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Rotations of Rigid Body with Small Angles of Nutation

There are many problems which can be reduced to the analysis of rotation of rigid bodies under small angles of nutation. In such cases it is possible to obtain the simple linear equation with constant coefficients, solution of which gives all importent characteristics of the motion. In order to show the main futures of our method we shall consider only very simple example. However the method can be upplied to much more difficult problems.

1. Derivation of basic equation

Let us consider axially symmetric body with fixed point O. The unit vector $\underline{m}$ has vertical direction. The tensor of inertia of the body in reference position has a form $\underline{\underline{\theta}} = (\lambda - \mu)\, \underline{m}\otimes\underline{m} + \mu\underline{\underline{E}}$. In actual position tensor of inertia has a form $\underline{\underline{\theta}}^{(t)} = \underline{\underline{P}}\cdot\underline{\underline{\theta}}\cdot\underline{\underline{P}}^T = (\lambda - \mu)\, \underline{m}'\otimes\underline{m}' + \mu\underline{\underline{E}}$, where $\underline{m}' = \underline{\underline{P}}\cdot\underline{m}$, $\underline{\underline{P}}$ is a turn-tensor of body: $\underline{\underline{P}}\cdot\underline{\underline{P}}^T = \underline{\underline{E}}$. Let $\underline{\omega}$ be an angular velocity of the body $\underline{\omega} = -(\dot{\underline{\underline{P}}}\cdot\underline{\underline{P}}^T)_\times/2$ — see Zhilin P. A., ZAMM, 75, 1995, S.133-134. If we accept that the axis of the body is clamped in elastic circular plate, then the reactive moment $\underline{M}_e$ can be found by means of approximate formula $\underline{M}_e = c\underline{m}'\times\underline{m}$, where c is a stiffness of the plate. Of course, the vector $\underline{m}'\times\underline{m}$ is supposed to be small. The turn-tensor $\underline{\underline{P}}$ can be represented in terms of eulerian angles in the form

$$\underline{\underline{P}} = \underline{\underline{Q}}(\psi\underline{m})\cdot\underline{\underline{Q}}(\vartheta\underline{e})\cdot\underline{\underline{Q}}(\varphi\underline{m})\,, \qquad \underline{e}\cdot\underline{m} = 0\,, \qquad |\underline{e}| = 1\,; \tag{1}$$

where $\psi,\ \vartheta,\ \varphi$ are the angles of precession, nutation and own rotation respectively; $\beta = \psi + \varphi$. Let us assume, that angle of nutation ϑ is small one: $|\vartheta| \ll 1$. Then approximate formula for $\underline{\underline{P}}$ takes a form

$$\underline{\underline{P}} = \Big((1 - \tfrac{1}{2}\gamma^2)\underline{\underline{E}} + \underline{\gamma}\times\underline{\underline{E}} + \tfrac{1}{2}\underline{\gamma}\otimes\underline{\gamma}\Big)\cdot\underline{\underline{Q}}(\beta\underline{m})\,, \qquad \underline{\gamma} = \vartheta\underline{\underline{Q}}(\psi\underline{m})\cdot\underline{e}\,, \tag{2}$$

where $\gamma^2 = \underline{\gamma}\cdot\underline{\gamma} = \vartheta^2$. In the (2) quantities of an order $O(\gamma^3)$ were rejected. An angular velocity $\underline{\omega}$ and unit vector $\underline{m}'$ corresponding to the turn-tensor (2) can be found out by means of expressions

$$\underline{\omega} = (\dot\beta - \tfrac{1}{2}\dot\varphi\gamma^2)\,\underline{m} + \dot{\underline{\gamma}} + \dot\beta\,\underline{\gamma}\times\underline{m} + O(\gamma^3)\,, \qquad \underline{m}' = (1 - \tfrac{1}{2}\gamma^2)\,\underline{m} + \underline{\gamma}\times\underline{m} \tag{3}$$

Now we are able to write expression for angular momentum $\underline{L}$

$$\underline{L} = \underline{\underline{P}}\cdot\underline{\underline{\theta}}\cdot\underline{\underline{P}}^T\cdot\underline{\omega} = L_m\underline{m} + \mu\,\dot{\underline{\gamma}} + \lambda\dot\beta\,\underline{\gamma}\times\underline{m} + O(\gamma^3)\,, \tag{4}$$

where the quantity L_m will be given below. The second law of dynamics by Euler in our case has a form

$$\dot{\underline{L}} = \underline{M}_e - lq\,\underline{m}\times\underline{m}' = -(c - lq)\underline{\gamma}\,, \qquad (\underline{m}'\times\underline{m} = -\underline{\gamma})\,, \tag{5}$$

where q is the weight of the body, l is a distance from O to the center of mass of the body. The projection of the equation (5) on a direction of $\underline{m}$ gives

$$L_m = \lambda\dot\beta + \gamma^2\Big((\mu - \lambda)\dot\psi - \tfrac{1}{2}\lambda\dot\varphi\Big) = \text{const} \tag{6}$$

If we project the equation (5) on a plane orthogonal to $\underline{m}$ and take into account the expression (6), then we shall get the next equation

$$\mu\ddot{\underline{\gamma}} - L_m\underline{m}\times\dot{\underline{\gamma}} + (c - lq)\underline{\gamma} = \underline{0} \tag{7}$$

Solution of (7) gives to us the angles ψ and ϑ. The angle φ can be found from (6), where $\beta = \varphi + \psi$.

2. Determination of eulerian angles

The particular solution of (7) can be found in a form

$$\underline{\gamma} = \underline{\underline{Q}}(pt\underline{m})\cdot\underline{a}, \quad \underline{a}\cdot\underline{m} = 0, \quad \underline{a} = \text{const}, \quad p = \text{const} \quad \Rightarrow \quad \dot{\underline{\gamma}} = p\underline{m}\times\underline{\gamma}, \quad \ddot{\underline{\gamma}} = p\underline{m}\times(p\underline{m}\times\underline{\gamma}) = -p^2\underline{\gamma}$$

If we substitute these expressions into (7), then we shall get

$$\mu p^2 - L_m p - c + lq = 0 \quad \Rightarrow \quad p_{1,2} = \Big(L_m \pm \sqrt{L_m^2 + 4\mu(c - lq)}\Big) \Big/ 2\mu$$

A general solution of (7) has a form

$$\underline{\gamma} = \underline{\underline{Q}}(p_1 t \underline{m}) \cdot \underline{a}_1 + \underline{\underline{Q}}(p_2 t \underline{m}) \cdot \underline{a}_2 \tag{8}$$

Thus a vector $\underline{\gamma}$ is a sum of two vectors rotating around $\underline{m}$ with angular velocities p_1 and p_2. The expression (8) contains four arbitrary constants (vectors $\underline{a}_1$ and $\underline{a}_2$ are orthogonal $\underline{m}$). In addition we have a constant L_m and one more constant will be found after integration (6). So we are able to satisfy any initial conditions. By using (8) and (2) we can find a vector of nutation

$$\vartheta \underline{e} = \underline{\underline{Q}}_1 \cdot \underline{a}_1 + \underline{\underline{Q}}_2 \cdot \underline{a}_2 \,, \qquad \underline{\underline{Q}}_\alpha \equiv \underline{\underline{Q}}\Big((p_\alpha t - \psi)\, \underline{m}\Big) \,, \quad \alpha = 1, 2 \tag{9}$$

From here we see that a regular precession may take place only in two cases

$$a) \;\; \psi = p_1 t, \;\; \underline{a}_2 = 0 \,, \qquad b) \;\; \psi = p_2 t, \;\; \underline{a}_1 = 0$$

From (9) we have two equations to find the angles ϑ and ψ

$$\vartheta = \underline{e} \cdot \underline{\underline{Q}}_1 \cdot \underline{a}_1 + \underline{e} \cdot \underline{\underline{Q}}_2 \cdot \underline{a}_2 \,, \qquad 0 = (\underline{e} \times \underline{m}) \cdot \Big(\underline{\underline{Q}}_1 \cdot \underline{a}_1 + \underline{\underline{Q}}_2 \cdot \underline{a}_2\Big) \tag{10}$$

3. An example

Let us consider following initial conditions

$$t = 0: \quad \psi = \varphi = 0, \;\; \vartheta = \vartheta_0 \,; \qquad \dot{\psi} = \dot{\vartheta} = 0, \;\; \dot{\varphi} = \omega_0 \tag{11}$$

From here it follows

$$t = 0: \quad \underline{\gamma} = \vartheta_0 \, \underline{e}, \;\; \underline{\dot{\gamma}} = \underline{0}, \;\; L_m = (1 - \tfrac{1}{2}\vartheta_0^2)\lambda\omega_0$$

Using these conditions and an expression (8) we have

$$\underline{a}_1 = -\frac{p_2 \vartheta_0}{p_1 - p_2} \, \underline{e} \,, \qquad \underline{a}_2 = \frac{p_1 \vartheta_0}{p_1 - p_2} \, \underline{e}$$

Expressions (10) in such case take form

$$\vartheta = \frac{\vartheta_0}{p_1 - p_2} \Big((p_1 \cos p_2 t - p_2 \cos p_1 t) \cos\psi + (p_1 \sin p_2 t - p_2 \sin p_1 t) \sin\psi \Big) \,,$$

$$(p_1 \sin p_2 t - p_2 \sin p_1 t) \cos\psi - (p_1 \cos p_2 t - p_2 \cos p_1 t) \sin\psi = 0$$

Hence we have the final result

$$\vartheta(t) = \frac{\vartheta_0}{p_1 - p_2} \sqrt{\Big(p_1 - p_2\Big)^2 + 2 p_1 p_2 \Big(1 - \cos(p_1 - p_2) t\Big)} \,, \qquad \operatorname{tg} \psi = \frac{p_1 \sin p_2 t - p_2 \sin p_1 t}{p_1 \cos p_2 t - p_2 \cos p_1 t}$$

In order to find velocity $\dot{\varphi}$ we have to use (6) and expression

$$\dot{\psi} = \frac{p_1 p_2 \, (p_1 + p_2) \Big(1 - \cos(p_1 - p_2) t\Big)}{\Big(p_1 - p_2\Big)^2 + 2 p_1 p_2 \Big(1 - \cos(p_1 - p_2) t\Big)}$$

Unfortunately there is no place to give an analysis of the solution.

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