# On the Painleve Paradoxes*

## 1 Introduction

The friction is one of the most widespread phenomena in a Nature. The manifestations of friction are rather diverse. The laws, with which the friction in concrete situations is described, are diverse as well. Most popular in the applications are two laws of friction: the linear law of viscous friction and so-called dry friction. The viscous friction is well investigated, and its manifestations are clear and are easily predicted. It cannot be said about the laws of dry friction, though they are studied and are used in practice already more than two hundred years. Note that the friction, arising at sliding of one rigid body on another at absence of greasing, is called the dry friction. The relative sliding of bodies in contact, as a rule, is accompanied by occurrence of forces of friction, which render significant influence on dynamic processes in different sorts technical devices. Coulomb carried out the first researches of the dry friction in the end of XVIII century. The schematic of the Coulomb experiment is submitted in a Fig.1.

In 1791 Coulomb has published the first formulation of the law of dry friction in the following simple form.

$$F_{fr} = -\mu N \operatorname{sign} \dot{x}, \quad \text{if} \quad \dot{x} \neq 0, \tag{A}$$

The external simplicity of this law rather deceptive. As a matter of fact the Coulomb law of friction is the most complicated constitutive equation in mechanics. This may be seen, for example, from the fact that up to now the general mathematical statement of the Coulomb law of friction is absent in literature. The formulation (A) is only small part of general statement. In experiments by Coulomb the force of squeezing $N$ of bodies was set and was known. However, this force is not known in the most of nontrivial problems and must be found in the process of a solution of the considered task. In some cases, the function $N(t)$ can have complex view and depends on many physical features of the task under consideration. Factor of friction $\mu$ is accepted to be the characteristic of bodies in contact. Now factors of friction for various pairs of bodies are resulted in the data books. In the simple situations the Coulomb law allows completely to solve the put task. During about one century it was considered, that the Coulomb law does not comprise any ambiguities from the theoretical point of view. At the same time,

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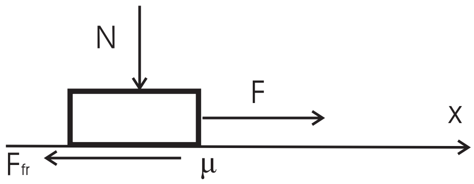


Figure 1: The Coulomb experiment

the rough development of machine-tool construction in second half of XIX century has revealed many cases, in which, on the first sight, the application of the Coulomb law leads to some contradictions. The special anxiety was caused by strange vibrations of machine tools (in some decades they were investigated and have received the name of frictional self-oscillations), processing, sharply lowering accuracy, of let out products. Sometimes the character of the movements arising in certain conditions was very strange, almost saltatory. Now such saltatory movements became object of intensive researches — see [1]. These circumstances, and also the theoretical needs, have forced the researchers again to address to the Coulomb law of dry friction. In 1895 Painleve has published the controversial book [2]. In what follows we shall cite the book [3], which contain other important works [4]–[14] on the subject. In [12] the opinion was expressed, that the Coulomb law is incompatible to the basic principles of the mechanics. Analyzing numerous examples of application of the Coulomb law in tasks of dynamics of systems with friction, Painleve comes to completely unexpected conclusion: "... While the marked special conditions are carried out, law by Coulomb is in the contradiction with dynamics of rigid bodies" [12], (see [3], p. 246) and further "... Between dynamics of a rigid body and the Coulomb law there is a logic contradiction under conditions, which can be carried out in the reality" [3], p. 248. As the logic a contradiction Painleve names situations, when the solution of the basic task of dynamics either does not exist, or is not unique. In modern literature these contradiction are known as the Painleve paradoxes. Now conclusions by Painleve even if they would be completely correct already anybody would not surprise. In continuum mechanics there is a chapter devoted to the theory of the constitutive equations, where the basic problem is the statement of conditions, at which those or other constitutive equations lead to the correctly put tasks. The Coulomb law is the typical constitutive equation, which, basically, can appear unacceptable. The merit of Painleve consists that he was the first who has pointed out at this central problem in mechanics. The Painleve results have called forth long discussion, in which such scientists as L. Prandtl, F. Klein, R. Von Mises, G. Hamel, L. Lecornu, de Sparre, F. Pfeifer and, of course, P. Painleve have taken part. The opinions of the participants of discussion were separated. L. Lecornu [7, 8], in essence, having recognized presence of paradoxes, offers to refuse from the model of rigid body. F. Klein [6] has come to a conclusion: "The Coulomb law of friction is not in the contradiction neither with principles of mechanics, nor with the phenomena observable in a nature: they need only correctly to be interpreted". An originality of results by F. Klein is caused by that he for the first time in tasks of a considered type used "hypothesis" of the instant stopping.

In this occasion the discussion has found new features, and at its center there was a hypothesis of F. Klein, which F. Klein did not consider as a hypothesis, but also has not deduced it on a level of a fact in evidence. R. Von Mises [9] concerning a hypothesis of F. Klein has expressed so: "1. F. Klein explains the phenomenon not from the point of view of the Coulomb law, but using a new rule obtained from experience. 2. This new skilled rule can be represented in the form of some modification of the Coulomb law". Further R. Von Mises results rather interesting reasons and gives the formulation adding the Coulomb law and allowing to combine sights of Painleve and Klein. Nevertheless, final conclusion by R. Von Mises is those: "Thus, not logic, but the methodology of the Newton mechanics compels us to refuse from the Coulomb law". G. Hamel [5] has joined the point of view by L. Lecornu about failure of the rigid body model. L. Prandtl [14] has expressed rather definitely: "In the statements of Mises and Hamel the speech goes about" to a hypothesis "of instant stopping. As opposed to this I emphasize, that in this case it is possible to speak only about result obtained through limiting transition. The research of elastic systems shows, generally speaking, something greater: it may be established, that from two possible movements, which the conventional theory gives for positive pulses, one, namely, accelerated motion will be steady, and another, slowed down, will be, on the contrary, unstable. In a limit we obtain the indefinitely large instability. So it is quietly possible to tell, that this second movement is practically impossible. From this it follows, that it is impossible by no means to expose of logic doubts against the Coulomb law". Under the Prandtl offer, F. Pfeifer made the large research [13]. However, the clear confirmation of such point of view was not carried out. Thus, in discussion the Painleve position has not found a convincing refutation, as was marked in three notes by Painleve [10, 11, 12] during the discussion. Even those authors, which disagree with the Painleve position, have not specified in which items of the Painleve reasoning is mistaken, and, hence, the position of Painleve remains not challenged. There was an opinion, which P. Appell [15], p. 117, has expressed in the following words: "it is not necessary to think, that only in exclusive cases there can be possible such difficulties. On the contrary, they arise in the most common cases, at least, at enough large value of factor of friction $\mu$. Because of this new experiments for a finding of the laws of friction, which is not resulting more in these difficulties, are necessary". Nevertheless, some ways of an exit from paradoxical situations were shown. The basic way of an exit is refusal of the rigid body model. Other way is application if necessary "hypotheses" of the instant stopping. However, its substantiation remained behind frameworks of the carried out researches. For decades, past from time of end of discussion, the interest to the Painleve paradoxes that faded, again grew. N.V. Butenin [16] showed fruitfulness of the Klein hypothesis in the large work. The significant development of ideas connected to partial refusal of the rigid body model was made in works of Le Suan Anh [17], in which the references to many other works can be found.

From told follows that it is necessary, firstly, to show features of the Coulomb law of friction, not complicated by any other circumstances, and, secondly, it is necessary to consider those conditions, which were investigated by Painleve. Only after that it will be possible either to recognize a position by Painleve, or to reject it partially or completely. It is well known that the tasks with the Coulomb friction have the not unique solution even in the elementary cases. F. Klein marked the importance of this fact for the first time. Namely, F. Klein has found out the existence of discontinues solutions, which should be taken into account for avoidance of the Painleve paradoxes. However majority of the scientists have not accepted the result of F. Klein. It is easy to understand the main reason of this. In the offer by Klein we deal with instant stopping of a body of nonzero weight. It is well known, that in such a case it is necessary to apply the infinitely large force, what is impossible in a reality. In works [8, 17] the physical sense of the discontinues solutions was shown and is specified as to choose the necessary solution from two possible ones. Nevertheless, as it became clear from the subsequent discussions, there is a necessity to consider the solution by F. Klein more carefully.

In given paper the authors are going to show the following. The authors agree that the laws of dry friction, similarly to all experimentally established laws, require the further researches and specifications. It is necessary, for example, if we wish to construct the satisfactory theory of frictional auto vibrations. At the same time, the authors resolutely object to the established opinion that the law of friction by Coulomb is the reason of certain paradoxical results contradicting to the experimental facts or common sense. If to consider cases, known in the literatures under the name of the Painleve paradoxes, then it is easy to see that all of them concern to dynamic tasks for systems of rigid bodies. It is well known that these tasks very frequently appear incorrectly put, though the law of friction by Coulomb in them can not be applied. Nonuniqueness or nonexisting of the solution are typical manifestations of the incorrectly put tasks. If we want to work with rigid bodies, we should be ready that the not unique solutions can appear which, in addition, can be non-smooth. The question, hence, consists not in getting rid of them, but in giving them correct interpretation. The significant part of given paper is devoted to this. Let's note, that in tasks of dynamics of systems with the Coulomb friction frequently shows features, characteristic for dynamics of systems at shock loading. Sometimes this shock loading appears larvae. Let's show told on an example of a task shown in a Fig. 1. We assume, that the body moved at $t < 0$ with constant speed. At the moment of time $t = 0$ all active forces stop the action, and the body goes on inertia. Actually at t=0 occurs shock loading of a body by force of friction. Really, at $t < 0$ on a body any forces did not act, as the active force was counterbalanced by force of friction. When the active force has disappeared, the shock loading of a body by force of friction has taken place. In other words, the collision of rigid bodies has taken place at absence of seen attributes of impact.

## 2 The Coulomb Law of Friction

The conventional formulation of the Coulomb law of dry friction in textbooks has a form

$$F_{fr} = -\mu N \operatorname{sign} \dot{x}, \quad \text{if} \quad \dot{x} \neq 0, \tag{1}$$

where the notation of the Fig.1 are used. Let us consider the task shown on the Fig.2 Making use Eq.(1) one may write the next equation of motion

$$m\ddot{x} + \mu m g \operatorname{sign}\left(\dot{x} - x_0 \omega \cos \omega t\right) = 0. \tag{2}$$

Initial conditions have a form

$$t = 0: \quad x = 0, \quad \dot{x} = 0. \tag{3}$$

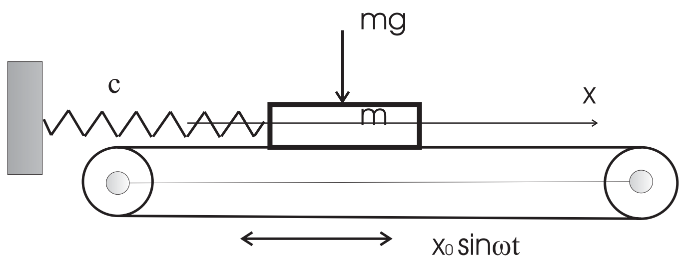


Figure 2: The body on vibro-transveyer

The Cauchy problem (2)–(3) may be solved, but its solution will not correspond to the real motion of the mass $m$. The reason is that the equality (1) expresses only part of the Coulomb low of dry friction. The general statement of this law can be represented in the form

$$F_{fr} = \begin{cases} -\mu N \operatorname{sign} \dot{x}, & \text{if} \quad \dot{x} \neq 0, \\ f_{st}, \ |f_{st}| \leqslant \mu N & \text{if} \quad \dot{x} = 0, \end{cases} \tag{4}$$

where $f_{st}$ must be determined from the static equation. More exact expression of the Coulomb law of friction is given by representation

$$F_{fr} = \begin{cases} -\mu N \operatorname{sign} \dot{x}, & \text{if} \quad \tau^2 \ddot{x}^2 + \dot{x}^2 \neq 0, \\ f_{st}, \ |f_{st}| \leqslant \mu N & \text{if} \quad \tau^2 \ddot{x}^2 + \dot{x}^2 = 0, \end{cases} \tag{5}$$

where $\tau$ is the time-like parameter. One has to remember that the force of squeezing $N$ must be nonnegative $N \geqslant 0$. If we have the two-sided contact then equality (5) must be replaced by the expression

$$F_{fr} = \begin{cases} -(\mu_1 N_1 + \mu_2 N_2) \operatorname{sign} \dot{x}, & \text{if} \quad \tau^2 \ddot{x}^2 + \dot{x}^2 \neq 0, \\ f_{st}, \ |f_{st}| \leqslant \mu_1 N_1 + \mu_2 N_2, & \text{if} \quad \tau^2 \ddot{x}^2 + \dot{x}^2 = 0, \end{cases}, \quad \begin{array}{l} N_1 \geqslant 0, \\ N_2 \geqslant 0, \end{array} \tag{6}$$

where $\mu_1, \ \mu_2$ are the factors of friction of downside and upside of the contact respectively, $N_1, \ N_2$ are the forces of squeezing on downside and upside of contact respectively, sometimes it is necessary to accept $N_1 N_2 = 0$.

Thus for the task shown on Fig.2 we have the next Cauchy problem

$$\begin{array}{r} m\ddot{x} - F_{fr} = 0, \\ y = x - x_0 \sin \omega t, \end{array} \qquad F_{fr} = \begin{cases} -\mu m g \ \operatorname{sign} \dot{y}, & \text{if} \quad \tau^2 \ddot{y}^2 + \dot{y}^2 \neq 0, \\ f_{st}, \ |f_{st}| \leqslant \mu m g & \text{if} \quad \tau^2 \ddot{y}^2 + \dot{y}^2 = 0. \end{cases} \tag{7}$$

To this system initial conditions (3) must be added. The main difficulty of the problem investigation is that it is necessary to look for nonsmooth solutions of (7), (3). For example, the function $\dot{y}(t)$ may be discontinuus. In order to see this fact more clearly let us consider the simple task shown on Fig.1 at $F = 0$. In this case we have the equation

$$m\ddot{x} - F_{fr} = 0 \tag{8}$$

and initial conditions

$$t = 0: \qquad x = 0, \quad \dot{x} = \nu. \tag{9}$$

If the friction force $F_{fr}$ is determined by expression (1), then we have the unique solution

$$\dot{x} = \begin{cases} \nu - \mu Nt/m, & 0 < t < \tau_{cl} \triangleq m\nu/\mu N, \\ 0, & \tau_{cl} \leqslant t. \end{cases} \tag{10}$$

This is the classical solution. If the friction force $F_{fr}$ is defined by (4) or (5), then we have two solutions

$$a)\ \dot{x}_1 = \begin{cases} \nu - \mu Nt/m, & 0 < t < \tau_{cl}, \\ 0, & \tau_{cl} \leqslant t. \end{cases}, \quad b)\ \dot{x}_2 = \begin{cases} \nu, & t = 0, \\ 0 & t > 0 \end{cases} \tag{11}$$

The second solution in (11) is an exact solution of the task (8), (9) and (5). However, it is discontinuus solution. Because of this it was ignored by the most of researches. F. Klein was the first who had pointed out the importance of the discontinuus solution in order to avoid the Painleve paradoxes [6]. N.V. Butenin [16] had used this discontinuus solution in order to solve the number of tasks. The physical meaning of discontinuus solution was shown in [18, 19]. The new reinterpret of solutions 11 will be given below in section 6.

The given above formulations of the Coulomb law of friction are not sufficient in order to apply them formally in any cases. One can say that the correct application of this law requires a thorough insight into the details of considered problem. Without this it is impossible to avoid all difficulties only by means of new experiments or new theoretical considerations.

## 3 The Painleve-Klein Problem. Conventional Approach

Let us consider the task that was studied by P. Painleve [2] and after that was discussed by F. Klein [6]. The task is shown on Fig.3. Namely in this problem P. Painleve had found at the first time the paradoxical situations. Let us show the way of reasoning by Painleve and Klein. Below the improved and enlarged analysis by Painleve and Klein is given, but the basic results are practically the same.

The equation of motion can be represented in the form

$$M_1\ddot{x} = R + P_1 + S\cos\alpha, \quad M_2\ddot{x} = P_2 - S\cos\alpha, \quad 0 < \alpha < \pi/2 \tag{12}$$

where $R$ is the friction force, $S$ is a longitudinal force in the rod. In this task we have to use the Coulomb law in the form (6), where

$$|N| = |S\sin\alpha| = |S|\sin\alpha = S\varepsilon_2\sin\alpha, \quad \varepsilon_2 = \operatorname{sign} S \tag{13}$$

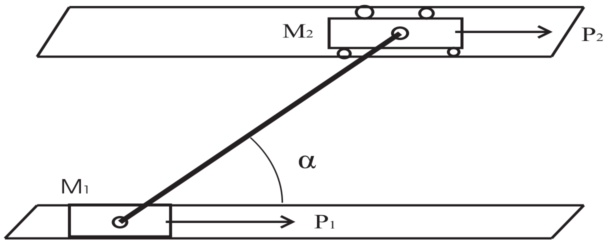


Figure 3: The Painleve-Klein Problem

Thus the friction force R is defined as

$$R = \begin{cases} -\mu S \varepsilon_1 \varepsilon_2 \sin\alpha, & \text{if} \quad \dot{x} \neq 0, \\ f, \ |f| \leqslant \mu |S| \sin\alpha, & \text{if} \quad \dot{x} = 0, \quad \ddot{x} = 0 \end{cases} \tag{14}$$

where $\varepsilon_1 = \operatorname{sign}\dot{x}$. The initial conditions have a form

$$t = 0: \qquad x = 0, \quad \dot{x} = v, \tag{15}$$

where $v$ is an initial velocity.

Let us suppose that at $t > 0$ the masses $M_1$ and $M_2$ are not moving: $\dot{x} = 0$. Then instead of equations of motion (12) we have the equation of statics

$$f + P_1 + S\cos\alpha = 0, \quad P_2 - S\cos\alpha = 0. \tag{16}$$

From (16) it follows

$$f = -(P_1 + P_2), \qquad S = P_2/\cos\alpha, \quad |P_1 + P_2| \leqslant \mu |P_2| \tan\alpha \tag{17}$$

The last inequality determines the domain on the plane $(P_1, P_2)$ where the statical solution exists. Thus from the theoretical point of view the static solution exists always, when inequality (17) holds. If $P_2 = 0$, then the system can be at rest only when $P_1 = 0$. But the initial velocity may be different from zero! If $P_1 = 0$, then the statical solution is possible at $v \neq 0$ only if $\mu \tan\alpha \geqslant 1$.

Let us suppose that at $t > 0$ the system is moving with $\dot{x} = \text{const} \neq 0$, $\ddot{x} = 0$. Then we have

$$-\mu S \varepsilon_1 \varepsilon_2 \sin\alpha + P_1 + S\cos\alpha = 0, \quad P_2 - S\cos\alpha = 0, \tag{18}$$

This system has solution if and only if

$$\mu \tan\alpha = 2. \tag{19}$$

The solution has a form

$$P_1 = P_2, \qquad \operatorname{sign} P_1 = \operatorname{sign} v.$$

If $\mu \tan\alpha \neq 2$, then the case $\dot{x} = \text{const} \neq 0$ is impossible.

Let us suppose that at $t > 0$ the masses $M_1$ and $M_2$ are moving: $\dot{x} \neq \text{const}$, $\ddot{x} \neq 0$. Then the force $S$ is determined by the expression

$$S = \frac{\gamma P_2 - P_1}{(1+\gamma)\cos\alpha - \mu\varepsilon_1\varepsilon_2\sin\alpha}, \qquad \gamma = \frac{M_1}{M_2}. \tag{20}$$

In the Painleve-Klein analysis the restriction

$$\gamma = 1, \qquad P_2 = 0 \tag{21}$$

were accepted. Multiplying expression (20) by $\varepsilon_2$ and taking into account the equality $|S| = S\varepsilon_2$ we obtain

$$\frac{\varepsilon_2\,(\gamma P_2 - P_1)}{(1+\gamma)\cos\alpha - \mu\varepsilon_1\varepsilon_2\sin\alpha} > 0. \tag{22}$$

For small $\mu > 0$ we have

$$(1+\gamma)\cos\alpha - \mu\sin\alpha > 0, \quad \Rightarrow \quad \mu\tan\alpha < 1+\gamma \tag{23}$$

Then from (22) it follows

$$\varepsilon_2 = \operatorname{sign}(\gamma P_2 - P_1). \tag{24}$$

Thus for small $\mu$ we have two solutions: one is given by (17) and another is determined by

$$S = \frac{\gamma P_2 - P_1}{d}, \qquad \begin{array}{r} R = -\mu\,\operatorname{sign}\nu\,|S|\sin\alpha, \\ d = (1+\gamma)\cos\alpha - \mu\,\operatorname{sign}\nu\,\operatorname{sign}(\gamma P_2 - P_1)\sin\alpha. \end{array} \tag{25}$$

This case was not considered by Painleve, since from the Painleve point of view in this case there is no problem. As we see, it is not so. At the moment we have no reasons in order to choose one of two possible solution. However, the Coulomb law of friction is not responsible for such a situation. In fact, our model is not adequate to the reality in many important aspects. In the next section we show why it is important and how to solve the problem of choice.

Let us suppose that the inequality

$$(1+\gamma)\cos\alpha - \mu\sin\alpha < 0 \tag{26}$$

is valid. This inequlity determines a domain of paradoxes accordingly Painleve. Here we have to consider the different cases.

1. The case when $\gamma P_2 - P_1 = 0$.

In such a case the friction is absent and there is nothing to discuss.

2. The case when $\varepsilon_1 = \operatorname{sign}\dot{x} = \operatorname{sign}\nu = 1$ and $\gamma P_2 - P_1 < 0$.

Then the inequality (22) may be rewritten as

$$\frac{\varepsilon_2\,(\gamma P_2 - P_1)}{(1+\gamma)\cos\alpha - \mu\varepsilon_2\sin\alpha} > 0. \tag{27}$$

In such a case we have two different solutions

$$S = \frac{\gamma P_2 - P_1}{(1+\gamma)\cos\alpha + \mu\sin\alpha}, \qquad \varepsilon_2 = -1 \tag{28}$$

and

$$S = \frac{\gamma P_2 - P_1}{(1+\gamma)\cos\alpha - \mu\sin\alpha}, \qquad \varepsilon_2 = +1. \tag{29}$$

Thus in this case we have three different solutions: (17), (28) and (29).

3. The case when $\varepsilon_1 = 1, \quad \gamma P_2 - P_1 > 0$.

In such a case inequality (27) is not valid for any value of $\varepsilon_2$. Thus in this case we have the unique solution (17). The system is instantly stopping.

4. The case when $\varepsilon_1 = -1, \quad \gamma P_2 - P_1 < 0$.

Then inequality (22) takes a form

$$\frac{\varepsilon_2 (\gamma P_2 - P_1)}{(1+\gamma)\cos\alpha + \mu\varepsilon_2\sin\alpha} > 0. \tag{30}$$

There is no value of $\varepsilon_2$ to satisfy this inequality. We have the unique solution (17).

5. The case when $\varepsilon_1 = -1, \quad \gamma P_2 - P_1 > 0$. In this case we have two solutions for $S$

$$S = \frac{\gamma P_2 - P_1}{(1+\gamma)\cos\alpha + \mu\sin\alpha}, \qquad \varepsilon_2 = +1 \tag{31}$$

and

$$S = \frac{\gamma P_2 - P_1}{(1+\gamma)\cos\alpha - \mu\sin\alpha}, \qquad \varepsilon_2 = -1 \tag{32}$$

Again we have three solutions (17), (31), (32).

Painleve presumed that the Coulomb law of friction is responsible for such unsatisfactory situation. Much later the Painleve analysis was confirmed by P. Appell [15]: "it is not necessary to think, that only in exclusive cases there can be possible such difficulties. On the contrary, they arise in the most common cases, at least, at enough large value of factor of friction $\mu$. Because of this new experiments for a finding of the laws of friction, which is not resulting more in these difficulties, are necessary". However, L. Prandtl [14] and F. Klein [6] were not agree with the conclusions by Painleve. F. Klein had pointed out that if $\nu < 0$, then there is a unique solution $\dot{x} = 0$. In order to avoid contradictions in the case $\nu > 0$, F. Klein offer to accept the statical solution $\dot{x} = 0$ as well. But F. Klein did not explain why we have to do this. The Painleve considerations are supposed to be right even in modern books — see, for example, [17].

## 4 The Painleve-Klein problem. Alternative approach

It is not difficult to see that the Painleve analysis, shown in previous section, is needed in some additions and improvements, especially from the physical point of view, since the task shown on Fig.3 is undefined in some important aspects. There is useful heuristic principle in mechanics: if one accepts some assumptions, then it is necessary to show the model in which these assumptions may be exactly realised. Only in such a case one can be shure that accepted assuptions have physical meaning. In the Painleve-Klein problem the model is described by equations (12) and (14). From the pure mathematical point of

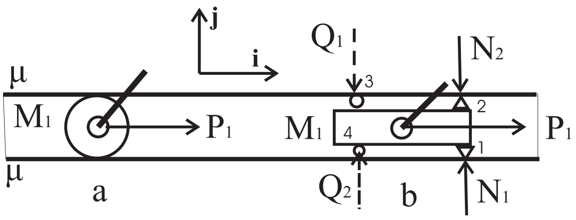


Figure 4: The structure of the downslipper

view this model is certain mathematical object and we have to study its properties. What these properties are, we can not speak about paradoxes, for the mathematical object can have the most fantastic properties. From the physical point of view the situation varies. We speak about paradoxes when the results of the decision of this or that task contradict common sense. But in such a case the physical statement of the task itself should not contradict common sense. Let's discuss the model discribed by equations (12) and (14). Equations (12) show that there are no moments on the ends of rod. This means that the masses $M_1$ and $M_2$ may be rotated with respect to the rod fluently. It is easy to see that equations (12) and (14) correspond to the case shown on Fig.4a. Let us underline that the mass $M_1$ is touching either the upside or downside of the gap. From Fig.4a it is seen that in considered case the Coulomb law of friction can't be used since we have rolling motion of $M_1$ instead of the sliding, which is possible only if we exclude the turn of the body $M_1$ with respect to the rod. However, in such a case the force in the rod will not be a longitudinal force any more and equations (12) must be changed. Thus the statement of the problem considered in the previous section is physically meaningless and the Coulomb law of friction is not responsible for the paradoxes. More realistic structure of the slipper is shown on Fig.4b. The forces acting on the slipper $M_1$ from the foundation are shown on Fig.4b. Some other cases are shown on Fig.5a-e. The difference between cases Fig.5a and Fig 5b is that in the second case the slipper can't rotate with respect to the rod. Below we deduce the equations for the case on Fig.4b. Let us write down the equations of motion

$$M_1\ddot{x}\mathbf{i} = P_1\mathbf{i} + \mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_4 + S(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}), \tag{33}$$

$$M_2\ddot{x}\mathbf{i} = P_2\mathbf{i} - S(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}) + R_5\mathbf{j}, \tag{34}$$

where $R_5 j$, $R_k$, $(k = 1,2,3,4)$ are the reactions acting on the body $M_2$ and points $1,2,3,4$ of the slipper $M_1$ (see Fig.4b) respectively, $S$ is a longitudenal force in the rod, $S > 0$ when the rod is stretched. For the reactions $R_k$ we have

$$\mathbf{R}_1 = R_1\mathbf{i} + N_1\mathbf{j}, \quad \mathbf{R}_2 = R_2\mathbf{i} - N_2\mathbf{j}, \quad \mathbf{R}_3 = -Q_1\mathbf{j}, \quad \mathbf{R}_4 = Q_2\mathbf{j} \tag{35}$$

and the restrictions

$$N_1 \geqslant 0, \quad N_2 \geqslant 0, \quad N_1N_2 = 0, \quad Q_1 \geqslant 0, \quad Q_2 \geqslant 0, \quad Q_1Q_2 = 0 \tag{36}$$

are valid. Besides, the functions $R_1$, $R_2$ are defined by the Coulomb law of friction. From equations (33)–(35) it follows

$$M_1\ddot{x} = P_1 + R_1 + R_2 + S\cos\alpha, \qquad M_2\ddot{x} = P_2 - S\cos\alpha, \tag{37}$$

$$N_1 - N_2 - Q_1 + Q_2 + S\sin\alpha = 0, \qquad R_5 - S\sin\alpha = 0,$$
$$N_1N_2 = 0, \quad Q_1Q_2 = 0. \tag{38}$$

The Coulomb law of friction can be written in the form

$$R_1 = \begin{cases} -\mu N_1 \operatorname{sign}\dot{x}, & \text{if} \quad \dot{x} \neq 0,\ \ddot{x} \neq 0, \\ f_1,\ |f_1| \leqslant \mu N_1, & \text{if} \quad \dot{x} = 0,\ \ddot{x} = 0, \end{cases} \tag{39}$$

$$R_2 = \begin{cases} -\mu N_2 \operatorname{sign}\dot{x}, & \text{if} \quad \dot{x} \neq 0,\ \ddot{x} \neq 0, \\ f_2,\ |f_2| \leqslant \mu N_2, & \text{if} \quad \dot{x} = 0,\ \ddot{x} = 0. \end{cases} \tag{40}$$

We need to know the sum

$$R_1 + R_2 = \begin{cases} -\mu\,(N_1 + N_2) \operatorname{sign}\dot{x}, & \text{if} \quad \dot{x} \neq 0,\ \ddot{x} \neq 0, \\ f_1 + f_2,\ |f_1| \leqslant \mu N_1,\ |f_2| \leqslant \mu N_2, & \text{if} \quad \dot{x} = 0,\ \ddot{x} = 0, \end{cases} \tag{41}$$

where the restrictions (36) must be taking into account. Now we are able to compare the statements (12), (14) and (37), (38), (41). If $Q_1 = Q_2 = 0$, then both statements are the same. Let us pay attention that the Coulomb law of friction is applying in both statements in the same manner. If $Q_1 \neq 0$ or $Q_2 \neq 0$, then these statements are different very much. First of all, the system (36)–(41) with initial conditions

$$t = 0: \qquad x = 0, \quad \dot{x} = v \tag{42}$$

is incomplete one. We need one more equation. There exist different ways. The most reliable way is to take into account the elasticity of the gap walls. Strictly speaking in such a case we must not only add a new equation but replace the first equation of system (38) by the next equation

$$M_1\ddot{y} = N_1 - N_2 - Q_1 + Q_2 + S\sin\alpha, \tag{43}$$

where $y$ is vertical coordinate of the mass center of $M_1$. Let us suppose that the slipper $M_1$ can be rotated by the small angle $\varphi$. In such a case the vertical coordinates of the points $1, 2, 3, 4$ may be found as

$$y_1 = y + l_1\varphi, \quad y_2 = y + l_1\varphi, \quad y_3 = y - l_1\varphi, \quad y_4 = y - l_1\varphi. \tag{44}$$

For the reactions $N_1$, $N_2$, $Q_1$, $Q_2$ the next constitutive equations may be accepted

$$N_1 = -c\,[1 - \theta(y + l_1\varphi)]\,(y + l_1\varphi), \quad N_2 = c\theta(y + l_1\varphi)(y + l_1\varphi),$$

$$Q_1 = c\theta(y - l_1\varphi)(y - l_1\varphi), \quad Q_2 = -c\,[1 - \theta(y - l_1\varphi)]\,(y - l_1\varphi), \tag{45}$$

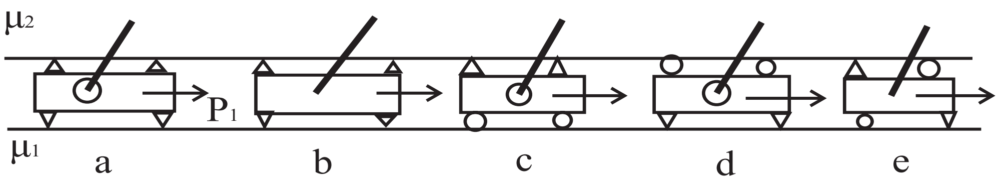


Figure 5: The slipper in the gap

where $l_1$ is a parameter of the length dimension, $c > 0$ is a stiffness of elastic foundation, and $\theta(z)$ is the characteristic function of the domain $z \geqslant 0$

$$\theta(z) = \left\{ \begin{array}{lll} 1, & \text{if} & z \geqslant 0, \\ 0, & \text{if} & z < 0. \end{array} \right. , \quad \begin{array}{l} \theta_+ \equiv \theta(y + l_1 \varphi), \\ \theta_- \equiv \theta(y - l_1 \varphi). \end{array} \tag{46}$$

The additional equation can be accepted in the next form

$$M_1 r^2 \ddot{\varphi} = l_2 \left( N_1 - N_2 + Q_1 - Q_2 \right) + l_3 (R_1 - R_2) - \varepsilon P_1, \tag{47}$$

where $l_2$, $l_3$, $\varepsilon$ are the parameters of the length dimension, $r$ is an inertia radius of the slipper. We obtain the closed system of equations (37), (39), (40), (43), (45), (47). To this system we have to add the initial conditions which can be taking, for example, in the next fom

$$t = 0: \qquad x = 0, \quad \dot{x} = v, \quad y = \dot{y} = \varphi = \dot{\varphi} = 0. \tag{48}$$

Only now we have the well-defined task from the physical point of view. The final statement can be represented as

$$M_1 \ddot{x} = P_1 + R_1 + R_2 + S \cos \alpha, \qquad M_2 \ddot{x} = P_2 - S \cos \alpha,$$

$$M_1 \ddot{y} + 2cy = S \sin \alpha, \quad M_1 r^2 \ddot{\varphi} + 2c l_2 l_1 \varphi = l_3 (R_1 - R_2) - \varepsilon P_1, \tag{49}$$

where

$$R_1 + R_2 = \left\{ \begin{array}{rll} -\mu c \, |y + l_1 \varphi| \operatorname{sign} \dot{x}, & \text{if} & \dot{x} \neq 0, \ \ddot{x} \neq 0, \\ f_1 + f_2, & \text{if} & \dot{x} = 0, \ \ddot{x} = 0, \end{array} \right. \tag{50}$$

where

$$|f_1| \leqslant \mu c (1 - \theta_+) \left( y + l_1 \varphi \right), \qquad |f_2| \leqslant \mu c \theta_+ (y + l_1 \varphi). \tag{51}$$

$$R_1 - R_2 = \left\{ \begin{array}{rll} \mu c (y + l_1 \varphi) \operatorname{sign} \dot{x}, & \text{if} & \dot{x} \neq 0, \ \ddot{x} \neq 0, \\ f_1 - f_2, & \text{if} & \dot{x} = 0, \ \ddot{x} = 0. \end{array} \right. \tag{52}$$

The Cauchy problem (49), (50), (52), (48) is a physically correct statement of the Painleve-Klein task. If we doubt in the Coulomb law of friction, then we have to show that the Cauchy problem is not well-defined. However, it is not so. Let us transform system (49)–(52). For this we accept the restriction $r^2 = l_1 l_2$. In such a case from (49)–(52) one can derive the equations

$$(M_1 + M_2) \ddot{x} = P_1 + P_2 + R_1 + R_2,$$

$$M_1\ddot{y} + 2cy = (\frac{\gamma}{1+\gamma}P_2 - \frac{1}{1+\gamma}P_1)\tan\alpha - \frac{\tan\alpha}{1+\gamma}(R_1 + R_2),$$

$$M_1\ddot{z} + 2cz = \frac{l_3}{l_2}(R_1 - R_2) - \frac{\varepsilon}{l_2}P_1, \quad z = y + l_1\varphi. \tag{53}$$

The force $S$ may be found from the equation

$$S = (P_2 - M_2\ddot{x})/\cos\alpha. \tag{54}$$

Let's note that the friction forces $R_1 + R_2$, $R_1 - R_2$ are expressed in terms of the variable $z$ by means of (50) and (52). Initial conditions for system (53) has a form

$$t = 0: \qquad x = 0, \quad \dot{x} = v, \quad y = z = \dot{y} = \dot{z} = 0. \tag{55}$$

For small $t > 0$ the system is moving. So making use (50) and (52) we can transform system (53) to the next form

$$(M_1 + M_2)\ddot{x} = -\mu\varepsilon_1 c\,|z| + P_1 + P_2, \quad \varepsilon_1 \equiv \operatorname{sign}\dot{x} = \operatorname{sign} v,$$

$$M_1\ddot{y} + 2cy = \mu\varepsilon_1 c\,|z|\,\frac{\tan\alpha}{1+\gamma} + (\frac{\gamma}{1+\gamma}P_2 - \frac{1}{1+\gamma}P_1)\tan\alpha,$$

$$M_1\ddot{z} + 2c(1 - \frac{\mu\varepsilon_1}{2}\frac{l_3}{l_2})z = -\frac{\varepsilon}{l_2}P_1, \tag{56}$$

The Cauchy problem (56)–(55) is well-defined and obviously has unique solution. Of course, we must have in mind that this problem has a meaning only when

$$\begin{array}{ll} 0 \leqslant \dot{x} \leqslant v, & \text{if} \quad \varepsilon_1 = +1, \\ v \leqslant \dot{x} \leqslant 0, & \text{if} \quad \varepsilon_1 = -1. \end{array} \tag{57}$$

We see that there are no problems when the Coulomb law of friction is using in the Painleve-Klein task. While we were forced to take into account an elasticity of the gap walls, nevertheless this was not connected with the law of friction but due to physical requirements only.

If we wish to use the rigid body model, then we have to make the passage to the limit $c \to \infty$. In such a case we obtain the rigid body model. If $c \to \infty$, then $z \to 0$ and $y \to 0$, but

$$\lim_{c\to\infty} cy = T \neq 0, \qquad \lim_{c\to\infty} cz = Z \neq 0. \tag{58}$$

Instead of the Cauchy problem (53) we shall get

$$(M_1 + M_2)\ddot{x} = P_1 + P_2 + R_1 + R_2, \quad 2Z = \frac{l_3}{l_2}(R_1 - R_2) - \frac{\varepsilon}{l_2}P_1,$$

$$2T = (\frac{\gamma}{1+\gamma}P_2 - \frac{1}{1+\gamma}P_1)\tan\alpha - \frac{\tan\alpha}{1+\gamma}(R_1 + R_2). \tag{59}$$

Equalities (50) and (52) take a form

$$R_1 + R_2 = \begin{cases} -\mu\,|Z|\operatorname{sign}\dot{x}, & \text{if} \quad \dot{x} \neq 0,\ \ddot{x} \neq 0, \\ f_1 + f_2, & \text{if} \quad \dot{x} = 0,\ \ddot{x} = 0, \end{cases}$$

$$R_1 - R_2 = \begin{cases} \mu Z \operatorname{sign} \dot{x}, & \text{if} \quad \dot{x} \neq 0,\ \ddot{x} \neq 0, \\ f_1 - f_2, & \text{if} \quad \dot{x} = 0,\ \ddot{x} = 0, \end{cases} \tag{60}$$

where

$$|f_1| \leqslant \mu(1 - \eta_1)\,|Z|\,, \qquad |f_2| \leqslant \mu\eta_1\,|Z|\,. \tag{61}$$

In (60) and (61) the next notation

$$\eta_1 = \lim_{z\to 0} \theta_+ = \begin{cases} 1, & \text{if} \quad z \to +0, \\ 0, & \text{if} \quad z \to -0, \end{cases}$$

$$\eta_2 = \lim_{y - l_1\varphi \to 0} \theta_- = \begin{cases} 1, & \text{if} \quad y - l_1\varphi \to +0, \\ 0, & \text{if} \quad y - l_1\varphi \to -0 \end{cases} \tag{62}$$

is used. As initial conditions we have to accept equalities (42).

Now we are able to compare the statements (12)–(15) and (59)–(62), (42). We see that they are quite different, while both of them are using the model of rigid body and standard form of the Coulomb law of friction. Let's consider the solution of system (59)–(62), (42). The Coulomb law has different forms for the motion and for the rest. So we have to consider these cases separately.

Let's suppose that $\dot{x} = 0,\ \ddot{x} = 0$. Then we get

$$P_1 + P_2 + f_1 + f_2 = 0, \quad 2T = (\frac{\gamma}{1+\gamma}P_2 - \frac{1}{1+\gamma}P_1)\tan\alpha - \frac{\tan\alpha}{1+\gamma}(f_1 + f_2),$$

$$2Z = \frac{l_3}{l_2}(f_1 - f_2) - \frac{\varepsilon}{l_2}P_1, \quad |f_1| \leqslant \mu(1 - \eta_1)\,|Z|\,, \qquad |f_2| \leqslant \mu\eta_1\,|Z|\,. \tag{63}$$

Now we have to consider two cases

$$a) \quad \eta_1 = 1 \quad \Rightarrow \quad f_1 = 0, \quad |f_2| \leqslant \mu\,|Z|\,, \quad Z \geqslant 0 \tag{64}$$

and

$$b) \quad \eta_1 = 0 \quad \Rightarrow \quad |f_1| \leqslant \mu\,|Z|\,, \quad f_2 = 0, \quad Z \leqslant 0 \tag{65}$$

In case (64) we have

$$f_2 = -(P_1 + P_2),\ 2T = P_2\tan\alpha,\ 2Z = \frac{l_3 - \varepsilon}{l_2}P_1 + \frac{l_3}{l_2}P_2 \geqslant 0, \tag{66}$$

$$|P_1 + P_2| \leqslant \frac{\mu}{2}\left|\frac{l_3 - \varepsilon}{l_2}P_1 + \frac{l_3}{l_2}P_2\right|. \tag{67}$$

Inequality (67) determines the domain of the static solution existence under $\eta_1 = 1$.

In case (65) we get

$$f_1 = -(P_1 + P_2),\ 2T = P_2\tan\alpha,\ 2Z = -\frac{l_3 + \varepsilon}{l_2}P_1 - \frac{l_3}{l_2}P_2 \leqslant 0, \tag{68}$$

$$|P_1 + P_2| \leqslant \frac{\mu}{2l_2}\,|(l_3 + \varepsilon)P_1 + l_3P_2|\,. \tag{69}$$

It is necessary to have in mind that the existence of two cases (64) and (65) does not mean that there is non-uniqueness of solutuion. These cases correspond to different physical conditions. Let $P_2$ be absent as in the Painleve-Klein problem. In such a case the statical solution exist only when $P_1 > 0$ and

$$a)\quad \frac{\mu}{2}\frac{l_3-\varepsilon}{l_2} \geqslant 1, \qquad b)\quad \frac{\mu}{2}\frac{l_3+\varepsilon}{l_2} \geqslant 1 \tag{70}$$

Let's the case when $\dot{x} = \nu = \text{const}$ and $\ddot{x} = 0$. System (59)–(61) takes a form

$$-\mu\varepsilon_1 |Z| + P_1 + P_2 = 0, \quad 2T = \mu\varepsilon_1 |Z| \frac{\tan\alpha}{1+\gamma} + (\frac{\gamma}{1+\gamma}P_2 - \frac{1}{1+\gamma}P_1)\tan\alpha,$$

$$(2 - \mu\varepsilon_1 \frac{l_3}{l_2})Z = -\frac{\varepsilon}{l_2}P_1. \tag{71}$$

This system has a solution only when

$$P_2 = \frac{\mu\varepsilon_1 |\varepsilon P_1|}{|2l_2 - \mu\varepsilon_1 l_3|} - P_1. \tag{72}$$

At last, let's consider the case $\ddot{x} \neq 0$. Equation (59) take a form

$$(M_1 + M_2)\ddot{x} = -\mu\varepsilon_1 |Z| + P_1 + P_2, \quad \varepsilon_1 \equiv \operatorname{sign}\dot{x} = \operatorname{sign}\nu,$$

$$2T = (\mu\varepsilon_1 |Z| + \gamma P_2 - P_1)\frac{\tan\alpha}{1+\gamma}, \quad (2 - \mu\varepsilon_1\frac{l_3}{l_2})Z = -\frac{\varepsilon}{l_2}P_1. \tag{73}$$

It is easy to see that this system has a unique solution in all cases and paradoxes of any kind are absent. This means that paradoxes shown in previous section are result of unsatisfactory statement of task but not due to the Coulomb law of friction. However, we have to underline that absence of paradoxes does not mean that the task (73) is good from physical point of view. It is not so. As a matter of fact when making a limit passage $c \to \infty$ we have lost a number of important properties of considered system. For example, the Cauchy problem has a meaning not only for initial condition (55). It is possible to put, for example, the next condition at $t = 0$

$$x = 0, \quad \dot{x} = \nu, \quad y = a \neq 0, \quad z = 0, \quad \dot{y} = \dot{z} = 0. \tag{74}$$

In this case the limit passage $c \to \infty$ leads to contradiction since

$$\lim_{c\to\infty} cy \to \infty,$$

what is natural from physical point of view. Besides, system (56) shows that if

$$\frac{\mu\varepsilon_1}{2}\frac{l_3}{l_2} > 1, \tag{75}$$

then we have almost instant shut-down of the system. We can't see it from equation (73). So from practical point of view it is much better to use equations (56).

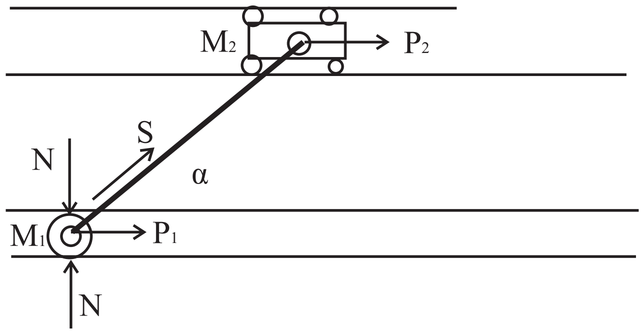


Figure 6: The modified task by Painleve

# 5 The modified Painleve-Klein problem

We saw that the classical task by Painleve, shown on Fig.3 and Fig.4a, has no physical meaning. Because of this we used the improved version shown on Fig.4b. There is another possibility to improve the statement by Painleve — see Fig.6.

# 6 The interpretation of instant shut-down of the body with finite mass

Let's turn back to the simplest task considered in section 2. Let's consider the body moving due to inertia along the rough surface — see Fig.1, where $F = 0$. We saw that in this task there are two solutions (11). At the first time an importance of this fact was marked by F. Klein. However the most of scientists do not accept the Klein result. It is easy to understand the main reason of this. In the Klein proposition we deal with the instantaneous stopping of a body with the finite mass. Everybody knows that in such a case the infinitely big force is needed what is impossible in reality. In paper [18] it was shown the physical sense of discontinuous solution and how to choose the necessary solution from two possible solutions. Nevertheless, as it is became clear from discussions with other peoples, there exists the necessity to consider the Klein proposition more carefully.

Let us consider the body moving due to an inertia along the rough surface. A friction is determined by the Coulomb law. Using notation shown in Fig.1 we can write the next system of equations

$$m\ddot{y} = F, \quad F = \begin{cases} -\mu m g \dot{x}/|\dot{x}|, & \text{if } \dot{x} \neq 0 \\ f_{st}, \ |f_{st}| \leqslant \mu m g, & \text{if } \dot{x} = 0 \end{cases} \tag{76}$$

where $y$ is a position of the mass center $C$, $x$ is a position of a point on the contact surface, $m$ is the body mass, $g$ is the gravity acceleration. Initial conditions can be chosen in the form

$$t = 0: \quad x = y = 0, \quad \dot{x} = \dot{y} = v \tag{77}$$

System (76) is theoretically exact. However in order to solve system (76)–(77) we must accept additional conventions that can be taken in different forms. The conventional way is to accept the model of rigid body. In such a case we have that $z = x = y$ and instead of system (76) we shall get the system

$$m\ddot{z} = F, \qquad F = \begin{cases} -\mu m g \dot{z}/|\dot{z}|, & \text{if } \dot{z} \neq 0, \\ f_{st},\ |f_{st}| \leqslant \mu m g, & \text{if } \dot{z} = 0, \end{cases} \tag{78}$$

that can be solved without any difficulties. It is easy to see that problem (77)–(78) has two solutions

$$a)\quad \dot{z}_1 = \begin{cases} \nu - \mu g t, & \text{if } t < \nu/\mu g, \\ 0, & \text{if } t > \nu/\mu g \end{cases} \quad \text{and} \quad b)\quad \dot{z}_2 = \begin{cases} \nu, & \text{if } t = 0, \\ 0, & \text{if } t > 0. \end{cases} \tag{79}$$

In the first of these solutions the acceleration has the discontinuity of the finite magnitude. In the second solution the acceleration has the discontinuity of the infinite magnitude that is considered to be impossible for the real body.

Now we have arrived to the point of discordance of opinions. First of all, we must understand the meaning of the function $z(t)$ in system (78) or in solutions (79). If $z(t)$ is a position of the mass center, then $z(t)$ is certainly determined by the first solution from expressions (79) whereas the second solution has no physical sense. Thus if we are able to prove that the function $z(t)$ in problem (77)–(78) has the only sense of the position of the mass center, then the Klein solution $b)$ from (6.4) must be eliminated. Is it possible to prove this presumption? Note that problem (77)–(78) considered from the mathematical point of view does not know our conceptualization of the meaning of function $z(t)$. Actually, in order to obtain system (78) from system (76) only the relation $x = y$ is important. It does not matter what kind of word explanations we shall use. This means that it is impossible to find the meaning of $z(t)$ in system (78) from formal considerations without additional investigations. From the physical point of view it is clear that the interpretations of functions $z(t)$ for solutions $a)$ and $b)$ in (79) must be different. For example, if $z(t)$ is the position of contact surface, then solution $b)$ in (79) can be realized in reality since the contact surface has no mass while the center of mass may keep its motion. Anyhow, in order to find the detailed answer we must investigate the problem more carefully.

## 6.1 Enlarged model

From equations (76) we see that a suitable model must have at least two degrees of freedom. The simplest model of such a kind can be taken in the form shown in Fig.7. The system consist of the rigid framework of the mass $m$ with a rigid horizontal rod inside and a body of mass $M$ that restrained to move along the rod. The spring of stiffness $c$ connects the body $M$ with the framework $m$. The latter simulates a contact surface whereas the body $M$ simulates the centre of mass. The enlarged model tends to the rigid body model as $c \to \infty$. It is not difficult to derive equations of motion for this model. They have the form

$$\ddot{x} + \omega_0^2 (x - y) = F/m, \qquad \ddot{y} + \omega^2 (y - x) = 0 \tag{80}$$

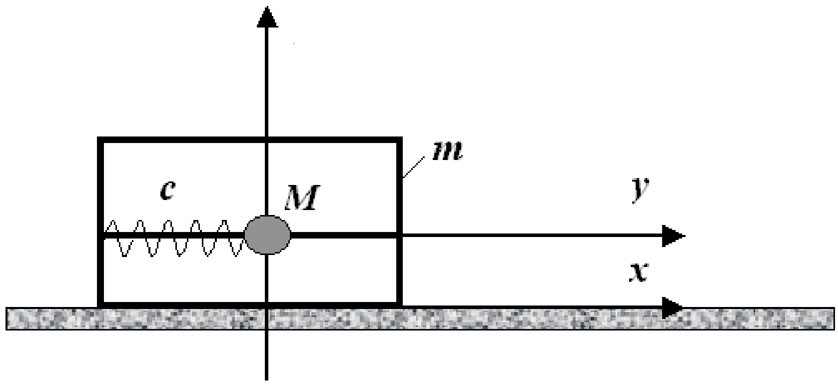


Figure 7: The enlarged model

where the friction force $F$ is defined by expression (76), $\omega_0^2 = c/m$, $\omega^2 = c/M$, $x$ and $y$ determine the positions of the framework and the body $M$ respectively. The initial conditions retain prior form (77). If $m \neq 0$, then for small $t > 0$ the solution of problem (80), (77) has the form

$$x = vt - \frac{1}{2}\mu g t^2 - \frac{M}{m}\frac{\mu g}{\Omega^2}\left(1 - \cos\Omega t\right), \tag{81}$$

$$y = vt - \frac{1}{2}\mu g t^2 + \frac{\mu g}{\Omega^2}\left(1 - \cos\Omega t\right), \tag{82}$$

where

$$\Omega^2 = \omega_0^2 + \omega^2 = c\left(\frac{1}{m} + \frac{1}{M}\right). \tag{83}$$

Solution (81), (82) is valid for such $t$ that

$$\dot{x} = v - \mu g t - \frac{M}{m}\frac{\mu g}{\Omega}\sin\Omega t > 0. \tag{84}$$

The moment of stopping $t = \tau$ must be found from the equation

$$v - \mu g \tau - \frac{M}{m}\frac{\mu g}{\Omega}\sin\Omega\tau = 0. \tag{85}$$

In what follows we shall assume that $m/M \ll 1$. Now we must consider solution (81)–(82) more carefully.

## 6.2 The model of rigid body

It is obvious that the enlarged model tends to the rigid body model as $c \to \infty$. This means that $\Omega \to \infty$. In such a case from expressions (81)–(82) it follows

$$x = y = vt - \frac{1}{2}\mu g t^2, \qquad t < \tau_{cl} \equiv \frac{v}{\mu g}, \tag{86}$$

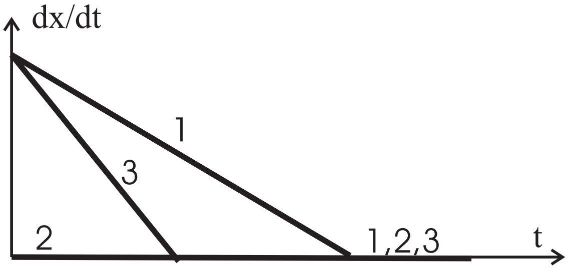


Figure 8: line 1 is the classical solution; line 2 is the solution for the massless framework; line 3 is the velocity of framework when m<<M

where $\tau_{cl}$ is the classical time of stopping. This solution is shown in Fig.8. Thus we see that solution $z_1$ from (79) is the limiting case

$$z_1 = \lim_{c\to\infty} x(t,c,m) = \lim_{c\to\infty} y(t,c,m) = \nu t - \frac{1}{2}\mu g t^2.$$

In this case the meaning of the solution $z_1$ is obvious.

## 6.3 The enlarged model with the massless framework

Let us consider the case such that

$$m \ll M$$

If $m \to 0$, then $\Omega \to \infty$ and $m\Omega^2 \to c$. The time of stopping $\tau_*$ must be found from equation (85). Let us suppose that $\Omega\tau_* \ll 1$. In such a case the approximate solution of equation (85) has the form

$$\tau_* = \frac{m}{m+M}\frac{\nu}{\mu g} \simeq \frac{m}{M}\tau_{cl} \ll \tau_{cl}. \tag{87}$$

We see that the strong inequality

$$\Omega\tau_* = \sqrt{\frac{m}{M}}\sqrt{\frac{c}{M}}\frac{\nu}{\mu g} \ll 1$$

is valid if $m \ll M$. For small intervals of time $t < \tau_*$ solution (81) and (82) can be rewritten in the next form

$$x = \nu t - \frac{1}{2}\left(1 + \frac{M}{m}\right)\mu g t^2, \qquad y = \nu t. \tag{88}$$

Now we can see the sense of solution $z_2$

$$\dot{z}_2 = \lim_{m\to 0} \dot{x}(t,c,m) = \begin{cases} \nu, & \text{if } t = 0, \\ 0, & \text{if } t > 0. \end{cases} \tag{89}$$

Here the function $z_2$ can be only treated as the position of the framework but not the position of the center of mass. The motion of the latter in case (89) is defined by the expression

$$y = \frac{v}{\omega}\sin\omega t, \qquad \omega^2 = \frac{c}{M}, \qquad t \geqslant 0. \tag{90}$$

This expression is valid if the force in the spring is less than $\mu M g$

$$|cy(t)|_{\max} \leqslant \mu(m+M)g \approx \mu M g. \tag{91}$$

This condition will be satisfied for all times if the initial velocity satisfies the inequality

$$|v| \leqslant \mu g\sqrt{\frac{M}{c}}, \quad v_c = \mu g\sqrt{\frac{M}{c}}, \tag{92}$$

where $v_c$ is a critical velocity. If this condition does not hold good, then inequality (91) the possibility to find the interval of time when solution (89)–(90) is valid. After that it is necessary to solve the Cauchy problem with new initial conditions.

## 6.4 Discussion

The second solution in (79) is called the Klein hypothesis. We saw that as a matter of fact it is not a hypothesis but the essential corollary of the conventional statement of all problems with Coulomb friction. The usual objection against the Klein hypothesis must be rejected if we accept the right interpretation for the function $z(t)$ in system (78). For the first solution in (79) it is quite possible to use two different interpretations, i.e. the function $z(t)$ can be considered as the position of the mass center or as the position of the contact surface. However for the second solution in (79) we have the only interpretation. This means that in general case the function $z(t)$ in (78) must be treated as the position of some point on the contact surface. Note that in many cases it is very important to take into account the second solution in (79) if we want to avoid contradictions of different kind.

Above we saw that exact statement of a task about movement of a body on the rough surface leads to unclosed system of equations. This is the direct indication on the singularity of the given task, for a problem of closing in many cases can not be solved by unique manner. In the given work the conventional closing of system (76) is used. As a result we have received the closed system, but in exchange we have got a new problem. Namely, the sense of function $z(t)$, strictly speaking, has remained uncertain. Maybe, the closing of system was made too roughly and rectilinearly, and the task revenges us, throwing up the senseless decision? Or, maybe, the Nature signals us about some important fact, which we should take into account? To the answers to these questions also is devoted the given subsection.

It is clear, that for the answer to the put above questions it is necessary to consider a task in the extended statement including additional the factors. It also was done above. By result of this analysis are two central conclusions.

First: the function $z(t)$ in system characterizes a position of point of a contact surface, but not a position of the center of mass; it at once removes traditional objection against use of the decision with instant shut-down, for instantly (or practically instantly) the body of infinitesimal mass stops.

The second conclusion: two solutions (79) cannot be understood so, that one of them is realized actually. As a matter of fact these solutions give us only top and bottom boundaries, between which a true solution is placed, but true movement of a body can not be found from the system (78).

As well as any theoretical statements given conclusions require experimental check. Let's carry out the following mental experiment. Let's take a bookcase with a number of horizontal shelves for the books. The case should be easy and rigid as much as possible. Besides we shall take a large load that can be placed on one shelf. Now it is possible to begin experiments. Previously we have to notice, that from a point of view of system (78) it has no importance on which shelf will be the load is located. Let's carry out a series of experiments, in each of which we shall use the same initial speed. In the first experiment the load is located on the bottom shelf. Let's measure the distance, gone the bookcase on inertia. If the center of mass of the bookcase with a load will be located enough close to a floor, the case will pass distance close to the predicted by the classical solution. The diagram for speed of a point on a contact surface will be close to the classical solution, i.e. to the line 1 on the Fig. 8. In the second experiment a load is arranged on the second shelf from below. The center of mass of a body thus will appear above, than in the first experiment. The measurements will show that the case will pass smaller distance rather than in the first experiment. The diagram for speed also moves to the left. Repeating these experiments and lifting a load higher and higher, we shall see, that the diagram of speed will nestle on an axis of ordinates, and gone distance became less and less. Certainly, all told carries speculative character. However it is difficult to doubt in told, for such behavior of a body is predicted by the analysis of behavior of the extended model.

Thus, we see, that the solutions (79) really give us only boundaries, in which there is a required solution, but itself the true solution, describing true movement of a body, is determined by the height parameter of the center of mass, which does not contain at all in system (78). From here also arises uncertainty in the solution of this system allowing defining only boundaries, in which there is a solution, but not solution.

The following picture of movement of extended model follows from all told. We admit, that its framework is opaque and has neglectably small mass. Admit that we observe only movement of a framework and do not know about content of this box. Let at $t < 0$ the system moved with constant velocity v under action of external forces, and the spring is considered not deformed. At $t = 0$ action of external forces suddenly stops, and body continues to move at action forces of friction. As the movement of this body will look from the point of view of the external observer? As the framework is considered to be inertialess, it instantly will stop, but the load $M$ will continue the movement stretching a spring. Force of elasticity of a spring, on the one hand slows down movement of a load M, but, on the other hand, it acts on the framework. Two variants further are possible. In the first variant, under action of force of elasticity the load $M$ will stop at some moment of time $t > 0$. It means, that force of elasticity is not bigger than the biggest possible force of friction of rest $\mu M g$. From the point of view of the external observer the body stands on a place, though invisible outside vibrations of a load M proceed inside a framework. The first variant is realized, if the initial speed was less than some critical velocity $\nu_c$, determined by an inequality (92).

The second variant is more interesting. Let initial speed exceeds the critical velocity.

Then the force of elasticity will exceed maximal force of friction of rest at the moment of time $t_1$, when the load $M$ will continue the movement. At this moment of time the framework will be broken from a place and instantly will catch up a load M, i.e. will restore the initial position with respect to the load $M$. Let's remind that the speech goes about the inertialess framework and spring. After that the framework will stop this process will be repeated, but already with smaller by initial velocity. The external observer thus will observe strange picture of movement of the framework. After cancellation external forces the framework will stop, some time will stand on a place, then will make the jump and again will stop. The number of such jumps depends on size initial velocity. After the appropriate number of jumps the framework is finally will stop, and the load, invisible to the observer, will make vibrations inside a framework. Basically, something similar can be observed in experiment. In this hypothetical case not only velocity, but also distance will be discontinuous functions of time. Certainly, if to a framework to attribute as much as small weight, the continuity of movement restores. But at very small weight this continuous movement will appear enough close to described above.

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