Phase Transitions and General Theory of Elasto-Plastic Bodies*

Abstract

The paper deals with a new theory of elastoplastic bodies based on a description of inelastic properties by means of the phase transitions in the material. The medium is assumed to be micropolar. The theory is applicable to the materials in any phase states. Besides, the theory takes into account the dry friction between the particles of the medium.

1   Introduction

A behavior of solid structures under an external loading has been studied during several centuries. However intensive and task-oriented investigations had begun in XIX century and are carried out till the present time. All known materials can be separated on two different classes: elastic materials and all others. In general, the fundamentals of the nonlinear theory of elasticity may be considered as completely developed [1]. For inelastic materials the situation is quite different. There exists a huge massive of experimental data. This data is widely used for practical purposes and normative documents for the engineering projects, but as a rule this data is not used in theoretical investigations. A lot of established experimental facts cannot be described by the existing theories of the elastoplastic bodies till now. Let us point out some of them [2]: 1. Under sufficiently high pressure all materials experience irreversible strains (Bridgman), which can be considered as phase transitions. The rate of these transitions is determined by the properties of the material and do not depend on the rate of change of the external loads. 2. At sufficiently high pressure all rigid bodies flow similarly to a fluid (Tresca, Bridgman). For example, the classical experiment by H. Tresca on extrude of lead shows absence of the stagnant zones in the material. On contrary, from any existing theory of plasticity it follows that the bands of the "dead" material should be present [3]. Thus we see a serious qualitative discrepancy between the theory and experiment. 3. The experiments on large inelastic deformation show essential influence of the size effect [4]. 4. In all experiments with a smooth loading the Savart – Masson effect is exhibited clearly. 5. Experiments with bulk materials show the necessity of taking into account the dry friction between particles of the medium. All these facts are of great importance because they are observed practically in all experiments. Nevertheless, the existing theories of plasticity are

not able to explain these facts. Moreover, the most of the known theories of plasticity are based on the yield criterions either by Saint-Venant – Tresca or by Mises. Both criterions were never strictly confirmed by experiments. While the existing theories cover almost all practical needs and extremely useful, nevertheless they are not able to explain some features of the material behavior.

The aim of the presentation is the attempt of build-up of such theory of inelastic materials, which would qualitatively feature the basic experimental facts. Besides, the theory should be sufficiently strict from the mathematical point of view. A novelty of the offered theory consists in the following. The experiments show that the inelastic materials cannot be modelled within framework of the material (Lagrangian) description. However the most of the known theories of the elastoplastic bodies are based on the material description. In what follows the spatial (Eulerian) description is used. The medium is assumed to be micropolar. Kinematics of the medium with rotational degrees of freedom is described. The fundamental laws are stated for open systems in a general form. The equation of the energy balance contains the term, which describes the formation of new particles or fragmentation of the initial particles. The concepts of internal energy, temperature and entropy are introduced by means of the pure mechanical arguments. The dry friction between the particles of the medium is introduced through the antisymmetric part of the stress tensor. The free energy is set in the form, which is suitable simultaneously for gases, fluids and solids. It is important to note that the material under the consideration has a finite tensile strength. That means that the constitutive equations can violate to the conditions of the strong ellipticity.

2 Fundamental Laws

2.1 Kinematic relations.

Let us consider a set of particles which are moving with respect to an inertial system. The set is not assumed to be a continuum. That means that the concept of a smooth differentiable manyfold cannot be used. Because of this a pure spatial (Eulerian) description will be used. Let a vector $\mathbf{V}(\mathbf{x}, t)$ be the velocity of a particle which at the actual instant of time t occupies the point $\mathbf{x}$ of a reference system. Let a quantity $\mathbf{K}(\mathbf{x}, t)$ be some property of the particle. In order to find the change of $\mathbf{K}(\mathbf{x}, t)$ during a motion of a particle we have to apply the material derivative [5]

$$\frac{\delta \mathbf{K}(\mathbf{x}, t)}{\delta t} \equiv \frac{d\mathbf{K}(\mathbf{x}, t)}{dt} + \left(\mathbf{V}(\mathbf{x}, t) - \frac{d\mathbf{x}}{dt} \right) \cdot \nabla \mathbf{K}(\mathbf{x}, t).$$

If the point $\mathbf{x}$ is moving with respect to the inertial reference system, then this definition does not coincide with conventional one and does not contradict with the Galilei principle of relativity. It is important to note that all used operators must be defined in the reference system rather than on smooth manyfold as at the material description. Besides let us point out that in the definition of a material derivative only the derivative $\mathbf{V} \cdot \nabla$ along the trajectory of a particle is used. Thus the continuity of $\mathbf{K}(\mathbf{x}, t)$ with respect to the space variable $\mathbf{x}$ is not assumed. For a vector of the particle acceleration we have

$$\mathbf{W}(\mathbf{x}, t) = \frac{d}{dt} \mathbf{V}(\mathbf{x}, t) + \left(\mathbf{V}(\mathbf{x}, t) - \frac{d\mathbf{x}}{dt} \right) \cdot \nabla \mathbf{V}(\mathbf{x}, t).$$

Let us introduce the displacement vector

$$\frac{\delta \mathbf{u}(\mathbf{x}, t)}{\delta t} = \mathbf{V}(\mathbf{x}, t) \;\Rightarrow\; \frac{d\mathbf{u}}{dt} = \mathbf{V} \cdot \mathbf{g}, \tag{1}$$

where

$$\mathbf{g}(\mathbf{x}, t) \equiv (\mathbf{E} - \boldsymbol{\nabla}\mathbf{u}(\mathbf{x}, t)), \quad \det \mathbf{g}(\mathbf{x}, t) > 0. \tag{2}$$

The tensor $\mathbf{g}(\mathbf{x}, t)$ will be termed the first measure of deformation. The Eq.(1) is a definition of the displacement vector. From (1) it follows

$$\boldsymbol{\nabla}\mathbf{V}(\mathbf{x}, t) = -\frac{\delta \mathbf{g}(\mathbf{x}, t)}{\delta t} \cdot \mathbf{g}^{-1}(\mathbf{x}, t). \tag{3}$$

Eqs.(1)–(3) may be found in [6] and will be used in the reduced equation of the energy balance. If a tensor $\mathbf{P}(\mathbf{x}, t)$ determines the rotation of a particle, then the angular velocity of the particle is defined by the modified Poisson equation [5]

$$\frac{\delta \mathbf{P}(\mathbf{x}, t)}{\delta t} = \boldsymbol{\omega}(\mathbf{x}, t) \times \mathbf{P}(\mathbf{x}, t). \tag{4}$$

Let us introduce the second measure of deformation $\mathbf{F}$ by means of equalities

$$\frac{\partial}{\partial x^s}\mathbf{P} = \mathbf{F}_s \times \mathbf{P}, \quad \mathbf{F} = \mathbf{g}^s \otimes \mathbf{F}_s, \tag{5}$$

where the vectors $\mathbf{g}_s$ are the basis vectors and the following conditions of integrability hold [5]

$$\frac{\partial \mathbf{F}_s}{\partial x^m} - \frac{\partial \mathbf{F}_m}{\partial x^s} = \mathbf{F}_m \times \mathbf{F}_s. \tag{6}$$

From Eq.(6) it follows

$$\boldsymbol{\nabla} \otimes \boldsymbol{\omega} = \frac{\delta \mathbf{F}}{\delta t} + \mathbf{F} \times \boldsymbol{\omega} + \boldsymbol{\nabla}\mathbf{V} \cdot \mathbf{F}.$$

2.2   Particles and mass balance.

Let us introduce two nonnegative functions: $\eta(\mathbf{x}, t)$ is the particle density and $\rho(\mathbf{x}, t)$ is the mass density. If the material has a tendency to a fragmentation, then the total mass is conserved, but the number of particles does not conserved. In such a case the following equations are valid

$$\frac{\delta \eta}{\delta t} + \eta \boldsymbol{\nabla} \cdot \mathbf{V} = \chi, \quad \frac{\delta \rho}{\delta t} + \rho \boldsymbol{\nabla} \cdot \mathbf{V} = 0, \tag{7}$$

where $\chi(\mathbf{x}, t)$ determines the production of new particles for the unit of time. From practical point of view the importance of $\eta(\mathbf{x}, t)$ is determined by the necessity to take into account the porosity of material. In such a case the function $\chi(\mathbf{x}, t)$ in Eqs.(7) depends on pressure. Using the identity [5]

$$\boldsymbol{\nabla} \cdot \mathbf{V}(\mathbf{x},t) = -\frac{1}{I_3(\mathbf{g})}\frac{\delta I_3(\mathbf{g})}{\delta t}, \quad I_3(\mathbf{g}) \equiv \det \mathbf{g}$$

the above equations can be written in the form

$$\frac{\delta}{\delta t}\left[\frac{\eta}{I_3(\mathbf{g})}\right] = \frac{\chi}{I_3}, \quad \frac{\delta}{\delta t}\left[\frac{\rho}{I_3(\mathbf{g})}\right] = 0. \tag{8}$$

Let us introduce a some characteristic of a particle called the particle volume v_p. The quantity $c = v_p\eta$, known as the compactness, determines the material volume $v_p\eta dV$ occupied by the material in the control volume dV. The quantity $c_p = 1 - v_p\eta$ is termed a porosity. Note that we apply the term "porosity" in unconventional sense because we do not consider the porous medium. We mean that any solid material has a several stable states corresponding to different magnitudes of the compactness. The transition of the material from one stable state to another stable state is a typical phase transition which we would like to take into account. For all known materials compactness satisfies an inequality $0 \le v_p\eta \le 0.74$. Thus for porosity we have $0.26 \le c_p \le 1$. The first equation from Eqs.(7) may be rewritten in terms of porosity

$$\frac{\delta c_p}{\delta t} + v_p\chi(c_p, p) = \nabla \cdot \mathbf{V}, \tag{9}$$

where p is a pressure and the function $v_p\chi(c_p, p)$ must be defined by the constitutive equation, for which there exist a many different possibilities, but the final results are not known. Because of this we are not able to give a short resume of these possibilities. As an example the following equation may be considered

$$\frac{dc_p}{dt} + \mathbf{V} \cdot \nabla c_p = \nabla \cdot \mathbf{V} - \frac{\epsilon^2 p}{\epsilon^2 + (p_c - p)^2},$$

where $\epsilon^2 \ll 1$ is a small parameter and p_c is some critical pressure. This equation shows the behavior of the porosity near one point of the phase transitions. The realistic equations should have a more complicated form.

2.3   Dynamics Laws.

Let us introduce the stress tensor $\mathbf{T}(\mathbf{x}, t)$ and the moment stress tensor $\mathbf{M}(\mathbf{x}, t)$. These tensors are defined in the space, but not in the material. For them the Cauchy formulae are valid

$$\mathbf{T}_{(n)} = \mathbf{n} \cdot \mathbf{T}, \quad \mathbf{M}_{(n)} = \mathbf{n} \cdot \mathbf{M}.$$

The first and the second laws by Euler have the well known form

$$\nabla \cdot \mathbf{T} + \rho\mathbf{F} = \rho\frac{\delta\mathbf{V}(\mathbf{x}, t)}{\delta t}, \tag{10}$$

$$\nabla\cdot\mathbf{M} + \mathbf{T}_\times + \rho\mathbf{L} = \rho\frac{\delta(\mathbf{J} \cdot \boldsymbol{\omega})}{\delta t}, \tag{11}$$

where the mass density of the inertia tensor $\mathbf{J}$ of a particle in the actual position is connected with the constant tensor $\mathbf{J}_0$ in the reference position by

$$\mathbf{J}(\mathbf{x}, t) = \mathbf{P}(\mathbf{x}, t) \cdot \mathbf{J}_0 \cdot \mathbf{P}^\mathsf{T}(\mathbf{x}, t). \tag{12}$$

2.4 Equation of the energy balance.

The equation of the energy balance in the local form can be written down as

$$\rho \frac{\delta \mathcal{U}}{\delta t} = \mathbf{T}^\mathsf{T} \cdot\cdot (\boldsymbol{\nabla}\mathbf{V} + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}^\mathsf{T} \cdot\cdot \boldsymbol{\nabla}\,\boldsymbol{\omega} + \boldsymbol{\nabla}\cdot \mathbf{h} + \rho q. \tag{13}$$

where $\mathcal{U}$ is the mass density of internal energy and the vector $\mathbf{h}$ is the vector of the heat flux. The right hand side of Eq.(13) contain the power of the stress tensor and of the moment stress tensor. One part of the power changes the specific internal energy. Another part partly remains in the body as a heat and partly radiates into the external medium. In order to separate these parts the stress tensor and the moment stress tensor must be represented as

$$\mathbf{T} = \mathbf{T}_e + \mathbf{T}_i, \quad \mathbf{M} = \mathbf{M}_e + \mathbf{M}_i, \tag{14}$$

where the quantities with the subscript "e" are independent of velocities and the quantities with the subscript "i" are the rest part of stresses. One may substitute Eq.(14) into Eq.(13) in order to get the final form of the energy balance equation. However in such a form the energy balance equation is almost useless. We have to transform this equation in order to obtain the reduced equation of the energy balance.

The forth fundamental law of mechanics is the second law of thermodynamics. The statement of this law will be given in the following section.

3 The heat conductivity equation

Let us introduce the concepts of the temperature, entropy and chemical potential by means of the following equation

$$\rho \vartheta \frac{\delta \mathcal{H}}{\delta t} + \rho \eta \frac{\delta \mathcal{C}}{\delta t} = \boldsymbol{\nabla} \cdot \mathbf{h} + \rho q + \mathbf{T}_i^\mathsf{T} \cdot\cdot (\boldsymbol{\nabla}\mathbf{V} + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}_i^\mathsf{T} \cdot\cdot \boldsymbol{\nabla}\boldsymbol{\omega}, \tag{15}$$

where the functions $\vartheta(\mathbf{x}, t)$, $\mathcal{H}(\mathbf{x}, t)$ and $\mathcal{C}(\mathbf{x}, t)$ are respectively termed the temperature, the specific entropy and the specific chemical potential. Let us underline that Eq.(15) is the definition for these functions. The only purpose of introduction of the specific entropy and the specific chemical potential (these functions by itself have no physical sense) is to define by an appropriate way the temperature ϑ and the particle density η or, what is the same, porosity of the material. Of course, we need some additional assumptions for a complete definition of those quantities. Now let us accept the second law of thermodynamics in the form of the following inequalities

$$\mathbf{T}_i^\mathsf{T} \cdot\cdot (\boldsymbol{\nabla}\mathbf{V} + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}_i^\mathsf{T} \cdot\cdot \boldsymbol{\nabla}\boldsymbol{\omega} \geq 0, \quad \mathbf{h} \cdot \boldsymbol{\nabla}\vartheta \geq 0. \tag{16}$$

Inequalities (16) are more strong than the consequences of the known inequality by Clausius – Duhem [7]. However from our point of view Eq.(16) are quite good for practical aims. The constitutive equation for the vector of the heat flux may be taken in the simplest form

$$\mathbf{h} = -\kappa \boldsymbol{\nabla}\vartheta, \quad \kappa \geq 0. \tag{17}$$

The substituting of Eq.(17) into Eq.(15) leads to the heat conduction equation.

4 Reduced equation of the energy balance, the Cauchy – Green relations

Let us introduce the specific free energy

$$\mathcal{F} = \mathcal{U} - \vartheta\mathcal{H} - \eta\mathcal{C}. \tag{18}$$

Making use Eqs.(14), (15) and (18) the equation (13) can be rewritten in the following form

$$\rho\frac{\delta\mathcal{F}}{\delta t} + \rho\mathcal{H}\frac{\delta\vartheta}{\delta t} + \rho\mathcal{C}\frac{\delta\eta}{\delta t} = \mathbf{M}_e^{\mathsf{T}} \cdot\cdot \frac{\delta\mathbf{F}}{\delta t} - \left(\mathbf{g}^{-1}\cdot\mathbf{T}_e^{\mathsf{T}} + \mathbf{g}^{-1}\cdot\mathbf{F}\cdot\mathbf{M}_e^{\mathsf{T}}\right)\cdot\cdot\frac{\delta\mathbf{g}}{\delta t} + $$
$$+ \frac{1}{2}\left[\left(\mathbf{M}_e^{\mathsf{T}}\cdot\mathbf{F} - \mathbf{T}_e\right)_{\times} \times \mathbf{P}\right]^{\mathsf{T}} \cdot\cdot\frac{\delta\mathbf{P}}{\delta t}. \tag{19}$$

The equation of the energy balance written in the form like Eq.(19) is termed the reduced equation of the energy balance. This equation involves only the intrinsic variables. From Eq.(19) we see that the free energy is a function of the following arguments

$$\mathcal{F} = \mathcal{F}\left(\vartheta, \eta, \mathbf{g}, \mathbf{F}, \mathbf{P}\right). \tag{20}$$

Taking into account this statement it is readily to derive the Cauchy–Green relations

$$\mathcal{H} = -\frac{\partial\mathcal{F}}{\partial\vartheta}, \quad \mathcal{C} = -\frac{\partial\mathcal{F}}{\partial\eta}, \quad \mathbf{M}_e = \rho\frac{\partial\mathcal{F}}{\partial\mathbf{F}}, \quad \mathbf{T}_e = -\rho\frac{\partial\mathcal{F}}{\partial\mathbf{F}}\cdot\mathbf{F}^{\mathsf{T}} - \rho\frac{\partial\mathcal{F}}{\partial\mathbf{g}}\cdot\mathbf{g}^{\mathsf{T}}. \tag{21}$$

Now Eq.(20) takes the form

$$\rho\left(\frac{\partial\mathcal{F}}{\partial\mathbf{P}}\right)^{\mathsf{T}} \cdot\cdot\frac{\delta\mathbf{P}}{\delta t} = \frac{1}{2}\left(\left(\mathbf{M}^{\mathsf{T}}\cdot\mathbf{F} - \mathbf{T}\right)_{\times} \times \mathbf{P}\right)^{\mathsf{T}} \cdot\cdot\frac{\delta\mathbf{P}}{\delta t}. \tag{22}$$

Here we have to take into account that the material derivative of the tensor $\mathbf{P}$ cannot be changed by an arbitrarily manner. Indeed, from the Poisson equation Eq.(4) it follows

$$\frac{\delta\mathbf{P}(\mathbf{x}, t)}{\delta t}\cdot\mathbf{P}^{\mathsf{T}}(\mathbf{x}, t) = \boldsymbol{\omega}(\mathbf{x}, t)\times\mathbf{E} \quad\Rightarrow\quad \left(\mathbf{A}\cdot\mathbf{P}\right)^{\mathsf{T}} \cdot\cdot\frac{\delta\mathbf{P}(\mathbf{x}, t)}{\delta t} = 0, \quad \forall\mathbf{A}: \mathbf{A} = \mathbf{A}^{\mathsf{T}}.$$

Hence we get the relation

$$\rho\frac{\partial\mathcal{F}}{\partial\mathbf{P}} - \frac{1}{2}\left(\mathbf{M}_e^{\mathsf{T}}\cdot\mathbf{F} - \mathbf{T}_e\right)_{\times} \times \mathbf{P} = \mathbf{A}\cdot\mathbf{P}.$$

In order to exclude the arbitrary symmetric tensor $\mathbf{A}$, we have to multiply both sides of this equation by the tensor $\mathbf{P}^{\mathsf{T}}$ and to calculate the vector invariants of both sides. As a result we have

$$\left[\rho\frac{\partial\mathcal{F}}{\partial\mathbf{P}}\cdot\mathbf{P}^{\mathsf{T}} + \mathbf{M}_e^{\mathsf{T}}\cdot\mathbf{F} - \mathbf{T}_e\right] \cdot\cdot\mathbf{C} = 0, \quad \forall\mathbf{C}: \ \mathbf{C} = -\mathbf{C}^{\mathsf{T}}. \tag{23}$$

The stress tensor $\mathbf{T}_e$ and the moment stress tensor $\mathbf{M}_e$ are defined by the Cauchy–Green relations Eqs.(21). That means that the condition Eq.(23) is the restriction superposed on the free energy.

Below we use the technics given in [8]. From Eq.(23) we see that the free energy must satisfy the following equation of first order partial differential equation

$$\left(\frac{\partial \mathcal{F}}{\partial \mathbf{g}}\right)^{\mathsf{T}} \cdot \cdot (\mathbf{C} \cdot \mathbf{g}) + \left(\frac{\partial \mathcal{F}}{\partial \mathbf{P}}\right)^{\mathsf{T}} \cdot \cdot (\mathbf{C} \cdot \mathbf{P}) + \left(\frac{\partial \mathcal{F}}{\partial \mathbf{F}}\right)^{\mathsf{T}} \cdot \cdot (\mathbf{C} \cdot \mathbf{F} - \mathbf{F} \cdot \mathbf{C}) = 0. \qquad (24)$$

The characteristic system for Eq.(24) has a form

$$\frac{d\mathbf{g}}{ds} = \mathbf{C} \cdot \mathbf{g}, \quad \frac{d\mathbf{P}}{ds} = \mathbf{C} \cdot \mathbf{P}, \quad \frac{d\mathbf{F}}{ds} = \mathbf{C} \cdot \mathbf{F} - \mathbf{F} \cdot \mathbf{C}. \qquad (25)$$

The free energy to satisfy Eq.(24) must be a function of the integrals of Eq.(25). The latter consists the system of the order 21 and has not more than 18 functionally independent integrals of Eq.(25).

5   Nonpolar medium with the Coulomb friction

Let us assume that the free energy is independent of the second deformation measure $\mathbf{F}$

$$\mathcal{F} = \mathcal{F}(\vartheta, \eta, \mathbf{g}), \quad \mathbf{M}_e = \mathbf{0}. \qquad (26)$$

We may rewrite Eq.(19) as

$$\rho \frac{\delta \mathcal{F}}{\delta t} + \rho \mathcal{H} \frac{\delta \vartheta}{\delta t} + \rho \mathcal{C} \frac{\delta \eta}{\delta t} = - \left(\mathbf{g}^{-1} \cdot \mathbf{T}_e^{\mathsf{T}}\right) \cdot \cdot \frac{\delta \mathbf{g}}{\delta t}.$$

The stress tensor can be decomposed as

$$\mathbf{T}_e = -p \, \mathbf{E} + \boldsymbol{\tau}, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{T}}, \quad \mathrm{tr}\, \boldsymbol{\tau} = 0.$$

The representations for $\mathbf{T}_i$ and $\mathbf{M}_i$ will be given below. Making use the technics given in the previous section one can prove that in case under consideration the free energy has a form

$$\mathcal{F} = \mathcal{F}(\vartheta, \eta, \rho, \gamma, \mathbf{G}),$$

where

$$\gamma \equiv I_3^2(\mathbf{g}), \quad \mathbf{G} \equiv I_3^{-2/3} \mathbf{g}^{\mathsf{T}} \cdot \mathbf{g}, \quad \det \mathbf{G} = 1.$$

Following [6] the unimodular tensor $\mathbf{G}$ will be termed the strain of shape change. The constitutive equations for the pressure p and for the deviator $\boldsymbol{\tau}$ take a form

$$p = \rho^2 \frac{\partial \mathcal{F}}{\partial \rho} + \rho I_3(\mathbf{g}) \frac{\partial \mathcal{F}}{\partial I_3(\mathbf{g})}, \quad \boldsymbol{\tau} = -2\rho \left[\gamma^{-1/3} \mathbf{g} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{G}} \cdot \mathbf{g}^{\mathsf{T}} - \frac{1}{3} \mathbf{G} \cdot \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{G}} \mathbf{E} \right].$$

Let us introduce the new parameters

$$\zeta = \frac{1}{\sqrt{\rho I_3(\mathbf{g})}}, \quad z = \sqrt{\frac{\rho}{I_3(\mathbf{g})}}, \quad \frac{\delta z}{\delta t} = 0.$$

In such a case we have the final form of constitutive equations

$$\mathcal{H} = -\frac{\partial \mathcal{F}}{\partial \vartheta}, \quad \mathcal{C} = -\frac{\partial \mathcal{F}}{\partial \eta}, \quad \mathfrak{p} = -\frac{\partial z \mathcal{F}}{\partial \zeta}, \quad -\frac{\zeta \tau}{2} = \gamma^{-1/3} \mathbf{g} \cdot \frac{\partial z \mathcal{F}}{\partial \mathbf{G}} \cdot \mathbf{g}^\mathsf{T} - \frac{1}{3} \mathbf{G} \cdot \cdot \frac{\partial z \mathcal{F}}{\partial \mathbf{G}} \mathbf{E}, \quad (27)$$

where the free energy is a function of five arguments

$$\mathcal{F} = \mathcal{F}(\vartheta, \eta, \zeta, z, \mathbf{G}). \tag{28}$$

Now let us assume the following representations for the viscous stresses

$$\mathbf{T_i} = \mathbf{t} \times \mathbf{E}, \quad \mathbf{M_i} = \mathbf{m} \times \mathbf{E}. \tag{29}$$

With these assumptions the first inequality from Eq.(16) takes the form

$$-\mathbf{t} \cdot (2\boldsymbol{\omega} - \boldsymbol{\nabla} \times \mathbf{V}) - \mathbf{m} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\omega}) > 0.$$

For the moment vector $\mathbf{m}$ we take the viscous friction law and for the stress vector $\mathbf{t}$ we assume that the Coulomb dry friction law is valid

$$\mathbf{t} = -\mathrm{k} \, \mathrm{h}(\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n}) \, |\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n}| \frac{(2\boldsymbol{\omega} - \boldsymbol{\nabla} \times \mathbf{V})}{|2\boldsymbol{\omega} - \boldsymbol{\nabla} \times \mathbf{V}|}, \quad \mathbf{m} = -\mu_m (\boldsymbol{\nabla} \times \boldsymbol{\omega}), \, \mu_m \geq 0, \quad (30)$$

where the function $\mathrm{h}(\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n})$ is determined by

$$\mathrm{h}(\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n}) = \left\{ \begin{array}{ll} 1, & \mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n} < 0, \\ 0, & \mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n} \geq 0, \end{array} \right.$$

$\mathrm{k} \geq 0$ is the coefficient of the dry friction. The unit vector $\mathbf{n}$ in Eq.(30) must be found as a solution of the problem

$$\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{m} = \max, \quad \forall \mathbf{n}, \mathbf{m} : |\mathbf{n}| = |\mathbf{m}| = 1, \quad \mathbf{n} \cdot \mathbf{m} = 0.$$

It is easy to prove that the solution of this problem is unique. The Coulomb law in Eq.(30) is applicable if a sliding is present. Otherwise we have a condition

$$2\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{V}, \tag{31}$$

and the vector $\mathbf{t}$ has to be found from Eq.(11)

$$-\mu_m \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{V})] - 4\mathbf{t} = \rho \frac{\delta}{\delta t} [\mathbf{J} \cdot (\boldsymbol{\nabla} \times \mathbf{V})]. \tag{32}$$

Using Eq.(32) the vector $\mathbf{t}$ can be eliminated from the first law of dynamics.

6   Isotropic materials

Let us suppose that we deal with isotropic materials. In such a case the free energy depends on the invariants of the tensor $\mathbf{G}$

$$\mathcal{F} = \mathcal{F}(\vartheta, \eta, \zeta, z, I_1, I_2), \quad I_1(\mathbf{G}) \equiv \mathbf{E} \cdot \cdot \mathbf{G}, \quad I_2(\mathbf{G}) \equiv \mathbf{G} \cdot \cdot \mathbf{G}. \tag{33}$$

Making use of Eq.(33) we can rewrite Eqs.(20) as

$$\mathcal{H} = -\frac{\partial \mathcal{F}}{\partial \vartheta}, \quad \mathcal{C} = -\frac{\partial \mathcal{F}}{\partial \eta}, \quad p = -\frac{\partial z\mathcal{F}}{\partial \zeta},$$

$$\zeta \boldsymbol{\tau} = \frac{2}{3} \left(I_1 \frac{\partial z\mathcal{F}}{\partial I_1} + I_2 \frac{\partial z\mathcal{F}}{\partial I_2} \right) \mathbf{E} - 2 \left(\frac{\partial z\mathcal{F}}{\partial I_1} \boldsymbol{\Lambda} + \frac{\partial z\mathcal{F}}{\partial I_2} \boldsymbol{\Lambda}^2 \right), \quad (34)$$

where the tensor $\boldsymbol{\Lambda}$ is defined by

$$\boldsymbol{\Lambda} = I_3^{-2/3}(\mathbf{g}) \, \mathbf{g} \cdot \mathbf{g}^\mathsf{T}.$$

The invariants of the tensor $\boldsymbol{\Lambda}$ are given by

$$I_1(\mathbf{G}) = I_1(\boldsymbol{\Lambda}) = \Lambda_1 + \Lambda_2 + \frac{1}{\Lambda_1 \Lambda_2} \geq 3, \quad I_2(\mathbf{G}) = I_2(\boldsymbol{\Lambda}) = \Lambda_1^2 + \Lambda_2^2 + \frac{1}{\Lambda_1^2 \Lambda_2^2} \geq 3,$$

where Λ_1, Λ_2 are two independent eigenvalues of $\boldsymbol{\Lambda}$.

7   Constitutive equation for the pressure

Let us assume that the free energy may be represented as a composition

$$z\mathcal{F} = f(\vartheta, \eta, \zeta, z) + z\mathcal{F}_d(\vartheta, \eta, z, I_1, I_2).$$

In such a case the pressure is determined by

$$p = -\frac{\partial f}{\partial \zeta}. \tag{35}$$

Let the pressure p be a linear function of the temperature

$$p = f_1(\zeta, \eta, z) + f_2(\zeta, \eta, z)\,\vartheta. \tag{36}$$

The most popular in physics of solids the constitutive equations by van-der-Waals and by Mu–Grüneisen have namely this form. For example, the van-der-Waals equation can be written as

$$p(\zeta, \vartheta) = -\frac{a}{\zeta^2} + \frac{c\,\vartheta}{\zeta - b}, \tag{37}$$

where a, b and c are the characteristics of the material. However, in our case these quantities may depend on the parameters η, z. It is known that the van-der-Waals equation satisfactory predicts the behavior of the real gases. It seems obvious that Eq.(36) can be by corresponding choice of the functions f_1 f_2 used not only for liquids and gases but for solids with the phase transitions. The pressure at $\vartheta = 0$ is described by means of the function $f_1(\zeta, \eta, z)$, the possible form of which is shown in Figure 1. The material shown in Figure 1 has three stable equilibrium states. The transition from one state to another is a typical phase transition. It is easy to understand that the diagram, like shown in Figure 1, cannot be found by experiment. However, the envelops of the true diagram can be established in an experiment. The upper envelope describes the properties of a material under compression, and the lower envelope describes the properties of the material
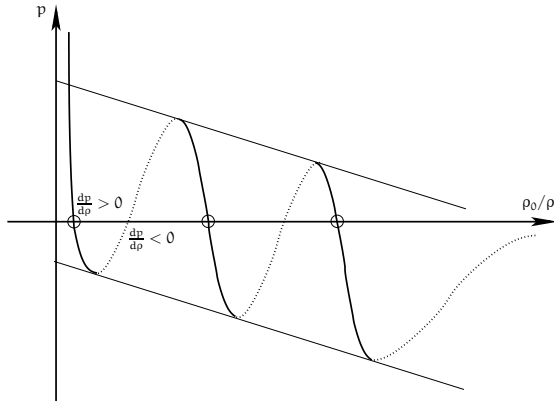
Figure 1: Constitutive equation for the pressure at zero temperature.

under extension. At the qualitative consideration the function $f_2(\zeta, \eta, z)$ may be chosen as in the van-der-Waals equation. The simplest example of the constitutive equation for the material with finite tensile strength is given by the expression

$$p = f_0 \left(\zeta^{-m} - \zeta^{-n} \right) + \frac{c\,\vartheta}{\zeta - b}, \tag{38}$$

where $m > n$, $\zeta > b(\eta)$. The pressure dependence on ζ at different temperatures is shown in Figure 2. Using Eq.(36) and Eq.(38) we find the expression for the free energy
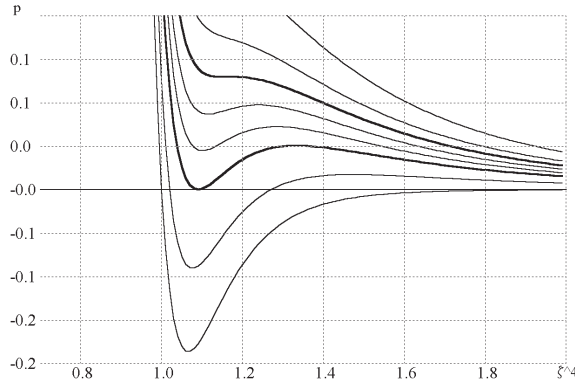

Figure 2: The pressure dependence on ζ^4 for different temperatures.

$$\rho_0\,\mathcal{F}(\zeta, \vartheta, \mathbf{E}) = f_0 \left(\frac{\zeta^{-m+1}}{m-1} - \frac{\zeta^{-n+1}}{n-1} \right) - c\,\vartheta\,\ln(\zeta - b) + \psi(\vartheta),$$

where $\psi(\vartheta)$ is some function of temperature. More general form of the constitutive equation for the pressure is given by

$$p = \sum_{k=0}^{N} a_k \zeta^{-k} + \frac{c\,\vartheta}{\zeta - b},$$ (39)

where parameters (a_k, N, c, b) are characteristics of material. All of them may depend on the structural parameter η. Besides, maybe it will be useful to take more general form of the function f_2. In general, Eq.(39) corresponds to the material with N solid phase states. If we desire to take into account the phase transition "solid–liquid" and "liquid–gas", then we have to construct the curve like shown in Figure 3. If it is desirable to take into account a several solid phase states, then we have to add to the constitutive equation the terms like the first term in the right hand side of Eq.(39).
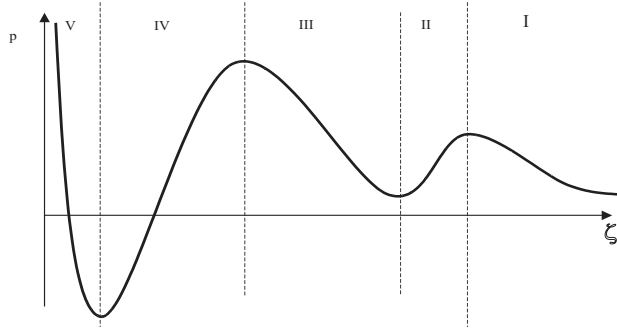

Figure 3: The three-phases medium: zones I, III, V correspond to stable gaseous, liquid and solid phases respectively; zones II, IV correspond the unstable states

8   Constitutive equation for the stress tensor deviator

From conventional point of view the state equation of solid is the relation between pressure, temperature and the mass density or volume. However the constitutive equation for the deviator τ of the stress tensor cannot be ignored. Let us underline that the most of the phase transitions in solid are connected with the fact that the maximal shear stress in material has a rather low upper limit. When defining the function zF_d we have first of all to take into account this fact. Let the values $0 < \lambda_1 \le \lambda_2 \le \lambda_3$ $(\lambda_1\lambda_2\lambda_3 = 1)$ be eigenvalues of the tensor $\mathbf{G}$. Let us introduce the quantity

$$\sigma \equiv 3I_2(\mathbf{G}) - I_1^2(\mathbf{G}) = (\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_3)^2.$$ (40)

If $\sigma = 0$, then $\mathbf{G} = \mathbf{E}$. Now let us assume that the deviatoric part of free energy $z\mathcal{F}_d$ depends on the parameters σ and I_1 rather than invariants I_1, I_2. In such a case from Eqs.(34) one may obtain the equation

$$\zeta\tau = -\frac{\partial z\mathcal{F}}{\partial I_1}\left(\mathbf{\Lambda} - \frac{1}{3}I_1\mathbf{E}\right) + 2\frac{\partial z\mathcal{F}}{\partial \sigma}\left[\frac{(\sigma - I_1^2)}{3}\mathbf{E} + 2I_1\mathbf{\Lambda} - 3\mathbf{\Lambda}^2\right].$$ (41)

If we consider the case of small deformations when $\|(\boldsymbol{\nabla}\mathbf{u})\| \ll 1$, then instead of Eq.(41) we get

$$\zeta\boldsymbol{\tau} = 2\mu\,\mathrm{dev}\,\boldsymbol{\varepsilon} + O(\varepsilon^2), \quad \mu \equiv \frac{\partial z\mathcal{F}}{\partial I_1}, \tag{42}$$

where ε is the tensor of linear deformations, the parameter μ may be termed the shear modulus. From Eq.(42) we see that in linear theory the dependence of free energy on a parameter σ is not important. By this reason and for the sake of simplicity we assume that the free energy does not depend on the parameter σ and Eq.(41) takes a form

$$\zeta\boldsymbol{\tau} = 2\mu\left(\frac{1}{3}I_1\mathbf{E} - \boldsymbol{\Lambda}\right), \quad \mu \equiv \frac{\partial z\mathcal{F}}{\partial I_1}. \tag{43}$$

The shear modulus μ is a function like $\mu = \mu(\vartheta, \eta, z, I_1)$. In order to define the function $\mu(\vartheta, \eta, z, I_1)$ we have many possibilities. But at the moment it is difficult to understand by unique manner what possibility is used by the Nature. The shear modulus depends on four different parameters ϑ, η, z, I_1. For all of them we have governing equations. We thing that the dependence μ from the temperature is not crucial one. The same may be said with respect to the variable z. However, the parameters η and I_1 have a crucial influence on the shear modulus. The problem is that from physical point of view both η and I_1 influences on the shear modulus in the almost similar way. As far as we know in mechanics of solids the parameter η has never been used and the behavior of the shear modulus is determined by deformations. In such a case it is possible to use, for example, the following representation

$$\mu = \mu_0(\vartheta, z)\left[1 - \cos\left(\frac{\pi(I_1 - 3)}{2l_*}\right)\right] \tag{44}$$

where l_* is a some characteristic of the material. The representation Eq.(44) corresponds to the free energy which looks just like the potential by Frenkel – Kontorova [4] in dynamics of crystal lattice. We do not think that this representation is sufficiently good for practical needs. At the moment we would like to point out the qualitative behavior of the shear modulus. We have to remember that under high pressure the shear modulus must vanish. The dependence of shear modulus on I_1 is not monotone in order to describe the Savart – Masson effect.

Maybe, more realistic constitutive equation for shear modulus is given by representation like

$$\mu = \mu_0(\vartheta, z, I_1)(1 - c_p)^2(c_p - 0.26)^2, \tag{45}$$

where c_p is the porosity which must satisfy Eq.(9). Small values of $(1 - c_p)$ occur for the gases. For solids c_p is close to 0.26. The quantitative dependence of μ from c_p may be, of course, different from Eq.(45). It is quite possible that we will need some combination of the representations like Eq.(44) and Eq.(45). The future investigations have to clear the situation.

Conclusion

Above a general (maybe, superfluous general) theory of materials in any phase states is developed. The present state of the theory does not suit for those people who desire to obtain the practical results immediately. But what do we know? We know that during more than 150 years the applied theories of inelastic materials were developed in great extent. And in spite of this

there exist a lot of very old experimental results which cannot be described by the existing theories. Why? Maybe, it is time to go far from practical results and to develop the theory which is right from fundamental point of view. We think that any applied theory must be consistent with the theory of such a kind. At the moment we have made only initial steps. However even from these initial steps we see that some conventional statements are not valid. For example, everybody knows that $I_3(\mathbf{g})$ is responsible for a volume change of a material. But we saw that it is not so and we have to introduce the special object to characterize the material volume. From physical point of view it is clear that the important role in the description of inelastic properties of the material must play the chemical potential which is responsible for the structural transformation in the material. As far we know the chemical potential was used in continuum mechanics only in the case of the multi-component media.

We hope that the given above theory attracts the attention both physicists and mathematicians. The theory is needed in additional minds to create the true useful applied theory.

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