k}\right) + A_3 \mathbf{k} \otimes \mathbf{k}, \qquad \mathbf{d}_3 = \mathbf{k}, \qquad A_1 = A_2. \tag{40}$$

The position of the body at the instant $t$ we shall call the actual position of the body. A turn of the body can be defined by the turn-tensor $\mathbf{P}(t)$ or by the vector of turn $\boldsymbol{\theta}(t)$

$$\mathbf{P}(t) = \mathbf{Q}(\boldsymbol{\theta}(t)).$$

The tensor of inertia $\mathbf{A}^{(t)}$ in the actual position is determined by the formula

$$\mathbf{A}^{(t)} = \mathbf{P}(t) \cdot \mathbf{A} \cdot \mathbf{P}^T(t). \tag{41}$$

If the tensor $\mathbf{A}$ is transversally isotropic, then one can write down

$$\mathbf{A}^{(t)} = A_1 \left(\mathbf{E} - \mathbf{k}' \otimes \mathbf{k}'\right) + A_3 \mathbf{k}' \otimes \mathbf{k}', \qquad \mathbf{k}' = \mathbf{P} \cdot \mathbf{k}. \tag{42}$$

A kinetic moment of the body can be expressed in two forms. In terms of the left angular velocity

$$\mathbf{L} = \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{P}^T \cdot \boldsymbol{\omega} = A_1 \boldsymbol{\omega} + (A_3 - A_1)(\mathbf{k}' \cdot \boldsymbol{\omega})\,\mathbf{k}'. \tag{43}$$

Here the first sign of equality concerns to a general case, the second sign of equality is applied to the transversally isotropic tensor of inertia only. In terms of the right angular velocity the kinetic moment has the form

$$\mathbf{L} = \mathbf{P} \cdot \mathbf{A} \cdot \boldsymbol{\Omega} = \mathbf{P} \cdot \left[A_1 \boldsymbol{\Omega} + (A_3 - A_1)(\mathbf{k} \cdot \boldsymbol{\Omega})\, \mathbf{k}\right]. \tag{44}$$

Let us note that

$$\mathbf{k}' \cdot \boldsymbol{\omega} = \mathbf{k} \cdot \mathbf{P}^{\mathsf{T}} \cdot \boldsymbol{\omega} = \mathbf{k} \cdot \boldsymbol{\Omega}. \tag{45}$$

An external torque $\mathbf{M}$ acting on the body can be represented in the form

$$\mathbf{M} = \mathbf{M}_e + \mathbf{M}_{ext},$$

where $\mathbf{M}_e$ is a reaction of the elastic foundation and $M_{ext}$ is an additional external torque. The elastic torque $\mathbf{M}_e$ is supposed to be potential. Besides it is supposed to be positional. In such case we write — see equation (23)

$$\mathbf{M}_e = -\mathbf{Z}^{\mathsf{T}}(\boldsymbol{\theta}) \cdot \frac{dU(\boldsymbol{\theta})}{d\boldsymbol{\theta}}, \tag{46}$$

where the scalar function $U(\boldsymbol{\theta})$ will be called an elastic energy. In what follows the elastic foundation is supposed to be transversally isotropic. This means that the elastic torque can be represented in form (25)

$$\mathbf{M}_e(\boldsymbol{\theta}) = -C\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right) \boldsymbol{\theta} - D\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right) \mathbf{Z}^{\mathsf{T}}(\boldsymbol{\theta}) \cdot \mathbf{k}, \tag{47}$$

where the unit vector $\mathbf{k}$ is placed on the axis of isotropy of the body when the elastic foundation is in the undeformed state.

$$C = 2\frac{\partial}{\partial(\theta^2)} U\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right), \qquad D = \frac{\partial}{\partial(\mathbf{k} \cdot \boldsymbol{\theta})} U\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right). \tag{48}$$

Let us show one of possible expressions of the elastic energy

$$U = \frac{1}{2} \frac{\alpha^2 c \theta^2}{\alpha^2 - \theta^2 + (\mathbf{k} \cdot \boldsymbol{\theta})^2} + \frac{1}{2} \frac{\beta^2 (d - c)(\mathbf{k} \cdot \boldsymbol{\theta})^2}{\beta^2 - (\mathbf{k} \cdot \boldsymbol{\theta})^2}, \tag{49}$$

where $\alpha^2 > 0$, $\beta^2 > 0$, $c > 0$ and $d > 0$ are the constant parameters and also parameters $c$ and $d$ are called the bending rigidness and torsional rigidness of the elastic foundation respectively.

If the parameters $\alpha^2$ and $\beta^2$ tend to the infinity, then we shall get the simplest form of the elastic potential

$$U = \frac{1}{2} c \left(\theta^2 - (\mathbf{k} \cdot \boldsymbol{\theta})^2\right) + \frac{1}{2} d (\mathbf{k} \cdot \boldsymbol{\theta})^2. \tag{50}$$

In this case expression (47) takes the form

$$\mathbf{M}_e(\boldsymbol{\theta}) = -c\boldsymbol{\theta} - (d - c)\, \mathbf{k} \cdot \boldsymbol{\theta} \mathbf{Z}^{\mathsf{T}}(\boldsymbol{\theta}) \cdot \mathbf{k}. \tag{51}$$

For the external torque $\mathbf{M}_{ext}$ let us accept the expression

$$\mathbf{M}_{ext} = -\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{dV(\boldsymbol{\theta})}{d\boldsymbol{\theta}} + \mathbf{M}_{ex}, \tag{52}$$

where the first term describes the potential part of the external torque. The second law of dynamics of Euler can be represented in two equivalent forms. In terms of the left angular velocity it takes the form

$$\left[\mathbf{P}(\boldsymbol{\theta}) \cdot \mathbf{A} \cdot \mathbf{P}^T(\boldsymbol{\theta}) \cdot \boldsymbol{\omega}\right]^{\cdot} + \mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{d(U+V)}{d\boldsymbol{\theta}} = \mathbf{M}_{ex}. \tag{53}$$

To this equation we have to add the left Poisson equation in form (15)

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}). \tag{54}$$

System of equations (53) and (54) gives to us a general model of the rigid body oscillator. In terms of the right angular velocity this model can be represented in the form

$$\mathbf{A} \cdot \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbf{A} \cdot \boldsymbol{\Omega} + \mathbf{Z}(\boldsymbol{\theta}) \cdot \frac{d(U+V)}{d\boldsymbol{\theta}} = \mathbf{P}^T(\boldsymbol{\theta}) \cdot \mathbf{M}_{ex}, \tag{55}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\Omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\Omega}). \tag{56}$$

It is important that the model of rigid body oscillator is represented in terms of natural variables: the vector of turn and the vector of angular velocity. Besides significant merit of stated above equations is that they contain the first derivatives of the unknown vectors only. Thus it is possible to use standard methods of the numerical analysis.

The rest of the paper deals with applications of the derived equations.

# 4 The stability of equilibrium state of rigid body oscillator under the action of the follower torque. Paradox of Nikolai

Let us consider the classical problem that was investigated by E.L.Nikolai [3]. Later it was studied by many authors — see, for example, [4], [5], where another references can be found.

The inertia tensor of the body is supposed to be transversally isotropic and is defined by expression (40). An external torque is defined by the next expression

$$\mathbf{M}_{ex} = L\mathbf{P}(\boldsymbol{\theta}) \cdot \mathbf{k}, \qquad L = \text{const}, \tag{57}$$

where the unit vector $\mathbf{k}$ is placed on the axis of symmetry of the body in the reference position when the elastic foundation is undeformed.

Accepting the stated above assumptions we are able to write down equations (55) and(56) in the next form.

$$A_1\dot{\boldsymbol{\Omega}} + (A_3 - A_1)\left(\mathbf{k} \cdot \dot{\boldsymbol{\Omega}}\right)\mathbf{k} - (A_3 - A_1)(\mathbf{k} \cdot \boldsymbol{\Omega})\,\mathbf{k} \times \boldsymbol{\Omega} + C\boldsymbol{\theta} + D\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \mathbf{k} = L\mathbf{k}, \tag{58}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\Omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\Omega}), \qquad g = \frac{\theta \sin\theta}{2(1-\cos\theta)}, \tag{59}$$

where the functions C and D are defined by expressions (48). It is easy to find the equilibrium solution of system of equations (58) and (59)

$$\boldsymbol{\theta} = \theta\mathbf{k}, \qquad \theta = const, \qquad \boldsymbol{\Omega} = 0. \tag{60}$$

Substituting (60) into system (58)–(59) we shall get the scalar equation

$$C\left(\theta^2, \theta\right)\theta + D\left(\theta^2, \theta\right) = L. \tag{61}$$

If the elastic energy has form (50), then equation (61) takes the linear form

$$C\left(\theta^2, \theta\right) = c, D\left(\theta^2, \theta\right) = (d-c)\,\mathbf{k}\cdot\boldsymbol{\theta} \Longrightarrow \theta = L\mathbf{k}/d. \tag{62}$$

In order to investigate a stability of the solution of equation (61) we shall use the method of superposition of small perturbations on the state of equilibrium. To this end let us consider the perturbed quantities

$$\boldsymbol{\theta}_\varepsilon = \theta\mathbf{k} + \varepsilon\boldsymbol{\varphi}(t), \qquad \boldsymbol{\Omega}_\varepsilon = \varepsilon\boldsymbol{\eta}, \tag{63}$$

where $\theta$ is the solution of (61).

Now we have to write down perturbed equations (58) and (59). For this it is sufficiently to provide the vectors $\boldsymbol{\theta}$ and $\boldsymbol{\Omega}$ in these equations by the subscripts $\varepsilon$. After that it is necessary to differentiate these equations with respect to $\varepsilon$ and to accept $\varepsilon = 0$. As the result we shall get equations in variations $\boldsymbol{\varphi}$ and $\boldsymbol{\eta}$.

For the sake of simplicity let us consider case (62). In such case perturbed equations (58) and (59) take the form

$$A_1\dot{\boldsymbol{\Omega}}_\varepsilon + (A_3 - A_1)\left(\mathbf{k}\cdot\dot{\boldsymbol{\Omega}}_\varepsilon\right)\mathbf{k} - (A_3 - A_1)(\mathbf{k}\cdot\boldsymbol{\Omega}_\varepsilon)\,\mathbf{k}\times\boldsymbol{\Omega}_\varepsilon + c\boldsymbol{\theta}_\varepsilon +$$

$$+ (d-c)\,\mathbf{k}\cdot\boldsymbol{\theta}_\varepsilon\mathbf{Z}^T(\boldsymbol{\theta}_\varepsilon)\cdot\mathbf{k} = L\mathbf{k}, \tag{64}$$

$$\dot{\boldsymbol{\theta}}_\varepsilon = \boldsymbol{\Omega}_\varepsilon + \frac{1}{2}\boldsymbol{\theta}_\varepsilon \times \boldsymbol{\Omega}_\varepsilon + \frac{1-g_\varepsilon}{\theta_\varepsilon^2}\boldsymbol{\theta}_\varepsilon \times (\boldsymbol{\theta}_\varepsilon \times \boldsymbol{\Omega}_\varepsilon), g_\varepsilon = \frac{\theta_\varepsilon \sin\theta_\varepsilon}{2(1-\cos\theta_\varepsilon)}. \tag{65}$$

Expressions (63) take the form

$$\boldsymbol{\theta}_\varepsilon = \frac{L}{d}\mathbf{k} + \varepsilon\boldsymbol{\varphi}, \qquad \boldsymbol{\Omega}_\varepsilon = \varepsilon\boldsymbol{\eta}, \qquad \boldsymbol{\theta}_\varepsilon \times \mathbf{k} = \varepsilon\boldsymbol{\varphi}\times\mathbf{k}. \tag{66}$$

The equations in variations can be represented as

$$A_1\dot{\boldsymbol{\eta}} + (A_3 - A_1)(\mathbf{k}\cdot\dot{\boldsymbol{\eta}})\,\mathbf{k} + c\boldsymbol{\varphi} + (d-c)(\mathbf{k}\cdot\boldsymbol{\varphi})\,\mathbf{k} +$$

$$+L\left(1 - \frac{c}{d}\right)\left[\frac{1}{2}\boldsymbol{\varphi}\times\mathbf{k} + \frac{1-g}{\theta}(\boldsymbol{\varphi} - \mathbf{k}\cdot\boldsymbol{\varphi}\mathbf{k})\right] = 0,$$

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{\eta} + \frac{1}{2}\frac{L}{d}\mathbf{k}\times\boldsymbol{\eta} - (1-g)(\boldsymbol{\eta} - (\mathbf{k}\times\boldsymbol{\eta})\,\mathbf{k}).$$

These equations can be rewritten in more simple form with the help of substitution

$$\boldsymbol{\eta} = \zeta \mathbf{k} + \mathbf{y}, \qquad \mathbf{y} \cdot \mathbf{k} = 0; \qquad \boldsymbol{\varphi} = \gamma \mathbf{k} + \boldsymbol{\psi}, \qquad \boldsymbol{\psi} \cdot \mathbf{k} = 0. \tag{67}$$

After some transformations one can write

$$A_3 \ddot{\gamma} + d\gamma = 0, \qquad \zeta = \dot{\gamma}, \tag{68}$$

$$A_1 \ddot{\boldsymbol{\psi}} + \left[ c \left( g^2 + \frac{L^2}{4d^2} \right) - \frac{L^2}{4d} + (1 - g)\, g d \right] \boldsymbol{\psi} + \frac{L}{2} \mathbf{k} \times \boldsymbol{\psi} = \mathbf{0}, \tag{69}$$

where

$$g = \frac{\theta \sin \theta}{2\,(1 - \cos \theta)}, \qquad \theta = \frac{L}{d}.$$

If the quantity $|L|/d$ is small, i.e. $|L|/d \ll 1$, then equation (69) can be rewritten as

$$A_1 \ddot{\boldsymbol{\psi}} + \left[ c + \frac{(c - 2d)\, L^2}{12 d^2} \right] \boldsymbol{\psi} + \frac{L}{2} \mathbf{k} \times \boldsymbol{\psi} = 0. \tag{70}$$

Let us look for a particular solution of these equation in the form

$$\boldsymbol{\psi} = \mathbf{a} \exp(pt), \qquad \mathbf{a} = const, \qquad \mathbf{a} \cdot \mathbf{k} = 0.$$

For the vector $\mathbf{a}$ we have the system

$$\left[ A_1 \left( p^2 + c + \frac{(c - 2d)\, L^2}{12 d^2} \right) \mathbf{E}_* + \frac{L}{2} \mathbf{k} \times \mathbf{E}_* \right] \cdot \mathbf{a} = \mathbf{0}, \quad \mathbf{E}_* = \mathbf{E} - \mathbf{k} \otimes \mathbf{k}.$$

The determinant of this system must be equal to zero

$$\left[ A_1 \left( p^2 + c + \frac{(c - 2d)\, L^2}{12 d^2} \right) \right]^2 + \frac{L^2}{4} = 0.$$

It is easy to see that at least one root of this equation has a positive real part. From this it follows that the solution of equation (70) infinitely increases. This means that the state of equilibrium (62) or (61) is unstable for arbitrarily small quantity of external twisting moment $L$. This phenomenon is well known under the name of paradox of Nikolai.

From the pure theoretical point of view it is no wonder that the state of equilibrium is unstable. However, from the practical point of view the situation is very disagreeable. Really, if the external torque is small, then it is supposed that the linear theory is valid. In this case system of equations (58) and (59) can be rewritten in the form of equation

$$A_1 \ddot{\boldsymbol{\theta}} + (A_3 - A_1)\, (\mathbf{k} \cdot \ddot{\boldsymbol{\theta}}) \mathbf{k} + c \boldsymbol{\theta} + (d - c)\, (\mathbf{k} \cdot \boldsymbol{\theta})\, \mathbf{k} = L \mathbf{k}.$$

The solution of this equation has a small norm if the torque $L$ and the norm of initial conditions are small. Namely this way is used in the most of applied investigations. There was no doubts that such approach is quite accurate. However, as it was shown above, if we take into account the small quantities of the second order, then the solution will be unstable. Is it really so? It is well-known fact [6] that the equations in variations may give a faulty result in some cases. This means that in doubtful cases the nonlinear analysis have to be used.

# 5 Nonlinear analysis and rigorous justification of the paradox of Nikolai

Let us consider the external torque of the kind

$$\mathbf{M}_{ex} = \gamma L \left( l_1 \mathbf{k} + l_2 \mathbf{P} \cdot \mathbf{k} \right), \qquad \gamma = \left( l_1^2 + l_2^2 + 2 l_1 l_2 \mathbf{k} \cdot \mathbf{P} \cdot \mathbf{k} \right)^{-\frac{1}{2}}. \tag{71}$$

If $l_1 = 1,\ l_2 = 0$, then $\mathbf{M}_{ex}$ is a dead torque; if $l_1 = 0,\ l_2 = 1$, then $\mathbf{M}_{ex}$ is a followed (tangential) torque; if $l_1 = l_2 = 1$, then $\mathbf{M}_{ex}$ is a semitangential torque. For the elastic torque let us accept expression (47) , where $C\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right)$ and $D\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right)$ are the functions of a general kind. The tensor of inertia is supposed to be transversally isotropic with the axis of symmetry $\mathbf{k}$.

For the vector of kinetic moment we have formulas (43) and (44). Let us write down the equation of the energy balance when the external torque is defined by expression (71)

$$\dot{\varepsilon} = \gamma L \left( l_1 \mathbf{k} \cdot \boldsymbol{\omega} + l_2 \mathbf{k} \cdot \boldsymbol{\Omega} \right), \qquad \varepsilon = \frac{1}{2} A_1 \omega^2 + \frac{1}{2} \left( A_3 - A_1 \right) \left( \mathbf{k} \cdot \boldsymbol{\Omega} \right)^2 + U\left( \boldsymbol{\theta} \right). \tag{72}$$

From (72) it follows

$$\varepsilon - \varepsilon_0 = L \int_0^t \gamma\left(\tau\right) \mathbf{k} \cdot \left( l_1 \boldsymbol{\omega}\left(\tau\right) + l_2 \boldsymbol{\Omega}\left(\tau\right) \right) d\tau. \tag{73}$$

If the integral in the right side of equation (73) is bounded for all $t$, then for small $|L|$ the energy $\varepsilon$ is close to the value of the initial energy $\varepsilon_0$. In such a case the stability is possible. If integral (73) is infinitely increasing, then we have the accumulation of energy in the system and the stability is impossible for arbitrarily small $|L|$.

Let us write the equation of motion in two forms

$$\left[ A_1 \boldsymbol{\omega} + \left( A_3 - A_1 \right) \left( \boldsymbol{\omega} \cdot \mathbf{k}' \right) \mathbf{k}' \right]^{\cdot} + C \boldsymbol{\theta} + D \mathbf{Z}^{T} \cdot \mathbf{k} = \gamma L \left( l_1 \mathbf{k} + l_2 \mathbf{k}' \right), \tag{74}$$

$$\begin{gathered} \left[ A_1 \boldsymbol{\Omega} + \left( A_3 - A_1 \right) \left( \boldsymbol{\Omega} \cdot \mathbf{k} \right) \mathbf{k} \right]^{\cdot} + \left( A_3 - A_1 \right) \left( \mathbf{k} \cdot \boldsymbol{\Omega} \right) \boldsymbol{\Omega} \times \mathbf{k} + \\ + C \boldsymbol{\theta} + D \mathbf{Z} \cdot \mathbf{k} = \gamma L \left( l_1 \mathbf{P}^{T} \cdot \mathbf{k} + l_2 \mathbf{k} \right), \end{gathered} \tag{75}$$

where $\boldsymbol{\omega} \cdot \mathbf{k}' = \boldsymbol{\Omega} \cdot \mathbf{k},\ \ \mathbf{k}' = \mathbf{P} \cdot \mathbf{k}$.

Equations (74) and (75) are equivalent. Nevertheless from them the nontrivial result can be found. Subtracting equation (75) from equation (74) one can get

$$\begin{gathered} \left[ A_1 \left( \boldsymbol{\omega} - \boldsymbol{\Omega} \right) + \left( A_3 - A_1 \right) \left( \boldsymbol{\Omega} \cdot \mathbf{k} \right) \left( \mathbf{k}' - \mathbf{k} \right) \right]^{\cdot} + \\ + \left( A_3 - A_1 \right) \left( \mathbf{k} \cdot \boldsymbol{\Omega} \right) \mathbf{k} \times \boldsymbol{\Omega} + D \boldsymbol{\theta} \times \mathbf{k} = \gamma L \left[ \left( l_1 - l_2 \right) \mathbf{k} + l_2 \mathbf{k}' - l_1 \mathbf{P}^{T} \cdot \mathbf{k} \right]. \end{gathered}$$

Multiplying this equation by the vector $\mathbf{k}$ we shall obtain the next equation

$$\begin{gathered} \left[ A_1 \left( \boldsymbol{\omega} - \boldsymbol{\Omega} \right) \cdot \mathbf{k} + \left( A_1 - A_3 \right) \mathbf{k} \cdot \boldsymbol{\Omega} \left( 1 - \cos\vartheta \right) \right]^{\cdot} = \\ = \gamma L \left( l_1 - l_2 \right) \left( 1 - \cos\vartheta \right), \end{gathered} \tag{76}$$

where $\cos\vartheta = \mathbf{k}\cdot\mathbf{k}' = \mathbf{k}\cdot\mathbf{P}\cdot\mathbf{k}$.

Let us note that equation (76) does not contain the characteristics of the elastic foundation. Equation (76) can be rewritten in another form. From equations (19) and (20) it follows

$$\boldsymbol{\omega} - \boldsymbol{\Omega} = \left(\mathbf{Z}^{-1} - \mathbf{Z}^{-T}\right)\cdot\dot{\boldsymbol{\theta}} = 2\frac{1-\cos\theta}{\theta^2}\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}. \tag{77}$$

The vector of turn $\boldsymbol{\theta}$ can be represented in the form of the composition

$$\boldsymbol{\theta} = x\mathbf{k} + \mathbf{y}, \mathbf{y}\cdot\mathbf{k} = 0, \mathbf{y} = y\,(t)\,\mathbf{Q}\,(\psi\,(t)\,\mathbf{k})\cdot\mathbf{m}, \tag{78}$$
$$\mathbf{m}\cdot\mathbf{k} = 0, |\mathbf{m}| = 1, \theta^2 = x^2 + y^2.$$

One can prove the formulas

$$\mathbf{k}\cdot\left(\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}\right) = \mathbf{k}\cdot(\mathbf{y}\times\dot{\mathbf{y}}) = \dot{\psi}y^2, \quad 1-\cos\vartheta = \frac{y^2\,(1-\cos\theta)}{\theta^2}. \tag{79}$$

Taking into account relations (77), (78) and (79) equation (76) can be rewritten in the form

$$[(1-\cos\vartheta)\,F]^{\cdot} = \gamma L\,(l_1 - l_2)\,(1-\cos\vartheta)\,, \tag{80}$$

where

$$F = 2A_1\dot{\psi} + (A_1 - A_3)\,\mathbf{k}\cdot\boldsymbol{\Omega}.$$

Equality (80) was derived by another way and was shown to the author in the private talk by Dr. A. Krivtsov. In fact equality (80) is due to the existence of property (26) for the elastic torque. Let us note that the right side of equation (80) has the constant sign, which is defined by the sign of the number $L\,(l_1 - l_2)$. Let us suppose that $L\,(l_1 - l_2) > 0$. In such a case let us choose the initial conditions such that $F\,|_{t=0} > 0$. Equality (80) shows to us that the function $F\,(t)$ tends to infinity as $t \to \infty$. This means that the body will have an infinitely big velocity of precession $\dot{\psi}$, i.e. state of equilibrium (61) or (62) is unstable for arbitrarily small value of twisting torque and for any transversally isotropic elastic foundation. Therefore the analysis on the base of the equations in variations gives the right result. The paradox of Nikolai is due to an accumulation of energy in the system.

## 6 The simplest rigid body oscillator. The total integrability of the basic equations

Let us consider the simplest case of the rigid body oscillator. For this end let us accept the next restrictions

$$\mathbf{A} = A\mathbf{E}, \qquad U = u\left(\theta^2\right), \qquad \frac{d}{d\boldsymbol{\theta}}U = 2u'\left(\theta^2\right)\boldsymbol{\theta} = c\left(\theta^2\right)\boldsymbol{\theta}. \tag{81}$$

In addition let us introduce the torque of friction in the form

$$\mathbf{M}_{ex} = -b\boldsymbol{\omega}, \qquad b = \mathrm{const} \geqslant 0. \tag{82}$$

In such a case basic equations (55) and (56) can be written down in the form

$$A\dot{\boldsymbol{\Omega}} + b\boldsymbol{\Omega} + c\left(\theta^2\right)\boldsymbol{\theta} = \mathbf{0}, \tag{83}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta}\times\boldsymbol{\Omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\Omega}), \qquad g = \frac{\theta\sin\theta}{2(1-\cos\theta)}. \tag{84}$$

It is seen that even in this simplest case the basic system is rather complicated. The system can be simplified only in the case of the plane oscillations when

$$\boldsymbol{\omega} = \boldsymbol{\Omega} = \dot{\boldsymbol{\theta}}, \qquad \boldsymbol{\theta}\times\boldsymbol{\Omega} = \mathbf{0}.$$

If it is so, then system (83) and (84) takes the form

$$A\ddot{\boldsymbol{\theta}} + b\dot{\boldsymbol{\theta}} + c\left(\theta^2\right)\boldsymbol{\theta} = \mathbf{0}; \qquad t = 0: \boldsymbol{\theta} = \boldsymbol{\theta}_0, \qquad \boldsymbol{\Omega} = \boldsymbol{\Omega}_0, \qquad \boldsymbol{\theta}_0\times\boldsymbol{\Omega}_0 = \mathbf{0}. \tag{85}$$

This system can be investigated without any problems.

Let us discuss system of equations (83) and (84) in a general case. In order to underline the difference between conventional approach and our method let us consider both of them.

## 6.1 Conventional approach

Let us try to investigate system (83), (84) on the base of application of the Euler angles. The tensor of turn can be represented [1] in the form

$$\mathbf{P}(\boldsymbol{\theta}) = \mathbf{Q}(\psi\mathbf{k})\cdot\mathbf{Q}(\vartheta\mathbf{p})\cdot\mathbf{Q}(\varphi\mathbf{k}) = \mathbf{Q}(\vartheta\mathbf{p}')\cdot\mathbf{Q}(\beta\mathbf{k}), \tag{86}$$

where

$$\beta = \varphi + \psi, \qquad \mathbf{p}' = \mathbf{Q}(\psi\mathbf{k})\cdot\mathbf{p}, \qquad \mathbf{k}\cdot\mathbf{p} = \mathbf{k}\cdot\mathbf{p}' = 0. \tag{87}$$

The left angular velocity is determined by the formula

$$\boldsymbol{\omega} = \left(\dot{\psi} + \dot{\varphi}\cos\vartheta\right)\mathbf{k} + \dot{\vartheta}\mathbf{p}' + \dot{\varphi}\sin\vartheta\,\mathbf{p}'\times\mathbf{k}. \tag{88}$$

Making use expressions (7), (86), (88) and substituting them into equation (90) one can derive the system

$$A\left(\dot{\psi} + \dot{\varphi}\cos\vartheta\right)^{\cdot} + b\left(\dot{\psi} + \dot{\varphi}\cos\vartheta\right) + \frac{c\left(\theta^2\right)\theta}{2\sin\theta}\sin\beta\,(1+\cos\vartheta) = 0,$$

$$A\left(\ddot{\vartheta} + \dot{\psi}\dot{\varphi}\sin\vartheta\right) + b\dot{\vartheta} + \frac{c\left(\theta^2\right)\theta}{2\sin\theta}\sin\vartheta\,(1+\cos\beta) = 0,$$

$$A\left[(\dot{\varphi}\sin\vartheta)^{\cdot} - \dot{\varphi}\dot{\vartheta}\right] + b\dot{\varphi}\sin\vartheta + \frac{c\left(\theta^2\right)\theta}{2\sin\theta}\sin\beta\sin\vartheta = 0. \tag{89}$$

In addition to this system we have the relations

$$1 + 2\cos\theta = \cos\vartheta + \cos\beta + \cos\vartheta\cos\beta, \beta = \varphi + \psi.$$

It is not so easy to find the total solution of system (89). Let us note that representation (86) is completely admissible. However, there are many another possibilities and the most of them will lead to the complicated equations. If we want to find the best representation, then we have to look for this representation in the process of a solution rather then to guess it a priori. The latter circumstances was underlined in the paper [1].

## 6.2 The total integrability of the equations of the simplest rigid body oscillator

Multiplying equation (83) by the tensor $\mathbf{P}(\boldsymbol{\theta})$ from the left one can obtain

$$A\dot{\boldsymbol{\omega}} + b\boldsymbol{\omega} + c\left(\theta^2\right)\boldsymbol{\theta} = \mathbf{0}. \tag{90}$$

Here the identity

$$\mathbf{P}\cdot\dot{\boldsymbol{\Omega}} = \left(\mathbf{P}\cdot\boldsymbol{\Omega}\right)^{\cdot} - \dot{\mathbf{P}}\cdot\boldsymbol{\Omega} = \dot{\boldsymbol{\omega}} - \left(\mathbf{P}\times\boldsymbol{\Omega}\right)\cdot\boldsymbol{\Omega} = \dot{\boldsymbol{\omega}}$$

was taken into account.

Equation (90) is equivalent to equation (83). However from (83) and (90) the non-trivial result follows

$$A\left(\boldsymbol{\omega}-\boldsymbol{\Omega}\right)^{\cdot} + b\left(\boldsymbol{\omega}-\boldsymbol{\Omega}\right) = \mathbf{0} \Longrightarrow \boldsymbol{\omega}-\boldsymbol{\Omega} = \left(\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0\right)\exp\left(-\frac{bt}{A}\right), \tag{91}$$

where $\boldsymbol{\omega}_0$ and $\boldsymbol{\Omega}_0$ are the initial angular velocities. Expression (91) gives to us three integrals. Now it is necessary to consider two cases

$$a)\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0=\mathbf{0}, \qquad b)\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0=\left|\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0\right|\mathbf{e}\neq\mathbf{0}.$$

In the first case we deal with the plane vibrations of the oscillator. Really, in the first case from (91) it follows that

$$\boldsymbol{\omega}=\boldsymbol{\Omega}\Longrightarrow\boldsymbol{\Omega}\times\boldsymbol{\theta}=\mathbf{0}.$$

The latter fact follows from (15) and (16). Thus we have equation (85). It is more interesting to investigate the case b). From equations (15) and (16) the next relation can be derived.

$$g\left(\theta\right)\left(\boldsymbol{\omega}-\boldsymbol{\Omega}\right)=\frac{1}{2}\boldsymbol{\theta}\times\left(\boldsymbol{\omega}+\boldsymbol{\Omega}\right).$$

Taking into account integral (91) one can get

$$g\left(\theta\right)\exp\left(-\frac{bt}{A}\right)\left(\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0\right)=\frac{1}{2}\boldsymbol{\theta}\times\left(\boldsymbol{\omega}+\boldsymbol{\Omega}\right).$$

Besides let us take into account the identity

$$\frac{1}{2}\boldsymbol{\theta}\times\left(\boldsymbol{\omega}+\boldsymbol{\Omega}\right)=\frac{\sin\theta}{\theta}\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}.$$

The previous expression can be rewritten in the form

$$\frac{2\left(1-\cos\theta\right)}{\theta^2}\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}=\left(\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0\right)\exp\left(-\frac{bt}{A}\right). \tag{92}$$

From this equation one more integral follows

$$\boldsymbol{\theta}\left(t\right)\cdot\left(\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0\right)=\mathbf{0}\Longrightarrow\boldsymbol{\theta}\left(t\right)\cdot\mathbf{e}=\mathbf{0}, \tag{93}$$

where the vector $\mathbf{e}$ is the vector $(\boldsymbol{\omega}_0 - \boldsymbol{\Omega}_0)/|\boldsymbol{\omega}_0 - \boldsymbol{\Omega}_0|$. Equation (93) shows that the vector $\boldsymbol{\theta}(t)$ can be represented in the form

$$\boldsymbol{\theta}(t) = \theta(t)\,\mathbf{Q}(\psi\mathbf{e}) \cdot \mathbf{m}, \qquad \mathbf{m} = \boldsymbol{\theta}_0/\theta_0, \qquad \mathbf{m} \cdot \mathbf{e} = 0, \qquad \psi(0) = 0. \tag{94}$$

From this representation it follows

$$\boldsymbol{\theta} \times \dot{\boldsymbol{\theta}} = \dot{\psi}\theta^2 \mathbf{e}. \tag{95}$$

Substituting (95) into (92) one can get

$$\dot{\psi} = \frac{1-\cos\theta_0}{1-\cos\theta(t)}\dot{\psi}_0 \exp\left(-\frac{bt}{A}\right), \qquad \dot{\psi}_0 > 0. \tag{96}$$

Thus if we know the angle of nutation $\theta(t)$ then the angle of precession can be found from (96). Let us derive the equation for the angle $\theta$. For this end let us calculate the right angular velocity

$$\boldsymbol{\Omega} = \frac{\dot{\theta}}{\theta}\boldsymbol{\theta} + \frac{\sin\theta}{\theta}\dot{\psi}\mathbf{e} \times \boldsymbol{\theta} - (1-\cos\theta)\,\dot{\psi}\mathbf{e}. \tag{97}$$

Substituting expression (97) into equation (83) and projecting the obtained equation on the vectors $\boldsymbol{\theta}$, $\mathbf{e}$ and $\mathbf{e} \times \boldsymbol{\theta}$ one can get three scalar equations, where two of them (projections on $\mathbf{e}$ and $\mathbf{e} \times \boldsymbol{\theta}$) will be identities because of equality (96). Projection on the vector $\boldsymbol{\theta}$ gives

$$A\left[\ddot{\theta} - \sin\theta\left(\frac{1-\cos\theta_0}{1-\cos\theta}\right)^2 \left(\dot{\psi}_0\right)^2 \exp\left(-\frac{2bt}{A}\right)\right] + b\dot{\theta} + c\left(\theta^2\right)\theta = 0. \tag{98}$$

If the friction is absent $(b = 0)$, then this equation can be solved in terms of quadratures. The plane motions of the oscillator can be found from equation (98) when $\dot{\psi}_0 = 0$. In a general case equation (98) can be studied by conventional methods of nonlinear mechanics. Let us note that even for small $\theta$ equation (98) is nonlinear one.

$$A\ddot{\theta} + b\dot{\theta} + \left[c(0) - A\left(\frac{\theta_0}{\theta}\right)^4 \dot{\psi}_0^2 \exp\left(-\frac{2bt}{A}\right)\right]\theta = 0. \tag{99}$$

In contrast with it for small turns system of equations (83) and (84) can be linearized and we shall get the linear equation

$$A\ddot{\boldsymbol{\theta}} + b\dot{\boldsymbol{\theta}} + c(0)\,\boldsymbol{\theta} = 0. \tag{100}$$

Nonlinear equation (99) can be derived from equation (100) if one take into account that $\theta = |\boldsymbol{\theta}|$. If the friction is absent $(b = 0)$ then equation (98) has an exact solution

$$\theta = \theta_0 = \mathrm{const}, \qquad \dot{\psi} = \dot{\psi}_0 = \mathrm{const}, \qquad \left(\dot{\psi}\right)^2 = \frac{c\left(\theta_0^2\right)\theta_0}{A\sin\theta_0}. \tag{101}$$

This solution is called a regular precession, which will be considered in the next section. If the friction is present, then for the big times equation (98) transforms to equation (85).

Let us compare two described approaches. The first approach is defined by representation (86) of the turn-tensor, where the unit vectors $\mathbf{k}$ and $\mathbf{p}$ $(\mathbf{k}\cdot\mathbf{p}=0)$ were chosen a priori. This means that for angles $\psi, \vartheta, \varphi$ and the velocities $\dot\psi, \dot\vartheta, \dot\varphi$ we have to provide the arbitrary initial conditions. In other words we have to look for a general solution of the system (89). It is not known if it is possible.

In the second approach the representation of the turn-tensor has a special form

$$\mathbf{P} = \mathbf{Q}(\boldsymbol{\theta}) = \mathbf{Q}\left[\theta\mathbf{Q}(\psi\mathbf{e})\cdot\mathbf{m}\right] = \mathbf{Q}(\psi\mathbf{e})\cdot\mathbf{Q}(\theta\mathbf{m})\cdot\mathbf{Q}^{\mathsf{T}}(\psi\mathbf{e}). \tag{102}$$

Here we used representation (94) for the vector of turn and the unit vectors $\mathbf{e}$ and $\mathbf{m}$, which were found in the process of solution. Representation (102) contains only two angles $\theta$ and $\psi$, but the unit vectors $\mathbf{e}$, $\mathbf{m}$ are chosen by a special manner. Representation (86) contains three angles $\psi, \vartheta$ and $\varphi$ but the unit vectors $\mathbf{k}$ and $\mathbf{p}$ can be any orthogonal vectors. Let us accept the relation $\varphi = -\psi$, i.e. $\beta = 0$, in representation (86). In such case system (89) takes the form $(\beta = 0)$

$$A\left[\dot\psi\left(1-\cos\theta\right)\right]^{\cdot} + b\dot\psi\left(1-\cos\theta\right) = 0,$$

$$A\left(\ddot\theta - \dot\psi^2\sin\theta\right) + b\dot\vartheta + c\left(\theta^2\right)\theta = 0,$$

$$A\left[\left(\dot\psi\sin\theta\right)^{\cdot} - \dot\theta\dot\psi\right] + b\dot\psi\sin\theta = 0.$$

The first equation of this system gives to us integral (96). The third equation is an identity if we take into account the first equation. At last, the second equation coincides with equation (98). Thus system (89) has a particular solution coinciding with the found above solution. However when using representation (86) this solution does not allow to satisfy all initial conditions since the vectors $\mathbf{k}$ and $\mathbf{p}$ have the preassigned directions.

Let us turn back to equation (98). A general analysis of this equation can be made by means of conventional methods. Because of this there is no need to do it in this paper.

# 7 The regular precession and the equations in variations

Let us consider the body with the transversally isotropic tensor of inertia. The elastic foundation is supposed to be transversally isotropic as well. The equations of motion are given by expressions (53), (54) and expression (47) for the elastic torque.

One can write down

$$\left[A_1\boldsymbol{\omega} + (A_3 - A_1)(\mathbf{k}'\cdot\boldsymbol{\omega})\mathbf{k}'\right]^{\cdot} + C\boldsymbol{\theta} + D\mathbf{Z}^{\mathsf{T}}\cdot\mathbf{k} = \mathbf{0}, \qquad \mathbf{k}' = \mathbf{P}\cdot\mathbf{k}, \tag{103}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta}\times\boldsymbol{\omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\omega}), \tag{104}$$

where the function $C$ and $D$ are defined by expressions (48).

A particular solution of system (103), (104) can be represented in the form

$$\boldsymbol{\theta} = \vartheta\mathbf{p}', \qquad \mathbf{p}' = \mathbf{Q}(\psi\mathbf{k})\cdot\mathbf{p}, \qquad \mathbf{P} = \mathbf{Q}(\vartheta\mathbf{p}'), \qquad \mathbf{p}\cdot\mathbf{k} = 0. \tag{105}$$

Motion (105) is called a regular precession if the restrictions

$$\vartheta = \mathrm{const}, \qquad \dot{\psi} = \mathrm{const} \tag{106}$$

hold good. The left angular velocity is defined in such case by the formula

$$\boldsymbol{\omega} = \mathbf{Q}(\psi\mathbf{k}) \cdot \boldsymbol{\omega}_0, \qquad \boldsymbol{\omega}_0 = \dot{\psi}\left[(1-\cos\vartheta)\,\mathbf{k} + \sin\vartheta\mathbf{k}\times\mathbf{p}\right] = \mathrm{const}. \tag{107}$$

We see that the vector $\boldsymbol{\omega}$ is a precession of the vector $\boldsymbol{\omega}_0$ around the axis $\mathbf{k}$. Also there are properties

$$\boldsymbol{\theta}\cdot\boldsymbol{\omega} = \boldsymbol{\theta}\cdot\boldsymbol{\Omega} = 0, \qquad \mathbf{k}\cdot\boldsymbol{\theta} = 0.$$

This means that the vector of turn is orthogonal to the vector of angular velocity. In addition let us accept the restriction

$$D\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right)|_{\mathbf{k}\cdot\boldsymbol{\theta}=0} = \frac{\partial}{\partial(\mathbf{k}\cdot\boldsymbol{\theta})} U\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right)|_{\mathbf{k}\cdot\boldsymbol{\theta}=0} = 0,$$

which is satisfied for the most kinds of elastic energy. After substitution (105)–(107) into equations (103), (104) we shall get the identities if the equality

$$\dot{\psi}^2 = \frac{C\left(\vartheta^2, 0\right)\vartheta}{\sin\vartheta\left[A_3\left(1-\cos\vartheta\right) + A_1\cos\vartheta\right]} \tag{108}$$

is valid. If $A_1 = A_3 = A$, then we have expression (101). Thus expressions (105)–(108) give to us the exact solution of system (103)–(104).

Now we must investigate a stability of solution (105)–(108). Generally it is rather cumbersome process. In order to simplify it let us accept

$$A = A_1 = A_3, \qquad D\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right) = 0, \qquad C\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right) = c = \mathrm{const}. \tag{109}$$

This means that the tensor of inertia and the elastic foundation are supposed to be isotropic. Under these assumptions perturbed equations of motion (103)–(104) take the form

$$A\dot{\boldsymbol{\omega}}_\varepsilon + c\boldsymbol{\theta}_\varepsilon = \mathbf{0},$$
$$\dot{\boldsymbol{\theta}}_\varepsilon = \boldsymbol{\omega}_\varepsilon - \frac{1}{2}\boldsymbol{\theta}_\varepsilon\times\boldsymbol{\omega}_\varepsilon + \frac{1-g_\varepsilon}{\theta_\varepsilon^2}\boldsymbol{\theta}_\varepsilon\times\left(\boldsymbol{\theta}_\varepsilon\times\boldsymbol{\omega}_\varepsilon\right). \tag{110}$$

The perturbed quantities $\boldsymbol{\omega}_\varepsilon$ and $\boldsymbol{\theta}_\varepsilon$ can be represented in the form

$$\boldsymbol{\omega}_\varepsilon = \boldsymbol{\omega} + \varepsilon\boldsymbol{\eta}, \qquad \boldsymbol{\theta}_\varepsilon = \boldsymbol{\theta} + \varepsilon\boldsymbol{\varphi}, \qquad |\varepsilon| \ll 1, \tag{111}$$

where $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ are defined by expressions (105)–(108). The quantities $\boldsymbol{\eta}$ and $\boldsymbol{\varphi}$ are called the first variations of $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ respectively. If we shall use representation (111), then we get the equations for $\boldsymbol{\eta}$ and $\boldsymbol{\varphi}$ with the varying coefficients. Because of this it will be better to represent the functions $\boldsymbol{\omega}_\varepsilon$ and $\boldsymbol{\theta}_\varepsilon$ in the next form

$$\boldsymbol{\omega}_\varepsilon = \mathbf{Q}(\psi\mathbf{k})\cdot\left(\boldsymbol{\omega}_0 + \varepsilon\boldsymbol{\eta}\right), \qquad \boldsymbol{\theta}_\varepsilon = \mathbf{Q}(\psi\mathbf{k})\cdot\left(\vartheta\mathbf{p} + \varepsilon\boldsymbol{\varphi}\right), \tag{112}$$

where the function $\psi$ is defined by (108).

It is easy to calculate

$$\dot{\boldsymbol{\omega}}_\varepsilon = \mathbf{Q}(\psi\mathbf{k}) \cdot \left[\dot{\psi}\mathbf{k} \times \boldsymbol{\omega}_0 + \varepsilon\left(\dot{\boldsymbol{\eta}} + \dot{\psi}\mathbf{k} \times \boldsymbol{\eta}\right)\right],$$

$$\dot{\boldsymbol{\theta}}_\varepsilon = \mathbf{Q}(\psi\mathbf{k}) \cdot \left[\dot{\psi}\vartheta\mathbf{k} \times \mathbf{p} + \varepsilon\left(\dot{\boldsymbol{\varphi}} + \dot{\psi}\mathbf{k} \times \boldsymbol{\varphi}\right)\right].$$

From equations (110) the next equations for variations $\boldsymbol{\eta}$ and $\boldsymbol{\varphi}$ can be derived

$$A\left(\dot{\boldsymbol{\eta}} + \dot{\psi}\mathbf{k} \times \boldsymbol{\eta}\right) + c\boldsymbol{\varphi} = 0,$$

$$\dot{\boldsymbol{\varphi}} + \dot{\psi}\mathbf{k} \times \boldsymbol{\varphi} = \frac{\vartheta \sin\vartheta}{2(1-\cos\vartheta)}\boldsymbol{\eta} - \frac{\vartheta - \sin\vartheta}{2(1-\cos\vartheta)}(\mathbf{p}\cdot\boldsymbol{\varphi})\,\boldsymbol{\omega}_0 - \frac{1}{2}\boldsymbol{\varphi} \times \boldsymbol{\omega}_0 -$$
$$-\frac{1}{2}\vartheta\mathbf{p} \times \boldsymbol{\eta} + \frac{2(1-\cos\vartheta) - \vartheta\sin\vartheta}{2\vartheta(1-\cos\vartheta)}(\boldsymbol{\varphi}\cdot\boldsymbol{\omega}_0 + \vartheta\mathbf{p} \times \boldsymbol{\eta})\,\mathbf{p},$$

where $\dot{\psi}$ is determined by (108) and $\vartheta = \mathrm{const}$. This system of linear differential equations with constant coefficients can be investigated by conventional methods. Our aim was only to show the derivation of the equations in variations.

## Appendix 1. Elastic energy of foundation

In the section 3 there was given the definition of an elastic energy in terms of potential function $U(\boldsymbol{\theta})$. This function was interpreted as the elastic energy of foundation. However in the nonlinear theory of elasticity the concept of elastic energy has a uniquely determined meaning. Thus it is necessary to show that there is no contradiction between these two concepts.

The foundation is supposed to be an elastic body. The boundary of the foundation is the surface $S = S_1 \bigcup S_2 \bigcup S_3$. The part $S_1$ of the surface $S$ is fixed. The part $S_2$ is a free surface. The part $S_3$ is the interface between the foundation and the rigid body.

Let us write the equation of the energy balance for the system "foundation plus rigid body"

$$\dot{K} + \dot{U} = 0, \tag{113}$$

where $K$ is the kinetic energy of rigid body, since the foundation is supposed to be inertialess; $U$ is the total intrinsic energy, i.e. elastic energy or energy of deformation, of the elastic foundation, since the intrinsic energy of rigid body has a constant value. The right side of (113) is equal to zero because the power of external forces is absent.

Now let us write the equation of the energy balance for rigid body only. The external forces, acting on the body, are generating by the vector of stress acting on the part $S_3$ of the boundary. Thus one can write

$$\dot{K} = -\int \mathbf{N}(P) \cdot \boldsymbol{\tau}(P) \cdot \dot{\mathbf{R}}(P)\, dS(P), \qquad P \in S_3, \tag{114}$$

where $\mathbf{R}(P)$ is the vector of position of the point $P$ of the surface $S_3$; the integration is going over the surface $S_3$; the vector $\mathbf{N}$ is the external unit normal to the surface $S_3$; the tensor $\boldsymbol{\tau}$ is the Cauchy stress tensor.

In according with the basic theorem of kinematics of rigid body we have

$$\mathbf{R}(P) = \mathbf{R}(Q) + \mathbf{P}(t) \cdot (\mathbf{r}(P) - \mathbf{r}(Q)), \tag{115}$$

where Q is the pole, $\mathbf{r}(P)$ and $\mathbf{r}(Q)$ are the vectors of position of points P and Q in the reference position. From equation (115) it follows

$$\mathbf{v}(P) = \mathbf{v}(Q) + \boldsymbol{\omega}(t) \times [\mathbf{R}(P) - \mathbf{R}(Q)]. \tag{116}$$

Substituting expression (116) into equation (114) one can get

$$\dot{K} = \mathbf{F} \cdot \mathbf{v}(Q) + \mathbf{M}_e \cdot \boldsymbol{\omega}, \tag{117}$$

where

$$\mathbf{F} = -\int \mathbf{N}(P) \cdot \boldsymbol{\tau}(P)\, dS(P),$$

$$\mathbf{M}_e = -\int [\mathbf{R}(P) - \mathbf{R}(Q)] \times \boldsymbol{\tau}(P) \cdot \mathbf{N}(P)\, dS(P).$$

Making use of (113) equation (117) can be rewritten in the form

$$\mathbf{F} \cdot \mathbf{v}(Q) + \mathbf{M}_e \cdot \boldsymbol{\omega} = -\dot{U}(\mathbf{R}(Q), \boldsymbol{\theta}), \tag{118}$$

where the vector $\boldsymbol{\theta}$ is the vector of turn of the rigid body and henceforth of the surface $S_3$. If the point Q is fixed, then we have definition (21) or (46). Thus the potential $U$ in expression (46) is the elastic energy of foundation.

# Appendix 2. A derivation of the representation for the integrating tensor

Calculating the trace from the both sides of the Poisson equation (9) one can obtain

$$(\mathrm{tr}\mathbf{Q})^{\cdot} = \mathrm{tr}(\boldsymbol{\omega} \times \mathbf{Q}) = -2\frac{\sin\theta}{\theta}\boldsymbol{\theta} \cdot \boldsymbol{\omega}, \qquad \mathrm{tr}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}.$$

Taking into account the equality

$$\mathrm{tr}\mathbf{Q} = 1 + 2\cos\theta$$

from the previous equation it is easy to derive

$$\theta\dot{\theta} = \boldsymbol{\theta} \cdot \dot{\boldsymbol{\theta}} = \boldsymbol{\theta} \cdot \boldsymbol{\omega}. \tag{119}$$

Multiplying equation (9) by the vector $\boldsymbol{\theta}$ one can get

$$\dot{\mathbf{Q}} \cdot \boldsymbol{\theta} = \boldsymbol{\omega} \times \boldsymbol{\theta} = -\mathbf{R} \cdot \boldsymbol{\omega}$$

Making use the identity

$$\dot{\mathbf{Q}} \cdot \boldsymbol{\theta} = (\mathbf{Q} \cdot \boldsymbol{\theta})^{\cdot} - \mathbf{Q} \cdot \dot{\boldsymbol{\theta}} = -(\mathbf{Q} - \mathbf{E}) \cdot \dot{\boldsymbol{\theta}}$$

and equation (3) the previous equation can be rewritten in the form

$$\left(\frac{\sin\theta}{\theta}\mathbf{R} + \frac{1-\cos\theta}{\theta^2}\mathbf{R}^2\right)\cdot\dot{\boldsymbol{\theta}} = \mathbf{R}\cdot\boldsymbol{\omega}.$$

A general solution of this equation has the form

$$\boldsymbol{\omega} = \lambda\boldsymbol{\theta} + \left(\frac{\sin\theta}{\theta}\mathbf{E} + \frac{1-\cos\theta}{\theta^2}\mathbf{R}\right)\cdot\dot{\boldsymbol{\theta}}, \tag{120}$$

where the scalar function $\lambda$ must be found.

Multiplying equation (120) by the vector $\boldsymbol{\theta}$ and taking into account equality (119) we have

$$\lambda = \frac{\theta - \sin\theta}{\theta^3}\boldsymbol{\theta}\cdot\dot{\boldsymbol{\theta}}.$$

Equation (120) takes the form

$$\boldsymbol{\omega} = \left[\mathbf{E} + \frac{1-\cos\theta}{\theta^2}\mathbf{R} + \frac{\theta-\sin\theta}{\theta^3}\mathbf{R}^2\right]\cdot\dot{\boldsymbol{\theta}} = \mathbf{Z}^{-1}\cdot\dot{\boldsymbol{\theta}}. \tag{121}$$

Here we use the identity

$$\mathbf{R}^2 = \boldsymbol{\theta}\otimes\boldsymbol{\theta} - \theta^2\mathbf{E}.$$

Expression (121) gives to us representation (20). Thus we had found the tensor $\mathbf{Z}^{-1}$. In order to calculate the tensor $\mathbf{Z}$ we must take into account that the tensor $\mathbf{Z}$ is the isotropic tensor function of the tensor $\mathbf{R}$. This means that the next representation is valid

$$\mathbf{Z} = \alpha\mathbf{E} + \beta\mathbf{R} + \gamma\mathbf{R}^2, \qquad \mathbf{Z}\cdot\mathbf{Z}^{-1} = \mathbf{E}.$$

From this it follows

$$\alpha = 1, \qquad \beta = -\frac{1}{2}, \qquad \gamma = \frac{1-g}{\theta^2}, \qquad g = \frac{\theta\sin\theta}{2(1-\cos\theta)}.$$

That is expression (11).

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