Nonlinear Theory of Thin Rods*

Abstract

Nonlinear theory of the rods is the oldest and, maybe, the most important theory in continuum mechanics of solids. However there are some theoretical problems which have no solution up to now. The report is devoted to discussion of the dynamic theory of the thin spatially bent and naturally twisted rods. The suggested theory includes all known variants of the theory of rods, but possesses wider branch of applicability. A new method of construction of the elasticity tensors is offered and their structure is established. To this end a new theory of the tensor symmetry in space with two independent orientations is essentially used. For plane elastic curves all modules of elasticity are determined. The significant attention in the report is given to the analysis of some classical problems, including those from them, solution of which leads to paradoxical results. In particular, it is in detail considered well-known elastica by Euler and it is shown, that alongside with known equilibrium configurations there are also dynamic equilibrium configurations. In this case the form of an elastic curve does not vary, but the bent rod makes rotations around of a vertical axis. Energy of deformation in this case does not vary. Let us note that these movements are not movements of a rod as the rigid whole for the clamped end face of the rod remains motionless. From this it follows, that the bent equilibrium configuration in the Euler elastica is, in contrast to the conventional point of view, unstable. On the other hand, this conclusion is not confirmed by experimental data. Therefore there is a paradoxical situation which demands the decision. The similar situation known under the name of the Nikolai paradox arises at torsion of a rod by the boundary twisting moment. In this case experiment shows that twisting moment produces stabilizing effect that is in the sharp contradiction with the theoretical data. In the report it is shown what to avoid the specified paradoxes it is possible at a special choice of the constitutive equation for the moment. It appears that the moment should depend in the special form on angular velocity. Last dependence is not connected with the presence (or absence) internal friction in the rod.

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1 The rod theory and modern mechanics

The theory of thin rods has played outstanding role in the history of development of mechanics and mathematical physics. In order to show the contribution of the theory of thin rods to the development of natural sciences more clearly, let us point out only some facts.

Birth of the ordinary differential equations. In 1691 Jacob Bernoulli has derived the differential equation of equilibrium of a rope (string)

$$\mathbf{N}' + \rho\mathbf{F} = \mathbf{0}. \tag{1}$$

The equation (1) was the first differential equation in the history of a science.

Birth of the equations in partial derivatives. In 1742 Jacque D'Alembert has derived the equation of vibrations of a string

$$\frac{\partial^2 u}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial^2 t^2} = 0. \tag{2}$$

The equation (2) was the first differential equation in partial derivatives. Development of the methods of its solution has led to the creation of ***the theory of decomposition of functions in series*** — Daniel Bernoulli and Leonard Euler.

Birth of the theory of bifurcation of the solutions of nonlinear differential equations. In 1744 L. Euler has solved a problem on a longitudinal bending of the rod, named later ***Euler's Elastica,*** and found the beginning of the theory of bifurcations and the theory of the eigenvalues of nonlinear operators.

Birth of a new mechanics and the proof of incompleteness of the Newton mechanics. In 1771 L. Euler has derived a general equations of equilibrium of rods

$$\mathbf{N}' + \rho\mathbf{F} = \mathbf{0}, \quad \mathbf{M}' + \mathbf{R}' \times \mathbf{N} + \rho\mathbf{L} = \mathbf{0}. \tag{3}$$

To derive the equations (3) it was required to Euler about 50 years of reflections. As a result Euler has made one of the greatest opening in mechanics and physics, which to the full extent is not realized by the majority of mechanics and physicists up to present time. Namely, Euler has realized the necessity of the introduction of moments as independent objects, which can not be in terms of the moment of force. That means, firstly, necessity of the introductions of the new fundamental law of physics, expressed by the second equation (3) and, secondly, the fundamental incompleteness of the Newton Mechanics. Though L. Euler has made the determining step for introduction of the moments, independent of forces, but the general definition of the moment has been given rather recently by P.A. Zhilin.

Birth of the theory of stability of the nonconservative systems. In 1927 E.L. Nikolai has reported the results of the analysis of stability of the equilibrium configuration of the rod under the action of the twisting moment. He has shown, that this configuration is unstable at any as much as small value of the twisting moment (the Nikolai Paradox). The scientists of that time were shocked by this result for it was in sharp contrast with the conventional Euler's concept of critical forces. Then P.F. Papkovitch has specified, that the Nikolai problem deals with the nonconservative system. Therefore it is not necessary to be surprised to the obtained result because it is possible of accumulation of the energy in system. The subsequent development of the theory of stability of nonconservative systems has revealed also others surprising facts, for example, destabilizing role of internal friction. In the report it will be shown, that paradox of Nikolai is due to reasons which has no direct relation with the nonconservativeness of system. Nevertheless, the theory of stability of nonconservative systems now is one of the important branches of mechanics.

Birth of the symmetry theory in multi-oriented spaces. In 1977 P.A. Zhilin at construction of the constitutive equations in the theory of rods and shells has found out, that the application of the classical theory of symmetry leads to the absurd results. The analysis has shown, that the reason of the impasse is that fact, that the theory of rods and shells contain tensor's objects that is defined in spaces with two independent orientations. Therefore in such space there exist the tensors of four various types. The classical theory of symmetry is applicable only to the so-called polar tensors, i.e. to objects, independent of a choice of orientations in space. Thus it was necessary to develop the generalized theory of symmetry, which is valid for tensors of any types. Let us note that without this generalized theory of symmetry the correct construction of a general theory of rods and shells is impossible.

Above only those facts have been marked which have affected and continue to influence on the theoretical foundations of modern mechanics and mathematical physics. In the report there is no need to speak about enormous value of the rod theory for decision of actual problems of technics. Unfortunately, frameworks of the report do not allow to tell about remarkable achievements of many researchers at the decision of the very much interesting specific problems.

Unsolved questions of the rod theory. In the rod theory it is obtained a lot of surprising and even paradoxical results which demand clear explanations. Spatial forms of the rod motions are not almost investigated. Within the framework of the existing theory of rods it is very difficult to investigate the important problems for related dynamics of rods and, for example, rigid bodies as these two two important objects of mechanics are stated on various and incompatible languages. The main obstacle in a way of overcoming of all these difficulties is absence a general nonlinear theory of rods stated in language convenient for applications. The first presentation of such theory is one of the purposes of the report. Another, not less important, the purpose of the report is the discussion, from positions of the submitted theory, of some classical problems of the rod theory and revealing in them of the new circumstances latent in existing decisions. In particular, the new interpretation of the Nikolai paradox based on the full analysis of the Euler elastica will be given in the report. The author has solutions of a several new problems, but, unfortunately, is forced to leave them behind frameworks of the report.

2 The model of rod

The model of thin rod is the directed curve, which is defined by fixation of the vector $\mathbf{r}(s)$ and triple $\mathbf{d}_m$

$$\{\mathbf{r}(s),\ \mathbf{d}_1(s),\ \mathbf{d}_2(s),\ \mathbf{d}_3(s)\},\quad \mathbf{d}_m\cdot\mathbf{d}_n=\delta_{mn},\quad 0\le s\le l, \tag{4}$$

where s is the length of the curve arc, l — the length of curve. The vector $\mathbf{r}(s)$ in (4) determines the carrying curve with natural triple $\{\mathbf{t}_1\equiv\mathbf{t},\ \mathbf{t}_2\equiv\mathbf{n},\ \mathbf{t}_3\equiv\mathbf{b}=\mathbf{t}\times\mathbf{n}\}$, where the vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are unit vectors of the tangent, normal and binormal respectively. For natural triple one has

$$\mathbf{t}_i' = \boldsymbol{\tau} \times \mathbf{t}_i, \quad \boldsymbol{\tau}(s) = R_t^{-1}(s)\mathbf{t}(s) - R_c^{-1}(s)\mathbf{b}(s), \tag{5}$$

where R_c is the radius of curvature and R_t is the radius of twisting, $\boldsymbol{\tau}$ is the Darboux vector. Thus, in each point of the curve the two triples are given: natural triple $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and additional triple $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3 = \mathbf{t}\}$. The vectors $(\mathbf{n}, \mathbf{b})$ and $(\mathbf{d}_1, \mathbf{d}_2)$ are placed in the cross plane to the undeformed curve and determines the cross-section of the undeformed rod, but, in general, does not coincide. In what follows the vectors $(\mathbf{d}_1, \mathbf{d}_2)$ are principal axes of inertia of the cross-section. The changing of the triple $\mathbf{d}_k(s)$ under the motion along the curve is determined by the vector $\mathbf{q}(s)$ such that

$$\mathbf{d}_k'(s) = \mathbf{q}(s) \times \mathbf{d}_k(s). \tag{6}$$

It is easy to find the relation between $\mathbf{q}$ and $\boldsymbol{\tau}$

$$\mathbf{q} = \left(\varphi' + R_t^{-1}\right)\mathbf{t} - R_c^{-1}\mathbf{b} = \varphi'\mathbf{t} + \boldsymbol{\tau}, \tag{7}$$

where φ is called the angle the natural twisting of the rod.

The motion of the rod is defined by

$$\mathbf{r}(s) \quad \to \quad \mathbf{R}(s,t); \qquad \mathbf{d}_k(s) \quad \to \quad \mathbf{D}_k(s,t)$$

or

$$\mathbf{R}(s,t) = \mathbf{r}(s) + \mathbf{u}(s,t), \qquad \mathbf{D}_k(s,t) = \mathbf{P}(s,t)\cdot\mathbf{d}_k(s), \tag{8}$$

where $\mathbf{u}(s,t)$ is the displacement vector, $\mathbf{P}(s,t)$ is the turn-tensor. The translational velocity and angular velocity are defined by

$$\mathbf{V}(s,t) = \dot{R}(s,t), \quad \dot{\mathbf{P}}(s,t) = \boldsymbol{\omega}(s,t) \times \mathbf{P}(s,t), \quad \dot{f} \equiv df/dt. \tag{9}$$

If the turn-tensor $\mathbf{P}(s,t)$ is given, then

$$\boldsymbol{\omega}(s,t) = -\frac{1}{2}\left[\dot{\mathbf{P}}\cdot\mathbf{P}^T\right]_\times, \quad (\mathbf{a}\otimes\mathbf{b})_\times \equiv \mathbf{a}\times\mathbf{b}. \tag{10}$$

3 Fundamental laws of mechanics

The first and the second laws of dynamics by Euler have an almost conventional form

$$\mathbf{N}'(s,t) + \rho_0\mathcal{F}(s,t) = \rho_0\left(\mathbf{V} + \underline{\boldsymbol{\Theta}_1\cdot\boldsymbol{\omega}}\right)^{\cdot}, \tag{11}$$

$$\mathbf{M}' + \mathbf{R}'\times\mathbf{N} + \rho_0\mathcal{L} = \underline{\rho_0\mathbf{V}\times\boldsymbol{\Theta}_1\cdot\boldsymbol{\omega}} + \rho_0\left(\underline{\mathbf{V}\cdot\boldsymbol{\Theta}_1} + \boldsymbol{\Theta}_2\cdot\boldsymbol{\omega}\right)^{\cdot}, \tag{12}$$

where the underlined terms had been never taken into account, for the curved rods they are important.

Let us write down the energy balance equation (George Green, 1839)

$$\rho_0\dot{\mathcal{U}} = \mathbf{N}\cdot(\mathbf{V}' + \mathbf{R}'\times\boldsymbol{\omega}) + \mathbf{M}\cdot\boldsymbol{\omega}' + h' + \rho_0\mathcal{Q}, \tag{13}$$

Let the vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$ be the vector of extension-shear deformation and the vector of bending-twisting deformation respectively. They are defined as

$$\boldsymbol{\mathcal{E}} = \mathbf{R}' - \mathbf{P} \cdot \mathbf{t}, \quad \mathbf{P}' = \boldsymbol{\Phi} \times \mathbf{P}. \tag{14}$$

The Cartan equation

$$\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}} = \mathbf{V}' + \mathbf{R}' \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi} = \boldsymbol{\omega}'. \tag{15}$$

Putting (15) into (13), we obtain the energy balance equation in the next form

$$\rho_0 \dot{\mathcal{U}} = \mathbf{N} \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}} \right) + \mathbf{M} \cdot (\dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi}) + h' + \rho_0 \mathcal{Q}, \tag{16}$$

4 Reduced equation of the balance equation

The force $\mathbf{N}$ and the moment $\mathbf{M}$ in the rod may be represented as superposition of the elastic ($\mathbf{N}_e$, $\mathbf{M}_e$) and dissipative ($\mathbf{N}_d$, $\mathbf{M}_d$) terms

$$\mathbf{N} = \mathbf{N}_e(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \mathbf{P}) + \mathbf{N}_d(s,t), \quad \mathbf{M} = \mathbf{M}_e(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \mathbf{P}) + \mathbf{M}_d(s,t).$$

Let the parameter $\vartheta(s,t)$ is the temperature of the rod measured by some thermometer. That means that the temperature is the experimentally measured parameter. Let us introduce a new function η called the entropy. Let us define this function by the equation

$$\vartheta \dot{\eta} = h' + \rho_0 \mathcal{Q} + \mathbf{N}_d \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}} \right) + \mathbf{M}_d \cdot (\dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi}) \tag{17}$$

Let us point out that such definition of the entropy does not need in distinction between reversible and irreversible processes. Introduction of the entropy by the equality (17) is possible for any processes. The equality (17) is called the equation of the heat conduction.

Making use of (17) the energy balance equation (16) may be represented in the form

$$\rho_0 \dot{\mathcal{U}} = \mathbf{N}_e \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}} \right) + \mathbf{M}_e \cdot \left(\dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi} \right) + \vartheta \dot{\eta}. \tag{18}$$

The equation (18) is called ***the reduced energy balance equation***. Let us suppose that

$$\mathcal{U} = \mathcal{U}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \mathbf{P}, \eta).$$

It is clear that that the internal energy does not change under the superposition of rigid motions. Let us consider the two motions: $\mathbf{R}(s,t)$, $\mathbf{P}(s,t)$ and $\mathbf{R}_*(s,t)$, $\mathbf{P}_*(s,t)$, which are related by the equality

$$\mathbf{R}_*(s,t) - \mathbf{R}_*(\tilde{s},t) = \mathbf{Q}(\alpha) \cdot [\mathbf{R}(s,t) - \mathbf{R}(\tilde{s},t)], \quad \mathbf{P}_*(s,t) = \mathbf{Q}(\alpha) \cdot \mathbf{P}(s,t),$$

where $\mathbf{Q}(\alpha)$ is the set of properly orthogonal tensors, s and $\tilde{s}$ are two any points of the rod. It is easy to find that

$$\boldsymbol{\mathcal{E}}_*(s,t) = \mathbf{R}'_* - \mathbf{P}_* \cdot \mathbf{t} = \mathbf{Q}(\alpha) \cdot \boldsymbol{\mathcal{E}}(s,t),$$

$$\mathbf{\Phi}_*(s,t) = -\frac{1}{2}\left[\mathbf{P}'_* \cdot \mathbf{P}_*^{\mathsf{T}}\right]_\times = -\frac{1}{2}\left[\mathbf{Q} \cdot \mathbf{P}'_* \cdot \mathbf{P}_*^{\mathsf{T}} \cdot \mathbf{Q}^{\mathsf{T}}\right]_\times = \mathbf{Q}(\alpha) \cdot \mathbf{\Phi}(s,t).$$

Thus the internal energy must satisfy the next equality

$$\mathcal{U}(\boldsymbol{\mathcal{E}}_*, \mathbf{\Phi}_*, \mathbf{P}_*, \eta) = \mathcal{U}[\mathbf{Q}(\alpha) \cdot \boldsymbol{\mathcal{E}},\ \mathbf{Q}(\alpha) \cdot \mathbf{\Phi},\ \mathbf{Q}(\alpha) \cdot \mathbf{P},\ \eta] = \mathcal{U}(\boldsymbol{\mathcal{E}}, \mathbf{\Phi}, \mathbf{P}, \eta). \tag{19}$$

For the tensor $\mathbf{Q}(\alpha)$ we may accept

$$\frac{d}{d\alpha}\mathbf{Q}(\alpha) = \boldsymbol{\zeta}(\alpha) \times \mathbf{Q}(\alpha), \quad \mathbf{Q}(0) = \mathbf{E}, \quad \boldsymbol{\zeta}(0) = \boldsymbol{\omega}(t).$$

Differentiating the equality (19) with respect to α and accepting $\alpha = 0$, we have the equation

$$-\left(\frac{\partial \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}} \times \boldsymbol{\mathcal{E}} + \frac{\partial \mathcal{U}}{\partial \mathbf{\Phi}} \times \mathbf{\Phi}\right) \cdot \boldsymbol{\omega} + \left(\frac{\partial \mathcal{U}}{\partial \mathbf{P}}\right)^{\mathsf{T}} \cdot\cdot (\boldsymbol{\omega} \times \mathbf{P}) = 0. \tag{20}$$

Besides we have

$$\frac{d\mathcal{U}}{dt} = \frac{\partial \mathcal{U}}{\partial \eta}\dot{\eta} + \frac{\partial \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}} \cdot \dot{\boldsymbol{\mathcal{E}}} + \frac{\partial \mathcal{U}}{\partial \mathbf{\Phi}} \cdot \dot{\mathbf{\Phi}} + \left(\frac{\partial \mathcal{U}}{\partial \mathbf{P}}\right)^{\mathsf{T}} \cdot\cdot (\boldsymbol{\omega} \times \mathbf{P}) = 0.$$

Taking into account the equality (20) this equation may be rewritten as

$$\frac{d\mathcal{U}}{dt} = \frac{\partial \mathcal{U}}{\partial \eta}\dot{\eta} + \frac{\partial \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}} \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}\right) + \frac{\partial \mathcal{U}}{\partial \mathbf{\Phi}} \cdot \left(\dot{\mathbf{\Phi}} - \boldsymbol{\omega} \times \mathbf{\Phi}\right).$$

Putting this equality into (18) we obtain

$$\left(\frac{\partial \rho_0 \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}} - \mathbf{N}_e\right) \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}\right) + \left(\frac{\partial \rho_0 \mathcal{U}}{\partial \mathbf{\Phi}} - \mathbf{M}_e\right) \cdot \left(\dot{\mathbf{\Phi}} - \boldsymbol{\omega} \times \mathbf{\Phi}\right) + \\ + \left(\frac{\partial \rho_0 \mathcal{U}}{\partial \eta} - \vartheta\right)\dot{\eta} = 0. \tag{21}$$

The equation (21) must be valid for any processes and for arbitrary values of the vectors $\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}$ and $\dot{\mathbf{\Phi}} - \boldsymbol{\omega} \times \mathbf{\Phi}$. It is possible if and only if the Cauchy-Green formulas are valid

$$\mathbf{N}_e = \frac{\partial \rho_0 \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}}, \quad \mathbf{M}_e = \frac{\partial \rho_0 \mathcal{U}}{\partial \mathbf{\Phi}}, \quad \vartheta = \frac{\partial \rho_0 \mathcal{U}}{\partial \eta}. \tag{22}$$

Besides accepting in (19) $\mathbf{Q} = \mathbf{P}^{\mathsf{T}}$ we see that the intrinsic energy is the function of the next argument

$$\mathcal{U} = \mathcal{U}(\boldsymbol{\mathcal{E}}_\times,\ \mathbf{\Phi}_\times,\ \eta), \quad \boldsymbol{\mathcal{E}}_\times \equiv \mathbf{P}^{\mathsf{T}} \cdot \boldsymbol{\mathcal{E}}, \quad \mathbf{\Phi}_\times \equiv \mathbf{P}^{\mathsf{T}} \cdot \mathbf{\Phi}. \tag{23}$$

The vectors $\boldsymbol{\mathcal{E}}_\times$ and $\mathbf{\Phi}_\times$ are called the energetic vectors of deformation.

Axisymmetrical vibrations of a ring.

$$\mathbf{R}(s,t) = [a + w(t)]\mathbf{n}(s), \quad \mathbf{P}(s,t) = \mathbf{E} \quad \Rightarrow \quad \mathbf{V} = \dot{w}(t)\mathbf{n}(s),\ \boldsymbol{\omega} = \mathbf{0},$$

where a is the radius of undeformed ring. Let us suppose that

$$\mathcal{F} = \mathbf{N}_d = \mathbf{0}, \quad \mathcal{L} = \mathbf{M}_d = \mathbf{0}, \quad \mathcal{Q} = 0, \quad \vartheta = \text{const}, \quad \eta = \text{const}.$$

The principal axes of inertia of the cross-section of the ring do not coincide with the vectors $\mathbf{n}$ and $\mathbf{b}$

$$\mathbf{d}_1 = \cos\alpha\,\mathbf{n} + \sin\alpha\,\mathbf{b}, \quad \mathbf{d}_2 = -\sin\alpha\,\mathbf{n} + \cos\alpha\,\mathbf{b}. \tag{24}$$

Let us calculate the inertial terms in (11) and (12)

$$\rho_0\dot{\mathbf{V}} = \tilde{\rho}F\ddot{w}\mathbf{n} = -\tilde{\rho}Fa\ddot{w}\,\mathbf{t}', \quad \rho_0\dot{\mathbf{V}}\cdot\mathbf{\Theta}_1 = -\ddot{w}\,\mathbf{n}\times\mathbf{d} = -\lambda\ddot{w}\,\mathbf{t} = -a\lambda\ddot{w}\,\mathbf{n}',$$

where

$$\lambda = \tilde{\rho}\frac{\sin 2\alpha}{2a}\int\limits_{(F)}(x^2-y^2)dxdy.$$

The equations of motion (11) and (12) takes a form

$$[\mathbf{N}(s,t) + \tilde{\rho}Fa\ddot{w}(t)\mathbf{t}(s)]' = \mathbf{0},$$

$$\left[\mathbf{M} - \frac{\tilde{\rho}F}{24}(H^2-h^2)\sin 2\alpha\ddot{w}(t)\mathbf{n}(s)\right]' + \left(1+\frac{\ddot{w}(t)}{a}\right)\mathbf{t}\times\mathbf{N} = \mathbf{0}.$$

From this it follows

$$\mathbf{N}(s,t) = -\tilde{\rho}Fa\ddot{w}(t)\mathbf{t}(s), \quad \mathbf{M} = \frac{\tilde{\rho}F}{24}(H^2-h^2)\sin 2\alpha\ddot{w}(t)\mathbf{n}(s). \tag{25}$$

The first equation in (25) gives the equation of nonlinear oscillator

$$\ddot{w}(t) + f(w) = 0,$$

where the function $f(w)$ is determined by the intrinsic energy. From the eq. (25) the universal constraint

$$24a\,\mathbf{M}\cdot\mathbf{n} + (H^2-h^2)\sin 2\alpha\,\mathbf{N}\cdot\mathbf{t} = 0, \tag{26}$$

follows. This constraint must be valid for any definition of the intrinsic energy. The existing versions of the rod theory do not satisfy constraint (26).

Paradox. It is obvious that the tensor of mirror reflection $\mathbf{Q} = \mathbf{E} - 2\,\mathbf{t}\otimes\mathbf{t}$ must belong to the symmetry grope for all quantities in this problem. However for the vector $\mathbf{N}$ we have $\mathbf{Q}\cdot\mathbf{N} = -\mathbf{N} \neq \mathbf{N}$, i.e. $\mathbf{Q}$ does not belong to the symmetry grope of $\mathbf{N}$. The classical theory of symmetry does not work!

5 The specification of the internal energy

In what follow we shall consider the isothermal processes. The intrinsic energy may be defined as quadratic form

$$\rho_0\mathcal{U}(\mathcal{E}_\times,\,\mathbf{\Phi}_\times) = \mathcal{U}_0 + \mathbf{N}_0\cdot\mathcal{E}_\times + \mathbf{M}_0\cdot\mathbf{\Phi}_\times +$$
$$+\frac{1}{2}\underline{\underline{\mathcal{E}_\times\cdot\mathbf{A}\cdot\mathcal{E}_\times}} + \mathcal{E}_\times\cdot\mathbf{B}\cdot\mathbf{\Phi}_\times + \frac{1}{2}\underline{\mathbf{\Phi}_\times\cdot\mathbf{C}\cdot\mathbf{\Phi}_\times} + \mathbf{\Phi}_\times\cdot(\mathcal{E}_\times\cdot\mathbf{D})\cdot\mathbf{\Phi}_\times, \tag{27}$$

where the vectors $\mathbf{N}_0$, $\mathbf{M}_0$, second rank tensors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and third rank tensor $\mathbf{D}$ are defined in the reference configuration and are called the elasticity tensors.

The main problem is to find the elasticity tensors. If we take into account only underlined term in (27), then we obtain the classical theory of rods. In some modern versions of the rod theory the twice underlined term is taken into account. All other terms are absent in the existing theories. However, as it will be shown below, no one term in the representation (27) can not be omitted without contradictions. The representation (27) may be rewritten in terms of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$

$$\rho_0 \mathcal{U}(\boldsymbol{\mathcal{E}}_\times, \boldsymbol{\Phi}_\times) = \mathcal{U}_0 + \tilde{\mathbf{N}}_0 \cdot \boldsymbol{\mathcal{E}} + \tilde{\mathbf{M}}_0 \cdot \boldsymbol{\Phi} + \\ + \frac{1}{2}\boldsymbol{\mathcal{E}} \cdot \tilde{\mathbf{A}} \cdot \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}} \cdot \tilde{\mathbf{B}} \cdot \boldsymbol{\Phi} + \frac{1}{2}\boldsymbol{\Phi} \cdot \tilde{\mathbf{C}} \cdot \boldsymbol{\Phi} + \boldsymbol{\Phi} \cdot (\boldsymbol{\mathcal{E}} \cdot \tilde{\mathbf{D}}) \cdot \boldsymbol{\Phi}, \quad (28)$$

where

$$(\tilde{\mathbf{N}}_0,\, \tilde{\mathbf{M}}_0) = \mathbf{P} \cdot (\mathbf{N}_0,\, \mathbf{M}_0), \quad (\tilde{\mathbf{A}},\, \tilde{\mathbf{B}},\, \tilde{\mathbf{C}}) = \mathbf{P} \cdot (\mathbf{A},\, \mathbf{B},\, \mathbf{C}) \cdot \mathbf{P}^T, \quad \tilde{\mathbf{D}} = \overset{3}{\underset{1}{\otimes}} \mathbf{P} \odot \mathbf{D}$$

are defined in the actual configuration. Here and in what follows the notation

$$\overset{k}{\underset{1}{\otimes}} \mathbf{P} \odot \mathbf{S} \equiv \overset{k}{\underset{1}{\otimes}} \mathbf{P} \odot \left(S^{i_1 \ldots i_k} \mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_k} \right) \equiv S^{i_1 \ldots i_k} \mathbf{P} \cdot \mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{P} \cdot \mathbf{e}_{i_k}$$

is used for the tensor $\mathbf{S}$ of the rank k.

Let us consider the generalized theory of the tensor symmetry [1]. In our case oriented 3d-space $E_3^{(o)}$ is the direct sum of oriented 2d-space $E_2^{(o)}$ and oriented 1d-space $E_1^{(o)}$

$$E_3^{(o)} = E_1^{(o)} \oplus E_2^{(o)}.$$

Orientations in $E_3^{(o)}$ and $E_1^{(o)}$ may be chosen independently.

Definition: *objects that do not depend on the choice of orientation in $E_3^{(o)}$ and $E_1^{(o)}$ are called polar ones; objects that depend on the choice of orientation in $E_3^{(o)}$ and do not depend on the choice of orientation in $E_1^{(o)}$ are called axial ones; objects that do not depend on the choice of orientation in $E_3^{(o)}$ but depend on the choice of orientation in $E_1^{(o)}$ are called polar t-oriented ones; objects that depend on the choice of orientation both in $E_3^{(o)}$ and in $E_1^{(o)}$ are called axial t-oriented ones.*

In theory under consideration: ρ_0, ϑ, η, $\mathcal{U}$, $\mathbf{r}$, $\mathbf{R}$, $\mathbf{u}$, $\boldsymbol{\mathcal{F}}$, $\mathbf{a}_c$, $\mathbf{d}$, $\mathbf{P}$, $\boldsymbol{\Theta}_2$, $\mathbf{A}$, $\mathbf{C}$ are polar objects; R_t, $\boldsymbol{\psi}$, $\boldsymbol{\omega}$, $\boldsymbol{\mathcal{L}}$, $\boldsymbol{\Theta}_1$, $\mathbf{B}$ are axial objects; R_c, $\mathbf{N}_0, \mathbf{N}$, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{E}}_\times$, $\mathbf{D}$ are polar t-oriented objects; $\mathbf{q}$, $\boldsymbol{\tau}$, $\mathbf{M}_0, \mathbf{M}$, $\boldsymbol{\Phi}$, $\boldsymbol{\Phi}_\times$ are axial t-oriented objects. Let us note that the differentiation with respect to s changes the type of an object. For example, $\mathbf{N}$ is the polar t-oriented vector but $\mathbf{N}'$ is the polar vector.

Definition: *the k-rank tensor $\mathbf{S}'$ is called orthogonal transformation of the k-rank tensor $\mathbf{S}$ and is defined as*

$$\mathbf{S}' \equiv (\mathbf{t} \cdot \mathbf{Q} \cdot \mathbf{t})^\beta (\det\, \mathbf{Q})^\alpha \overset{k}{\underset{1}{\otimes}} \mathbf{Q} \odot \mathbf{S}, \quad (29)$$

where $\alpha = 0$, $\beta = 0$, *if* $\mathbf{S}$ *is a polar tensor;* $\alpha = 1$, $\beta = 0$, *if* $\mathbf{S}$ *is an axial tensor;* $\alpha = 0$, $\beta = 1$, *if* $\mathbf{S}$ *is a polar* t*-oriented tensor;* $\alpha = 1$, $\beta = 1$, *if* $\mathbf{S}$ *is an axial* t*-oriented tensor.*

Definition: *the set of the orthogonal solutions of the equation*

$$\mathbf{S}' = \mathbf{S}, \tag{30}$$

is called the symmetry grope of the tensor $\mathbf{S}$, *where* $\mathbf{S}$ *is given and orthogonal tensors* $\mathbf{Q}$ *must be found. The* $\mathbf{S}'$ *is defined by (29).*

Now we are able to explain paradox of the previous section. Vector $\mathbf{N}$ is a polar t-oriented vector. Therefore its symmetry grope must be found from the equation

$$(\mathbf{t} \cdot \mathbf{Q} \cdot \mathbf{t})\, \mathbf{Q} \cdot \mathbf{N} = \mathbf{N}.$$

It is easy to see that the tensor of mirror reflection $\mathbf{Q} = \mathbf{E} - 2\,\mathbf{t} \otimes \mathbf{t}$ belongs to the symmetry grope of $\mathbf{N}$ accordingly to the definition (30).

The requirements of symmetry are necessary tools. However they are not sufficient in order to construct the elasticity tensors. The latter depend on many factors. Even in the simplest case, when the rod made of isotropic material, the elasticity tensors depend on the shape of rod, i.e. on vectors Darboux $\boldsymbol{\tau}$ and $\mathbf{q}$ or, what is the same, on vector $\boldsymbol{\tau}$ and on the intensity of angle of natural twisting φ'. If the diameter of the cross-section is chosen as an unit of length, then the modulus of the vector $\boldsymbol{\tau}$ is a small quantity. By this reason it is possible to use the decomposition

$$\mathbf{f} = \mathbf{f}_0 + \mathbf{f}_1 \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{f}_2 \cdot \boldsymbol{\tau},$$

where $\mathbf{f}$ is any tensor of elasticity.

Making use of this technics one may obtain

$$\mathbf{A} = A_1 \mathbf{d}_1\, \mathbf{d}_1 + A_2 \mathbf{d}_2\, \mathbf{d}_2 + A_3 \mathbf{d}_3\, \mathbf{d}_3 + \frac{A_{12}}{R_t} (\mathbf{d}_1\, \mathbf{d}_2 + \mathbf{d}_2\, \mathbf{d}_1) + \\ + \frac{1}{R_c} \left[A_{13} (\mathbf{d}_1\, \mathbf{d}_3 + \mathbf{d}_3\, \mathbf{d}_1) \cos\alpha + A_{23} (\mathbf{d}_2\, \mathbf{d}_3 + \mathbf{d}_3\, \mathbf{d}_2) \sin\alpha\right], \tag{31}$$

where the meaning of the angle α is defined by (24), $\mathbf{d}_3 \equiv \mathbf{t}$, $\mathbf{a}\,\mathbf{b} \equiv \mathbf{a} \otimes \mathbf{b}$. The representation for $\mathbf{C}$

$$\mathbf{C} = C_1 \mathbf{d}_1\, \mathbf{d}_1 + C_2 \mathbf{d}_2\, \mathbf{d}_2 + C_3 \mathbf{d}_3\, \mathbf{d}_3 + \frac{C_{12}}{R_t} (\mathbf{d}_1\, \mathbf{d}_2 + \mathbf{d}_2\, \mathbf{d}_1) + \\ + \frac{1}{R_c} \left[C_{13} (\mathbf{d}_1\, \mathbf{d}_3 + \mathbf{d}_3\, \mathbf{d}_1) \cos\alpha + C_{23} (\mathbf{d}_2\, \mathbf{d}_3 + \mathbf{d}_3\, \mathbf{d}_2) \sin\alpha\right]. \tag{32}$$

If the natural twisting is absent, then

$$A_{12} = A_{13} = A_{23} = 0, \quad C_{12} = C_{13} = C_{23} = 0.$$

A general representation for $\mathbf{B}$

$$\mathbf{B} = \varphi' B_0 \mathbf{t}\,\mathbf{t} + \frac{1}{R_t}\left[B_1 \mathbf{d}_1 \mathbf{d}_1 + B_2 \mathbf{d}_2 \mathbf{d}_2 + B_3 \mathbf{t}\,\mathbf{t} + \varphi' \left(b_1 \mathbf{d}_1 \mathbf{d}_1 + b_2 \mathbf{d}_2 \mathbf{d}_2\right) \times \mathbf{t}\right] + \\ + \frac{1}{R_c}\left[\left(B_{13} \mathbf{d}_1 \sin\alpha + B_{23} \mathbf{d}_2 \cos\alpha\right) \mathbf{t} + \mathbf{t}\left(B_{31} \mathbf{d}_1 \sin\alpha + B_{32} \mathbf{d}_2 \cos\alpha\right)\right] + \\ + \frac{\varphi'}{R_c}\left[\left(b_{13} \mathbf{d}_1 \cos\alpha + b_{23} \mathbf{d}_2 \sin\alpha\right) \mathbf{t} + \mathbf{t}\left(b_{31} \mathbf{d}_1 \cos\alpha + b_{32} \mathbf{d}_2 \sin\alpha\right)\right]. \quad (33)$$

If the natural twisting is absent, then $\varphi' = 0$. Not all elastic modulus in (33) are important. In order to see that fact let us write down the expression

$$\mathcal{E} \cdot \mathbf{B} = \varphi' B_0 \,\epsilon \mathbf{t} + \frac{1}{R_t}\left[B_1 \Gamma_1 \mathbf{d}_1 + B_2 \Gamma_2 \mathbf{d}_2 + B_3 \epsilon\, \mathbf{t} + \varphi' \left(b_1 \Gamma_1 \mathbf{d}_1 + b_2 \Gamma_2 \mathbf{d}_2\right) \times \mathbf{t}\right] + \\ + \frac{1}{R_c}\left[\left(B_{13} \Gamma_1 \sin\alpha + B_{23} \Gamma_2 \cos\alpha\right) \mathbf{t} + \epsilon\left(B_{31} \mathbf{d}_1 \sin\alpha + B_{32} \mathbf{d}_2 \cos\alpha\right)\right] + \\ + \frac{\varphi'}{R_c}\left[\left(b_{13} \Gamma_1 \cos\alpha + b_{23} \Gamma_2 \sin\alpha\right) \mathbf{t} + \epsilon\left(b_{31} \mathbf{d}_1 \cos\alpha + b_{32} \mathbf{d}_2 \sin\alpha\right)\right]. \quad (34)$$

Because the shear deformations Γ_1, Γ_2 are as a rule small then instead of (34) one may write

$$\mathcal{E} \cdot \mathbf{B} = \varphi' B_0 \,\epsilon \mathbf{t} + \frac{\epsilon}{R_t} B_3 \,\mathbf{t} + \\ + \frac{\epsilon}{R_c}\left(B_{31} \mathbf{d}_1 \sin\alpha + B_{32} \mathbf{d}_2 \cos\alpha\right) + \frac{\epsilon\varphi'}{R_c}\left(b_{31} \mathbf{d}_1 \cos\alpha + b_{32} \mathbf{d}_2 \sin\alpha\right). \quad (35)$$

Thus we see that only modulus B_0, B_3, B_{31}, B_{32}, b_{31}, b_{32} may be important. More over it is clear from physical sense that modulus b_{31}, b_{32} may be ignored as well. Thus instead of (33) one may write down

$$\mathbf{B} = \varphi' B_0 \mathbf{t}\,\mathbf{t} + \frac{B_3}{R_t} \mathbf{t}\,\mathbf{t} + \frac{1}{R_c} \mathbf{t}\left(B_{31} \mathbf{d}_1 \sin\alpha + B_{32} \mathbf{d}_2 \cos\alpha\right).$$

This technology does not suit in order to find the vectors $\mathbf{N}_0$ and $\mathbf{M}_0$, which linearly depend on the external loads. As a rule these vectors are not important.

6 The determination of the elastic modulus

At the present time all elastic modulus have been found. Let us show how to find the elastic modulus A_1, A_2, A_3. It is easy to prove the representations

$$A_3 = E\,F, \quad A_1 = k_1\, G\, F, \quad A_2 = k_2\, G\, F, \quad (36)$$

where E is the Yang modulus, $G = E/2(1+\nu)$ is the shear modulus of the material of rod.

Dimensionless coefficients k_1 and k_2 in (36) are called the shear correction factors. There are many different values for these factors, but all of them must satisfy the inequality

$$\pi^2/12 \leq k_1,\, k_2 < 1.$$

In order to illustrate the determination of shear correction factor let us consider the next dynamics problem of 3d-theory of elasticity for the body occupying the domain: $-h/2 \leq x \leq h/2$, $-H/2 \leq y \leq H/2$, $0 \leq z \leq l$. Let us accept that $\mathbf{i} = \mathbf{d}_1$, $\mathbf{j} = \mathbf{d}_2$, $\mathbf{k} = \mathbf{t}$. Let the lateral surface of the body is free. The boundary conditions are determined as

$$z = 0,\, l: \quad \mathbf{u}_{(3)} \cdot \mathbf{d}_1 = \mathbf{u}_{(3)} \cdot \mathbf{d}_2 = 0, \quad \mathbf{t} \cdot \mathbf{T} \cdot \mathbf{t} = 0,$$

where $\mathbf{u}_{(3)}$ and $\mathbf{T}$ are the vector of displacement and the stress tensor respectively. Let us consider the shear vibrations of the form

$$\mathbf{u}_{(3)} = W e^{i\omega t} \sin \lambda x\, \mathbf{t}, \quad \mathbf{T} = G\lambda W e^{i\omega t} \cos \lambda x (\mathbf{t}\, \mathbf{d}_1 + \mathbf{d}_1\, \mathbf{t}), \quad \lambda = (2k+1)\pi/h,$$

where ω is the natural frequency of the body. These expressions satisfy the boundary conditions. To satisfy the equations of motion we have to accept

$$\boldsymbol{\nabla} \cdot \mathbf{T} = \tilde{\rho} \ddot{\mathbf{u}}_{(3)} \quad \Rightarrow \quad \omega^2 = \frac{G}{\tilde{\rho}} \frac{(2k+1)^2 \pi^2}{h^2}, \quad k = 0, 1, 2, \ldots \tag{37}$$

Let us consider this in the framework of the beam theory. We have

$$\mathbf{u} = \mathbf{0}, \quad \boldsymbol{\psi} = \psi_2 \mathbf{d}_2 = const, \quad \mathbf{N} = N_1 \mathbf{d}_1, \quad \mathbf{M} = \mathbf{0}, \quad \mathbf{N}_0 = \mathbf{0}, \quad \mathbf{M}_0 = \mathbf{0};$$

$$\mathbf{e} \equiv \mathbf{u}' + \mathbf{t} \times \boldsymbol{\psi} = -\psi_2 \mathbf{d}_1, \quad \boldsymbol{\kappa} \equiv \boldsymbol{\psi}' = \mathbf{0}, \quad \mathbf{N} = -A_1 \psi_2 \mathbf{d}_1, \quad \mathbf{M} = \mathbf{0}.$$

The equation of motion takes a form

$$\mathbf{N}'(s,t) = \mathbf{0}, \quad -A_1 \psi_2 \mathbf{d}_2 = \Theta_2 \ddot{\psi}_2 \mathbf{d}_2 \;\Rightarrow\; \omega^2 = A_1/\Theta_2, \quad \Theta_2 = \tilde{\rho} F\, h^2/12. \tag{38}$$

Comparing the frequencies found in terms of the three-dimensional theory (37), and the frequency found under the theory of beam (38), we see huge distinction. The three-dimensional theory gives the spectrum of the natural frequencies while the beam theory gives only one frequency. It is not surprising, for area of applicability of the three-dimensional theory is much more wider than area of applicability of the beam theory. The beam theory gives a good description only low-frequency vibrations. Let us note, that shift vibrations are already high-frequency vibrations, their frequencies trend to infinity at $h \to 0$. While frequencies of bending vibrations trend to zero at $h \to 0$, and frequencies of longitudinal vibrations are limited at $h \to 0$. Therefore it is quite natural, that the beam theory does not allow to describe all shift spectrum, but it can describe the lowest frequency from a spectrum (37). For this end it is enough to accept

$$\frac{A_1}{\Theta_2} = \frac{G}{\tilde{\rho}} \frac{\pi^2}{h^2} \quad \Rightarrow \quad A_1 = \frac{\pi^2}{12} GF \quad \Rightarrow \quad k_1 = \frac{\pi^2}{12}.$$

It may be proved that $k_1 = k_2$.

It is useful to consider the certain seeming paradox connected to definition of shear coefficient. We shall try to determine it from the exact decision of a static problem on pure shift of a beam, which is given by formulas

$$\mathbf{T} = \tau\,(\mathbf{t}\,\mathbf{d}_1 + \mathbf{d}_1\,\mathbf{t}), \quad G\mathbf{u}_{(3)} = \tau\, x\,\mathbf{t} \quad \Rightarrow$$

$$\Rightarrow \quad \mathbf{N} = \tau\, F\,\mathbf{d}_1, \quad \mathbf{M} = \mathbf{0}, \quad \mathbf{u} = \mathbf{0}, \quad G\,\boldsymbol{\psi} = -\tau\,\mathbf{d}_2.$$

From the other hand we have

$$\mathbf{N} = \mathbf{A}\cdot(\mathbf{t}\times\boldsymbol{\psi}) \quad \Rightarrow \quad \tau\, F = -A_1\,\mathbf{d}_2\cdot\boldsymbol{\psi} \quad \Rightarrow \quad k_1 = 1. \tag{39}$$

Namely this value of shear coefficient was obtained by M. Rubin (2003). His arguments are based on the solution (39). Thus we obtain a theoretical paradox: from two exact solution we obtain two different values of shear coefficient. The existing beam theory are not able to explain this paradox.

In fact the solution of this paradox is very simple. Let us consider the expression (27). It contains the vectors $\mathbf{N}_0$ and $\mathbf{M}_0$, which are linear functions of loads acting on lateral surface of beam. Because of this the equality (39) must be written as

$$\mathbf{N} = \mathbf{N}_0 + \mathbf{A}\cdot(\mathbf{t}\times\boldsymbol{\psi}) \quad \Rightarrow \quad \mathbf{N}_0 = \tau\, F\,(1-k_1)\mathbf{d}_1.$$

Therefore the problem on pure shear does not allow to calculate the shear coefficient.

The elastic modulus C_3, C_1, C_2 are well known

$$C_1 = E\,J_1, \quad C_2 = E\,J_2, \quad J_1 \equiv \int\limits_{(F)} y^2 dxdy, \quad J_2 \equiv \int\limits_{(F)} x^2 dxdy, \tag{40}$$

$$C_3 = G\,J_r, \quad J_r = 2\int\limits_{(F)} U(x,y)dxdy, \quad \triangle U = -2, \quad U = 0 \text{ on } \partial F. \tag{41}$$

Let us consider the tensor of elasticity $\mathbf{B}$. In known versions of the rod theory we have

$$B_{31} = 0, \quad B_{32} = 0, \quad B_3 = 0.$$

However the representation (34) and universal equality (26) it follows

$$B_{32} = EJ_4 + B_{31}, \quad J_4 \equiv \int (x^2 - y^2)dxdy \neq 0. \tag{42}$$

Thus the conditions $B_{32} = B_{31} = 0$ are impossible. The next formulas may be proved

$$B_0 = E\,(J_1 + J_2 - J_r) \geq 0, \quad B_{32} = C_2, \quad B_{31} = C_1. \tag{43}$$

The above presented rod theory is consistent nonlinear theory with very wide branch of applicability. At present author does not know the problems when this theory leads to some contradictions or mistakes.

7 The longitudinal-twisting waves in the rod

Let us consider the longitudinal-twisting waves in the naturally twisted beam.

$$\frac{\partial^2 u}{\partial s^2} - \frac{1}{c_l^2}\frac{\partial^2 u}{\partial t^2} + \frac{\varphi' B_0}{EF}\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{c_l^2}\mathcal{F}_t = 0, \qquad c_l^2 = \frac{E}{\rho}. \tag{44}$$

$$\frac{\partial^2 \psi}{\partial s^2} - \frac{1}{c_t^2}\frac{\partial^2 \psi}{\partial t^2} + \frac{\varphi' B_0}{GJ_r}\frac{\partial^2 u}{\partial s^2} + \frac{\rho F}{GJ_r}\mathcal{L}_t = 0, \quad c_t^2 = \frac{GJ_r}{\rho J_p}. \tag{45}$$

The solution of the system (44)–(45) may be represented in terms of solutions of the wave equations

$$\frac{\partial^2 \nu}{\partial s^2} - \frac{1}{\Omega_1}\frac{\partial^2 \nu}{\partial t^2} = 0, \quad \frac{\partial^2 \vartheta}{\partial s^2} - \frac{1}{\Omega_2}\frac{\partial^2 \vartheta}{\partial t^2} = 0, \tag{46}$$

where Ω_1 and Ω_2 are some parameters, which must be found. A general solution of the system (44)–(45) has a form

$$u(s,t) = \nu(s,t) + \frac{\gamma_1 c_l^2}{\Omega_2 - c_l^2}\vartheta(s,t), \quad \psi(s,t) = \vartheta(s,t) + \frac{\gamma_2 c_t^2}{\Omega_1 - c_t^2}\nu(s,t), \tag{47}$$

where

$$\gamma_1 \equiv \frac{\varphi' B_0}{EF}, \quad \gamma_2 \equiv \frac{\varphi' B_0}{GJ_r},$$

ν and ϑ are solutions of (46). The parameters Ω_1 and Ω_2 are the roots of equation

$$\Omega_1^2 - (c_l^2 + c_t^2)\Omega_1 + (1 - \gamma_1\gamma_2)c_l^2 c_t^2 = 0, \quad \Omega_2 < c_t^2, \quad \Omega_1 > c_l^2, \quad c_t^2 < c_l^2.$$

So, the presence of natural twisting in a beam does not change a character of wave process in the beam. It still waves without a dispersion, but the presence of natural twisting changes velocities of wave propagation in a beam. The longitudinal - torsional wave is the solution of the first equation from (46), and the velocity of its propagation $\sqrt{\Omega_1}$ is bigger than the velocity of propagation of longitudinal wave in a beam without natural twisting. The torsional-longitudinal wave is the solution of second equation from (46), and the velocity of its propagation $\sqrt{\Omega_2}$ appears below velocity of propagation of a wave of torsion in a beam without natural twisting.

8 The twisting of a beam by the dead moments

In order to show how to work with presented rod theory let us consider the task of twisting of beam by the dead moment when external surface loads in (11)–(12) are absent, i.e. $\mathbf{F} = \mathbf{0}$, $\mathbf{L} = \mathbf{0}$. The equation of equilibrium

$$\mathbf{N}'(s,t) = \mathbf{0}, \quad \mathbf{M}' + \mathbf{R}' \times \mathbf{N} = \mathbf{0}. \tag{48}$$

The boundary conditions

$$s = 0: \ \mathbf{R} = \mathbf{0}, \quad \mathbf{P} = \mathbf{E}; \qquad s = l: \ \mathbf{N} = \mathbf{0}, \quad \mathbf{M} = \mathbf{L} \equiv L\,\mathbf{m}, \tag{49}$$

where $\mathbf{L} = \mathsf{const}$ and $\mathbf{m}$ is unit constant vector. Solution of static equations (48) taking into account boundary conditions (49)

$$\mathbf{N} = \mathbf{0}, \quad \mathbf{M} = \mathbf{L} = L\,\mathbf{m}. \tag{50}$$

Cauchy-Green relations of naturally twisted beam

$$\mathbf{N} = \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{P}^{T} \cdot \mathcal{E} + \varphi' B_0 (\mathbf{t} \cdot \mathbf{P}^{T} \cdot \mathbf{\Phi}) \mathbf{P} \cdot \mathbf{t},$$

$$\mathbf{M} = \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^{T} \cdot \mathbf{\Phi} + \varphi' B_0 (\mathcal{E} \cdot \mathbf{P} \cdot \mathbf{t}) \mathbf{P} \cdot \mathbf{t},$$

where φ' is the natural twisting of beam: $\varphi = 2\pi s/a$, a is a length on which the cross-section is turning by the angle 2π. We see that

$$\mathcal{E} = -\left(\frac{\varphi' B_0}{A_3}\right) (\mathbf{\Phi} \cdot \mathbf{P} \cdot \mathbf{t}) \mathbf{P} \cdot \mathbf{t} \quad \Rightarrow \quad \mathbf{R}' = \left(1 - \frac{\varphi' B_0}{A_3} \mathbf{\Phi} \cdot \mathbf{P} \cdot \mathbf{t}\right) \mathbf{P} \cdot \mathbf{t}, \tag{51}$$

$$\mathbf{L} = \mathbf{P} \cdot [C_t \mathbf{t}\,\mathbf{t} + C_1 \mathbf{d}_1\,\mathbf{d}_1 + C_2 \mathbf{d}_2\,\mathbf{d}_2] \cdot \mathbf{P}^{T} \cdot \mathbf{\Phi}, \quad C_t \equiv C_3 \left(1 - \frac{\varphi'^2 B_0^2}{C_3\,A_3}\right). \tag{52}$$

Let us accept that $C_1 = C_2$. Then from (52) it follows

$$\mathbf{\Phi} = \mathbf{P} \cdot [C_t^{-1} \mathbf{t}\,\mathbf{t} + C_1^{-1} (\mathbf{E} - \mathbf{t}\,\mathbf{t})] \cdot \mathbf{P}^{T} \cdot \mathbf{L}, \quad \mathbf{P}' = \mathbf{\Phi} \times \mathbf{P}. \tag{53}$$

The system (53) has a first integral

$$\mathbf{\Phi} \cdot \mathbf{L} = \mathbf{L} \cdot \mathbf{P} \cdot [C_t^{-1} \mathbf{t}\,\mathbf{t} + C_1^{-1} (\mathbf{E} - \mathbf{t}\,\mathbf{t})] \cdot \mathbf{P}^{T} \cdot \mathbf{L} = \mathsf{const}. \tag{54}$$

The energy integral (54) is the constraint on the turn-tensor $\mathbf{P}$. A general solution of (54) has a form

$$\mathbf{P}(t) = \mathbf{Q}(\alpha\,\mathbf{m}) \cdot \mathbf{Q}(\beta\,\mathbf{t}), \tag{55}$$

where notation

$$\mathbf{Q}(\gamma\,\mathbf{p}) \equiv (1 - \cos\gamma) \mathbf{p}\ \mathbf{p} + \cos\gamma\,\mathbf{E} + \sin\gamma\ \mathbf{p} \times \mathbf{E}$$

is used for rotation by the angle γ around unit vector $\mathbf{p}$. For any $\alpha(s)$ and $\beta(s)$ the energy (54) keeps a constant value. Making use of (55), the system (53) rewrite in a form

$$\mathbf{\Phi} = \mathbf{Q}(\alpha\,\mathbf{m}) \cdot [C_t^{-1} \mathbf{t}\,\mathbf{t} + C_1^{-1} (\mathbf{E} - \mathbf{t}\,\mathbf{t})] \cdot \mathbf{L}, \quad \mathbf{\Phi} = \mathbf{Q}(\alpha\,\mathbf{m}) \cdot (\alpha'\,\mathbf{m} + \beta'\,\mathbf{t})$$

or

$$\alpha'(s)\,\mathbf{m} + \beta'(s)\,\mathbf{t} = L\,[C_t^{-1} \mathbf{t}\,\mathbf{t} + C_1^{-1} (\mathbf{E} - \mathbf{t}\,\mathbf{t})] \cdot \mathbf{m} = L\,[(C_t^{-1} - C_1^{-1}) \cos\sigma\ \mathbf{t} + C_1^{-1}\,\mathbf{m}].$$

The solution of this system

$$\alpha'(s) = L\,C_1^{-1}, \quad \beta'(s) = L\,(C_t^{-1} - C_1^{-1}) \cos\sigma, \quad \cos\sigma \equiv \mathbf{m} \cdot \mathbf{t}. \tag{56}$$

From (56) it follows

$$\alpha(s) = L\,C_1^{-1} s, \quad \beta(s) = L \cos\sigma\,(C_t^{-1} - C_1^{-1})\,s.$$

It is easy to calculate

$$\mathbf{\Phi}\cdot\mathbf{P}\cdot\mathbf{t} = \frac{L\cos\sigma}{C_t}. \tag{57}$$

This is a variation of twisting of the beam

$$\tilde{\mathbf{q}}\cdot\mathbf{P}\cdot\mathbf{t} = \varphi' + \mathbf{\Phi}\cdot\mathbf{P}\cdot\mathbf{t}.$$

The axis extension follows from (51)

$$\varepsilon = \boldsymbol{\mathcal{E}}\cdot\mathbf{P}\cdot\mathbf{t} = -\frac{\varphi' B_0}{A_3}\frac{L\cos\sigma}{C_t}. \tag{58}$$

If $\varphi' L > 0$, then $\varepsilon < 0$. If $\varphi' L < 0$, then $\varepsilon > 0$.

Let us calculate the Darboux vector, curvature and twisting of deformed beam

$$\tilde{\tau} = \alpha'\cos\sigma\,\tilde{\mathbf{t}} - \alpha'\sin\sigma\,\tilde{\mathbf{b}} = \alpha'\mathbf{m} \quad \Rightarrow \quad \tilde{R}_c^{-1} = \alpha'\sin\sigma, \quad \tilde{R}_t^{-1} = \alpha'\cos\sigma.$$

In order to find the actual configuration of the beam it is necessary to integrate (51)

$$\mathbf{R} = (1+\varepsilon)\left[s\cos\sigma\,\mathbf{m} + \frac{C_1}{L}\mathbf{Q}\left(\frac{Ls}{C_1}\mathbf{m}\right)\cdot(\mathbf{t}\times\mathbf{m}) - \frac{C_1}{L}(\mathbf{t}\times\mathbf{m})\right].$$

The vector in square brackets of this expression, describes a spiral on the cylinder of radius $R_0 = C_1\sin\sigma/|L|$. The axis of the cylinder is spanned on a vector $\mathbf{m}$ and passes through the point determined by a vector

$$(1+\varepsilon)C_1(\pi\cos\sigma\,\mathbf{m} - 2\mathbf{t}\times\mathbf{m})/L.$$

The length of one coil of a spiral is equal $2\pi C_1/|L|$. The step h of a spiral is equal $l\cos\sigma$.

9 Elastica of Euler (1744)

Mathematical statement

$$\mathbf{N}' = \mathbf{0}, \quad \mathbf{M}' + \mathbf{R}'\times\mathbf{N} = \mathbf{0}; \tag{59}$$

$$\mathbf{M} = \mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}\cdot\mathbf{\Phi} = (C_3 - C_1)(\mathbf{t}\cdot\mathbf{P}^T\cdot\mathbf{\Phi})\mathbf{P}\cdot\mathbf{t} + C_1\mathbf{\Phi}, \tag{60}$$

where $\mathbf{N}$ is defined by equation of equilibrium. Boundary conditions

$$s=0:\ \mathbf{R}=\mathbf{0}, \quad \mathbf{P}=\mathbf{E}; \qquad s=l:\ \mathbf{N}=-N\mathbf{t}, \quad \mathbf{M}=\mathbf{0}. \tag{61}$$

Kinematic relations

$$\mathbf{R}' = \mathbf{P}\cdot\mathbf{t}, \quad \mathbf{P}' = \mathbf{\Phi}\times\mathbf{P}, \quad |\mathbf{R}'| = 1.$$

The problem (59)–(61) has an obvious solution

$$\mathbf{R}(s) = s\,\mathbf{t}, \quad \mathbf{P}=\mathbf{E}, \quad \mathbf{N} = -N\mathbf{t}, \quad \mathbf{M}=\mathbf{0}. \tag{62}$$

As it was shown by Euler the solution (62) is unique solution if $N \le N_{cr}$. If $N > N_{cr}$, then there are another solutions. It is possible to prove that all this solutions are plain curves. Beside for the vector the next representation

$$\mathbf{\Phi} = \mathbf{R}'\times\mathbf{R}'' = -R_c^{-1}\mathbf{b} \equiv \psi'(s)\mathbf{b}, \quad \mathbf{b}\equiv\tilde{\mathbf{b}}, \quad \psi'(s) \equiv -R_c^{-1}(s) \tag{63}$$

may be found. In such a case the turn-tensor has a form

$$\mathbf{P} = \mathbf{Q}(\psi \mathbf{b}) \quad \Rightarrow \quad \mathbf{R}' = \cos\psi(s)\,\mathbf{t} + \sin\psi(s)\,\mathbf{b}. \tag{64}$$

Thus we have

$$\mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}\cdot\mathbf{\Phi} = C_1\,\psi'(s)\mathbf{b}. \tag{65}$$

For determination $\psi(s)$ we have well-known boundary value problem

$$C_1\psi'' + N\sin\psi = 0, \quad \psi(0) = 0, \quad \psi'(l) = 0. \tag{66}$$

If

$$N > N_{cr} \equiv \frac{\pi^2 C_2}{4l^2},$$

then (66) has nontrivial solutions. The exact solution of (66) is well-known. Let us show the approximate solution for small values of $\gamma \equiv 1 - \sqrt{N_{cr}/N} > 0$

$$\psi(s) = \psi_l \sin\vartheta = 4\left(\sqrt{\frac{N}{N_{cr}}} - 1\right)^{1/2} \sin\left[\frac{\pi s}{2l} + \frac{\gamma}{2}\sin\frac{\pi s}{l}\right]. \tag{67}$$

Let's sum up. If longitudinal stretching force is applied to the free end of a beam, then there is only one rectilinear equilibrium configuration. The situation varies, if on a beam acts compressing force. In this case always there is a rectilinear equilibrium configuration, which is determined by the following expressions

$$\mathbf{R}(s) = (1 - N/A)\,s\mathbf{t}, \quad \mathbf{P} = \mathbf{E}, \quad \mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{0}. \tag{68}$$

If the module of compressing force N exceeds value Euler's critical force N_{cr}, then there is one more solution submitted by the formula (67). Intuitively clearly, that at $N > N_{cr}$ the second solution is realized. The first solution will be unstable.

In the literature [2] at judgement about stability of an equilibrium configuration usually use the energetic reasons. Namely, the stable configuration is supposed to be those that has smaller energy. Strictly speaking, comparison of energies of equilibrium configurations have no the direct relation to concept of stability. An equilibrium configuration of conservative system is steady, if its potential energy has an isolated local minimum, which is not connected to energy of other equilibrium configuration. Nevertheless, from two possible equilibrium configurations the Nature if it is possible, chooses a configuration with smaller energy. Therefore in Euler's elastica it supposed that the bent configuration is stable, as potential energy in this case is less [2]. Nevertheless, a such arguments in Euler's elastica are not valid. The matter is that in a considered case a minimum of energy is not isolated. Actually we have family of the equilibrium bent configurations, and all of them possess the same energy. Really, the received decision allows to find an angle of turn unequivocally ψ around of a vector of a binormal $\mathbf{b}$, but the vector $\mathbf{b}$ has not been determined uniquely manner, for it is possible rotate $\mathbf{b}$ around $\mathbf{t}$

$$\mathbf{b} = \mathbf{Q}\left(\varphi(t)\mathbf{t}\right)\cdot\mathbf{b}_0,$$

where $\mathbf{b}_0$ is an arbitrary fixed vector orthogonal $\mathbf{t}$; $\varphi(t)$ is the arbitrary angle of turn around $\mathbf{t}$. From this it follows

$$\mathbf{P}(s) = \mathbf{Q}(\varphi\mathbf{t}) \cdot \mathbf{Q}(\psi\mathbf{b}_0) \cdot \mathbf{Q}^{\mathsf{T}}(\varphi\mathbf{t}) \quad \Rightarrow \quad \mathbf{P}\big|_{s=0} = \mathbf{E};$$

$$\mathbf{R}' = \mathbf{Q}(\varphi\mathbf{t}) \cdot [\cos\psi(s)\,\mathbf{t} + \sin\psi(s)\,\mathbf{b}_0]\,.$$

If φ depends on time, then an angular velocity may be calculated as [3]

$$\boldsymbol{\omega} = \dot{\varphi}\,[(1-\cos\psi)\,\mathbf{t} - \sin\psi\mathbf{b}\times\mathbf{t}]\,, \quad \boldsymbol{\omega}\big|_{s=0} = 0.$$

Thus, if in Euler's elastica we give to the bent beam small angular velocity, then it will slowly rotate around of the vector $\mathbf{t}$, running all set of equilibrium configurations. And for this it is not required of application of the external moment. It is necessary to emphasize, that we do not mean the rotations of the beam as the rigid whole. For example, the clamped end of a beam does not turn, for at $s = 0$ the turn-tensor becomes unit tensor for any value of φ. In fact the beam does not resist to special kinds of deformation, that for real beam does not correspond to the reality. Let's note that mentioned fact is present for any form of specific energy of a beam. The only important requirement is that the specific energy must be transversally isotropic. In particular, the marked feature explains the so-called Nikolai paradox [4]. Nikolai shows that the equilibrium configuration of a beam loaded by dead (or following) moment, is unstable for arbitrary small value of moment. This result is in sharp contradiction with experimental data. It is supposed that the Nikolai paradox is due to nonconservativity of problem. However this explanation is unsatisfactory, for it is easy to show, that the Nikolai paradox exists in a problem on twisting of a beam by the potential (conservative) moment.

10 Stationary rotations in the Euler elastica

Below the Euler elastica will be examined in dynamic statement. The equation of motion

$$\mathbf{N}'' = \rho F\ddot{\mathbf{R}}', \quad \mathbf{M}' + \mathbf{R}'\times\mathbf{N} = \mathbf{0}, \quad \mathbf{M} = C_1\,\mathbf{R}'\times\mathbf{R}'', \quad \mathbf{R}' = \mathbf{P}\cdot\mathbf{t}. \qquad (69)$$

The boundary conditions (61)

$$s = 0: \ \mathbf{R} = \mathbf{0}, \quad \mathbf{P} = \mathbf{E}, \quad \mathbf{N}' = \mathbf{0}; \qquad s = l: \ \mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{0}. \qquad (70)$$

Let's look for solution of the task (69)–(70) in a form

$$\mathbf{P}(s,t) = \mathbf{Q}\,[\varphi(t)\mathbf{t})] \cdot \mathbf{Q}\,[\psi(s)\mathbf{e})] \cdot \mathbf{Q}^{\mathsf{T}}\,[\varphi(t)\mathbf{t})]\,, \quad \mathbf{e}\cdot\mathbf{t} = 0, \qquad (71)$$

where $\mathbf{e}$ is the constant unit vector. The vector of bending-twisting $\boldsymbol{\Phi}$ corresponding to the turn-tensor (71)

$$\boldsymbol{\Phi} = \psi'(s)\mathbf{Q}\,[\varphi(t)\mathbf{t})] \cdot \mathbf{e} = \psi'(s)\mathbf{e}_*, \quad \mathbf{e}_*(t) \equiv \mathbf{Q}\,[\varphi(t)\mathbf{t})] \cdot \mathbf{e}. \qquad (72)$$

Besides there are formulas

$$\mathbf{R}' = \cos\psi(s)\,\mathbf{t} + \sin\psi(s)\,\mathbf{e}_*(t)\times\mathbf{t}, \quad \ddot{\mathbf{R}}' = \sin\psi(s)\left(\ddot{\varphi}\mathbf{e}_*(t) - \dot{\varphi}^2\mathbf{e}_*(t)\times\mathbf{t}\right).$$

For vector $\mathbf{N}$ may be used decomposition

$$\mathbf{N} = -N\mathbf{t} + Q_*\mathbf{e}_* + Q\mathbf{e}_* \times \mathbf{t}, \quad Q_*'(0,t) = Q'(0,t) = 0, \quad Q_*(l,t) = Q(l,t) = 0.$$

Substituting these expressions into the first equation from (69) one will get

$$Q'' = -\rho F \dot{\varphi}^2 \sin\psi, \quad Q_*'' = \rho F \ddot{\varphi} \sin\psi. \tag{73}$$

The vector of moment is expressed as

$$\mathbf{M} = C_1 \mathbf{R}' \times \mathbf{R}'' = C_1 \psi' \mathbf{e}_*.$$

The second equation from (69) is equivalent to

$$C_1\psi'' + N\sin\psi + Q\cos\psi - Q_*\sin\psi = 0, \quad Q_* = 0. \tag{74}$$

From this it follows that in the Euler elastica only the stationary rotations are possible

$$\ddot{\varphi} = 0 \quad \Rightarrow \quad \dot{\varphi} \equiv \omega = \text{const}. \tag{75}$$

Thus we obtain the next nonlinear boundary value problem

$$Q'' = -\rho F \omega^2 \sin\psi, \quad C_1\psi'' + N\sin\psi + Q\cos\psi = 0; \tag{76}$$

$$s = 0: \; Q'(0) = 0, \quad \psi(0) = 0; \qquad s = l: \; Q(l) = 0, \quad \psi'(l) = 0. \tag{77}$$

The problem (76)–(77) is difficult to find the exact solution. However it is easy to find the approximated solution for small value of ω^2. Let's use the decomposition

$$\psi(s) = \psi_{st}(s) + \vartheta(s), \quad |\vartheta(s)| \ll 1,$$

where $\psi_{st}(s)$ is the solution of the static task at $N > N_{cr}$. In such a case instead of (76)–(77) we obtain

$$Q'' = -\rho F \omega^2 \sin\psi_{st}, \quad C_1\vartheta'' + (N\cos\psi_{st})\,\vartheta = Q\cos\psi_{st}; \tag{78}$$

$$s = 0: \; Q'(0) = 0, \quad \vartheta(0) = 0; \qquad s = l: \; Q(l) = 0, \quad \vartheta'(l) = 0. \tag{79}$$

The problem (78)–(79) has unique solution.

Thus, the account of forces of inertia does not change a conclusion about presence rotating “equilibrium” configurations. That means, that the bent equilibrium configurations in the Euler elastica are unstable, for the turned bent configuration is not any more close to the original configurations.

It is necessary to emphasize, that experiment does not confirm a conclusion about presence rotating “equilibrium” configurations. The rough experiment which has been carried out by the author, has shown, that if bent equilibrium configuration slightly to push, low-frequency vibrations start, but not rotations.

11 The Nikolai paradox

11.1 Potential moment

Let us introduce a concept of potential moment. This concept is necessary for a statement and an analysis of many problems. Nevertheless a general definition of potential moment is absent in the literature.

***Definition**: A moment* $\mathbf{M}(t)$ *is called potential, if there exists a scalar function* $U(\boldsymbol{\theta})$ *depending on a turn-vector such that*

$$\mathbf{M} \cdot \boldsymbol{\omega} = -\dot{U}(\boldsymbol{\theta}) = -\frac{dU}{d\boldsymbol{\theta}} \cdot \dot{\boldsymbol{\theta}}. \tag{80}$$

One may obtain the equality

$$\dot{\boldsymbol{\theta}}(t) = \mathbf{Z}(\boldsymbol{\theta}) \cdot \boldsymbol{\omega}(t), \tag{81}$$

where

$$\mathbf{Z}(\boldsymbol{\theta}) = \mathbf{E} - \frac{1}{2}\mathbf{R} + \frac{1-g}{\theta^2}\mathbf{R}^2, \quad g = \frac{\theta \sin\theta}{2(1-\cos\theta)}, \quad \theta = |\boldsymbol{\theta}|. \tag{82}$$

The tensor $\mathbf{Z}(\boldsymbol{\theta})$ will be called the integrating tensor in the following. The equality (80) can be rewritten in the form

$$\left(\mathbf{M} + \frac{dU}{d\boldsymbol{\theta}} \cdot \mathbf{Z}\right) \cdot \boldsymbol{\omega} = 0.$$

From this it follows

$$\mathbf{M} = -\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{dU}{d\boldsymbol{\theta}} + \mathbf{f}(\boldsymbol{\theta}, \boldsymbol{\omega}) \times \boldsymbol{\omega}, \tag{83}$$

where $\mathbf{f}(\boldsymbol{\theta}, \boldsymbol{\omega})$ is an arbitrary function of vectors $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$.

***Definition**: A moment* $\mathbf{M}$ *is called positional, if* $\mathbf{M}$ *depends on the turn-vector* $\boldsymbol{\theta}$ *only. For the positional moment* $\mathbf{M}(\boldsymbol{\theta})$ *we have*

$$\mathbf{M}(\boldsymbol{\theta}) = -\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{dU(\boldsymbol{\theta})}{d\boldsymbol{\theta}}. \tag{84}$$

***Definition**: The potential* $U(\boldsymbol{\theta})$ *is called transversally isotropic with an axis of symmetry* $\mathbf{k}$*, if the equality*

$$U(\boldsymbol{\theta}) = U[\mathbf{Q}(\alpha\mathbf{k}) \cdot \boldsymbol{\theta}]$$

holds for any turn-tensor $\mathbf{Q}(\alpha\mathbf{k})$.

It can be proved that a general form of a transversally isotropic potential can be expressed as a function of two arguments

$$U(\theta) = F\left(\mathbf{k} \cdot \theta, \theta^2\right). \tag{85}$$

For this potential one can derive the expression

$$\mathbf{M}(\theta) = -2\frac{\partial F}{\partial(\theta^2)}\theta - \frac{\partial F}{\partial(\mathbf{k} \cdot \theta)}\mathbf{Z}^T \cdot \mathbf{k}. \tag{86}$$

There exists the obvious identity

$$(\mathbf{E} - \mathbf{Q}(\theta)) \cdot \theta = \left(\mathbf{E} - \mathbf{Q}^{\mathsf{T}}\right) \cdot \theta = \mathbf{0} \Longrightarrow (\mathbf{a} - \mathbf{a}') \cdot \theta = 0$$

for arbitrary $\mathbf{a}$, $\mathbf{a}' = \mathbf{Q} \cdot \mathbf{a}$. Taking into account this identity, we may obtain

$$\left(\mathbf{E} - \mathbf{Q}^{\mathsf{T}}(\theta)\right) \cdot \mathbf{M} = \frac{\partial F}{\partial (\mathbf{k} \cdot \theta)} \mathbf{k} \times \theta.$$

Multiplying this equality by the vector $\mathbf{k}$ we obtain

$$(\mathbf{k} - \mathbf{k}') \cdot \mathbf{M} = 0. \tag{87}$$

For the isotropic potential, equality (87) holds for any vector $\mathbf{a}$. Sometimes equality (87) is very important.

11.2 The equations of motion of a rigid body on elastic foundation

The inertia tensor is supposed to transversally isotropic

$$\mathbf{A} = A_1 (\mathbf{E} - \mathbf{k} \otimes \mathbf{k}) + A_3 \mathbf{k} \otimes \mathbf{k}, \quad \mathbf{d}_3 = \mathbf{k}, \qquad A_1 = A_2. \tag{88}$$

The position of a body at the instant t is called the actual position of a body. The motion of the body can be defined either by the turn-tensor $\mathbf{P}(t)$ or by the turn-vector $\boldsymbol{\theta}(t)$

$$\mathbf{P}(t) = \mathbf{Q}(\theta(t)).$$

The tensor of inertia $\mathbf{A}^{(t)}$ in the actual position is determined by

$$\mathbf{A}^{(t)} = \mathbf{P}(t) \cdot \mathbf{A} \cdot \mathbf{P}^{\mathsf{T}}(t). \tag{89}$$

If the tensor $\mathbf{A}$ is transversally isotropic, this results in

$$\mathbf{A}^{(t)} = A_1 (\mathbf{E} - \mathbf{k}' \otimes \mathbf{k}') + A_3 \mathbf{k}' \otimes \mathbf{k}', \qquad \mathbf{k}' = \mathbf{P} \cdot \mathbf{k}. \tag{90}$$

The kinetic moment of the body can be expressed in two forms. In terms of the left angular velocity we obtain

$$\mathbf{L} = \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{P}^{\mathsf{T}} \cdot \boldsymbol{\omega} = A_1 \boldsymbol{\omega} + (A_3 - A_1)(\mathbf{k}' \cdot \boldsymbol{\omega}) \mathbf{k}'. \tag{91}$$

In terms of the right angular velocity the kinetic moment has the form

$$\mathbf{L} = \mathbf{P} \cdot \mathbf{A} \cdot \boldsymbol{\Omega} = \mathbf{P} \cdot [A_1 \boldsymbol{\Omega} + (A_3 - A_1)(\mathbf{k} \cdot \boldsymbol{\Omega}) \mathbf{k}]. \tag{92}$$

Let us note that

$$\mathbf{k}' \cdot \boldsymbol{\omega} = \mathbf{k} \cdot \mathbf{P}^{\mathsf{T}} \cdot \boldsymbol{\omega} = \mathbf{k} \cdot \boldsymbol{\Omega}. \tag{93}$$

An external moment $\mathbf{M}$ acting on the body can be represented in the form

$$\mathbf{M} = \mathbf{M}_e + \mathbf{M}_{ext},$$

where $\mathbf{M}_e$ is a reaction of the elastic foundation and $\mathbf{M}_{ext}$ is an additional external moment. The elastic moment $\mathbf{M}_e$ is supposed to be positional one

$$\mathbf{M}_e = -\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{dU(\boldsymbol{\theta})}{d\boldsymbol{\theta}}. \tag{94}$$

The scalar function $U(\boldsymbol{\theta})$ is called the elastic energy. In the following, the elastic foundation is supposed to be transversally isotropic. Then the elastic moment can be represented in form (86), i.e.

$$\mathbf{M}_e(\boldsymbol{\theta}) = -C\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right)\boldsymbol{\theta} - D\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right)\mathbf{Z}^T(\boldsymbol{\theta})\cdot\mathbf{k}, \tag{95}$$

where the unit vector $\mathbf{k}$ is placed on the axis of isotropy of the body in the reference position, and

$$C = 2\frac{\partial}{\partial(\theta^2)}U\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right), \qquad D = \frac{\partial}{\partial(\mathbf{k}\cdot\boldsymbol{\theta})}U\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right). \tag{96}$$

For an external moment $\mathbf{M}_{ext}$ let us accept the expression

$$\mathbf{M}_{ext} = -\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{dV(\boldsymbol{\theta})}{d\boldsymbol{\theta}} + \mathbf{M}_{ex}, \tag{97}$$

where the first term describes the potential part of the external moment.

The second law of dynamics by Euler can be represented in two equivalent forms. In terms of the left angular velocity we find from $\dot{\mathbf{L}} = \mathbf{M}$

$$\left[\mathbf{P}(\boldsymbol{\theta})\cdot\mathbf{A}\cdot\mathbf{P}^T(\boldsymbol{\theta})\cdot\boldsymbol{\omega}\right]^{\cdot} + \mathbf{Z}^T(\boldsymbol{\theta})\cdot\frac{d(U+V)}{d\boldsymbol{\theta}} = \mathbf{M}_{ex}. \tag{98}$$

This equation has to be completed by the left Poisson equations

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta}\times\boldsymbol{\omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\omega}). \tag{99}$$

In terms of the right angular velocity, the model (98)–(99) can be represented as

$$\mathbf{A}\cdot\dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega}\times\mathbf{A}\cdot\boldsymbol{\Omega} + \mathbf{Z}(\boldsymbol{\theta})\cdot\frac{d(U+V)}{d\boldsymbol{\theta}} = \mathbf{P}^T(\boldsymbol{\theta})\cdot\mathbf{M}_{ex}, \tag{100}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta}\times\boldsymbol{\Omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\Omega}). \tag{101}$$

11.3 The regular precession

Let us consider a body with a transversally isotropic tensor of inertia. The elastic foundation is supposed to be transversally isotropic as well. The equations of motion are given by expressions (98), (99) and expression (95) for the elastic moment:

$$[A_1\boldsymbol{\omega} + (A_3 - A_1)(\mathbf{k}'\cdot\boldsymbol{\omega})\mathbf{k}']^{\cdot} + C\boldsymbol{\theta} + D\mathbf{Z}^T\cdot\mathbf{k} = \mathbf{0}, \quad \mathbf{k}' = \mathbf{P}\cdot\mathbf{k}, \tag{102}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta}\times\boldsymbol{\omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\omega}), \tag{103}$$

where the functions C and D are defined by (96). We assume a particular solution of system (102), (103) to be represented in the form

$$\theta = \vartheta \mathbf{p}', \quad \mathbf{p}' = \mathbf{Q}(\psi \mathbf{k}) \cdot \mathbf{p}, \quad \mathbf{p} \cdot \mathbf{k} = 0, \quad \mathbf{P} = \mathbf{Q}(\vartheta \mathbf{p}'), \tag{104}$$

where the motion (104) is called a regular precession if

$$\vartheta = \mathrm{const}, \quad \dot{\psi} = \mathrm{const} \quad \Rightarrow \quad \dot{\boldsymbol{\theta}} = \dot{\psi} \mathbf{k} \times \boldsymbol{\theta}. \tag{105}$$

The left angular velocity is given as

$$\boldsymbol{\omega} = \mathbf{Q}(\psi \mathbf{k}) \cdot \boldsymbol{\omega}_0, \quad \boldsymbol{\omega}_0 = \dot{\psi}\left[(1 - \cos\vartheta)\,\mathbf{k} + \sin\vartheta\,\mathbf{k} \times \mathbf{p}\right] = \mathrm{const}. \tag{106}$$

We see that the angular velocity vector ω is a precession of the vector ω_0 around the axis $\mathbf{k}$ orthogonal to the turn-vector:

$$\boldsymbol{\theta} \cdot \boldsymbol{\omega} = \boldsymbol{\theta} \cdot \boldsymbol{\Omega} = 0, \quad \mathbf{k} \cdot \boldsymbol{\theta} = 0.$$

In addition, let us accept the restriction

$$\mathsf{D}\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right)\big|_{\mathbf{k}\cdot\boldsymbol{\theta}=0} = \frac{\partial}{\partial\left(\mathbf{k} \cdot \boldsymbol{\theta}\right)} U\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right)\big|_{\mathbf{k}\cdot\boldsymbol{\theta}=0} = 0,$$

which is satisfied for most kinds of elastic energy. Then we obtain from Eq. (102) for the assumed solution

$$\dot{\psi}^2 = \frac{\mathsf{C}\left(\vartheta^2, 0\right)\vartheta}{\sin\vartheta\left[A_3\left(1 - \cos\vartheta\right) + A_1 \cos\vartheta\right]}. \tag{107}$$

11.4 The inertia elastic foundation.

Let us consider the inertia elastic foundation. In such a case instead of (95) we shall get

$$\mathbf{M}_e(\boldsymbol{\theta}) = -\mathsf{C}\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right)\boldsymbol{\theta} - \mathsf{D}\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right)\mathbf{Z}^{\mathsf{T}}(\boldsymbol{\theta}) \cdot \mathbf{k} - \mu\boldsymbol{\theta} \times \boldsymbol{\omega}. \tag{108}$$

Instead of (102)–(103) we obtain

$$\left[A_1 \boldsymbol{\omega} + (A_3 - A_1)(\mathbf{k}' \cdot \boldsymbol{\omega})\,\mathbf{k}'\right]^{\cdot} + \mathsf{C}\boldsymbol{\theta} + \mathsf{D}\mathbf{Z}^{\mathsf{T}} \cdot \mathbf{k} + \mu\boldsymbol{\theta} \times \boldsymbol{\omega} = \mathbf{0}, \tag{109}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1 - g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}), \quad \mathbf{k}' = \mathbf{P} \cdot \mathbf{k}. \tag{110}$$

It is to show that the regular precession is impossible. The angle of nutation tends to zero and the Nikolai paradox is impossible.

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