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Dynamics of the Two Rotors Gyrostat on a Nonlinear Elastic Foundation

Dynamics of multirotor gyrostats is important for the studying of many problems of mechanics and physics. However a general model of the multirotor gyrostat was not represented in publications up to now. In the report a particular case of such an object is shown. An extension to the general model can be made without any problems. The basic equations of two rotors gyrostat and some simple examples are represented in the paper.

1. Kinematics of two rotors gyrostat

Let there be given a rigid body A with the fixed point O. Let us choose in the body some materia axis OM clamped in an elastic foundation and passing through the point O. In what follows the position of the body A, when the elastic foundation is undeformed, will be accepted as the reference position. Let the unit vector $\mathbf{m}$ be situated on the axis OM when the body A is in the reference position. Inside the body A there are two axially symmetric bodies B and C that can be rotated by motors, stators of which are fixed with respect to the body A. The unit vectors $\mathbf{p}$ and $\mathbf{q}$ are situated on the axes of the bodies B and C respectively in the reference position. For the sake of simplicity the lines spanned on the vectors $\mathbf{p}$ and $\mathbf{q}$ are supposed to be passing through the point O. Let the tensors $\mathbf{P}$, $\mathbf{P_b}$ and $\mathbf{P_c}$ be the turn-tensors of the bodies A, B and C respectively. The next representations can be proved

$$\mathbf{P_b} = \mathbf{P}\cdot\mathbf{Q}\left(\alpha\mathbf{p}\right), \qquad \mathbf{P_c} = \mathbf{P}\cdot\mathbf{Q}\left(\beta\mathbf{q}\right), \tag{1}$$

where α and β are the angles of the turn of the rotors B and C with respect to the body A and the next notation was used

$$\mathbf{Q}\left(\varphi\mathbf{n}\right) = \left(\mathbf{1}-\cos\varphi\right)\mathbf{n}\otimes\mathbf{n}+\cos\varphi\mathbf{E}+\sin\varphi\mathbf{n}\times\mathbf{E}, \qquad \varphi=\varphi\mathbf{n} \tag{2}$$

where φ is called the vector of turn, $\mathbf{E}$ is the unit tensor. The left angular velocities are introduced by means of the equations by Poisson

$$\dot{\mathbf{P}} = \omega\times\mathbf{P}, \quad \dot{\mathbf{P}}_\mathbf{b} = \omega_\mathbf{b}\times\mathbf{P_b}, \quad \dot{\mathbf{P}}_\mathbf{c} = \omega_\mathbf{c}\times\mathbf{P_c}, \qquad \left(\dot{\mathbf{f}} = \mathbf{df/dt}\right) \tag{3}$$

One can prove the next relations

$$\omega_\mathbf{b} = \omega+\dot{\alpha}\mathbf{p}', \quad \omega_\mathbf{c} = \omega+\dot{\beta}\mathbf{q}', \;\; \mathbf{p}' = \mathbf{P}\cdot\mathbf{p}, \;\; \mathbf{q}' = \mathbf{P}\cdot\mathbf{q} \tag{4}$$

2. Kinetic moment of two rotors gyrostat and equations of motion

Let the tensors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be tensors of inertia of the bodies A, B, C respectively. These tensors are calculated in the reference position with respect to the point O. As it is known the kinetic moment of the gyrostat is the sum of the kinetic moments of the bodies A, B and C. It means that the next representation is valid

$$\mathbf{K} = \mathbf{P}\cdot\mathbf{A}\cdot\mathbf{P^T}\cdot\omega+\mathbf{P_b}\cdot\mathbf{B}\cdot\mathbf{P_b^T}\cdot\omega_\mathbf{b}+\mathbf{P_c}\cdot\mathbf{C}\cdot\mathbf{P_c^T}\cdot\omega_\mathbf{c} \tag{5}$$

The inertia tensors of rotors have to be transversally isotropic ones

$$\mathbf{B} = \left(\lambda_\mathbf{b}-\mu_\mathbf{b}\right)\mathbf{p}\otimes\mathbf{p}+\mu_\mathbf{b}\mathbf{E}, \;\; \mathbf{C} = \left(\lambda_\mathbf{c}-\mu_\mathbf{c}\right)\mathbf{q}\otimes\mathbf{q}+\mu_\mathbf{c}\mathbf{E}, \;\; \mathbf{Q}\left(\alpha\mathbf{p}\right)\cdot\mathbf{B}\cdot\mathbf{Q^T}\left(\alpha\mathbf{p}\right) = \mathbf{B}, \;\; \mathbf{Q}\left(\beta\mathbf{q}\right)\cdot\mathbf{C}\cdot\mathbf{Q^T}\left(\beta\mathbf{q}\right) = \mathbf{C} \tag{6}$$

Making use of $(1), (4), (6)$ the expression (5) can be rewritten in the form

$$\mathbf{K} = \mathbf{P}\cdot\mathbf{J}, \;\; \mathbf{J} = \mathbf{D}\cdot\mathbf{\Omega}+\lambda_\mathbf{b}\dot{\alpha}\mathbf{p}+\lambda_\mathbf{c}\dot{\beta}\mathbf{q}, \;\; \mathbf{D} = \mathbf{A}+\mathbf{B}+\mathbf{C} \tag{7}$$

where $\mathbf{\Omega} = \mathbf{P^T}\cdot\omega$ is the right angular velocity. Let us write down the second law of dynamics by Euler

$$\dot{\mathbf{K}} = \mathbf{P}\cdot\left(\dot{\mathbf{J}}+\mathbf{\Omega}\times\mathbf{J}\right) = \mathbf{M_{ext}}, \quad \mathbf{M_{ext}} = \mathbf{M_e}+\mathbf{M_c}+\mathbf{M_{ex}} \tag{8}$$

where $\mathbf{M_{ext}}$ is the total moment acting on the gyrostat, $\mathbf{M_e}$ is a reaction of an elastic foundation, $\mathbf{M_c}$ and $\mathbf{M_{ex}}$ are potential and nonpotential parts of an external moment. The next representations can be proved (see-author's report on GAMM-97)

$$\mathbf{M_e} = -\mathbf{Z^T}\cdot\frac{\mathbf{dU}}{\mathbf{d}\theta}, \;\; \mathbf{M_c} = -\mathbf{Z^T}\cdot\frac{\mathbf{dV}}{\mathbf{d}\theta}, \;\; \mathbf{Z} = \mathbf{E}-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{R}+\frac{\mathbf{1}-\mathbf{g}}{\theta^\mathbf{2}}\mathbf{R^2}, \quad \mathbf{R} = \theta\times\mathbf{E}, \;\; \mathbf{g} = \frac{\theta\sin\theta}{\mathbf{2}\left(\mathbf{1}-\cos\theta\right)} \tag{9}$$

where θ is a vector of turn of the body A, $\theta=|\theta|,$, $U(\theta)$ is an elastic energy of foundation, $V(\theta)$ is a potential of an external field. If the elastic foundation is transversally isotropic with the axis of isotropy $\mathbf{m}$ then we have $U(\theta)=F\left(\mathbf{m}\cdot\theta,\theta^{\mathbf{2}}\right)$ and

$$\mathbf{M_e}=-\mathbf{2}\frac{\partial\mathbf{F}}{\partial\left(\theta^{\mathbf{2}}\right)}\theta-\frac{\partial\mathbf{F}}{\partial\left(\mathbf{m}\cdot\theta\right)}\mathbf{Z}^{\mathbf{T}}\cdot\mathbf{m} \tag{10}$$

The function F is determined by physical properties of the foundation. For example, if the turn-vector θ is small then the function F can be taken in the simplest form

$$F(\theta)=\frac{1}{2}C\left[\theta^2-(\mathbf{m}\cdot\theta)^2\right]+\frac{1}{2}D(\mathbf{m}\cdot\theta)^2 \tag{11}$$

where C and D are called the bending rigidness and the torsional rigidness of the elastic foundation respectively. Now the equation (8) takes a form

$$\dot{\mathbf{J}}+\mathbf{\Omega}\times\mathbf{J}+\mathbf{2}\frac{\partial\mathbf{F}}{\partial\left(\theta^{\mathbf{2}}\right)}\theta+\frac{\partial\mathbf{F}}{\partial\left(\mathbf{m}\cdot\theta\right)}\mathbf{Z}\cdot\mathbf{m}+\mathbf{Z}\cdot\frac{\mathbf{dV}}{\mathbf{d}\theta}=\mathbf{P}^{\mathbf{T}}\cdot\mathbf{M_{ex}} \tag{12}$$

To this equation we have to add the kinematical relation

$$\dot{\theta}=\mathbf{\Omega}+\frac{\mathbf{1}}{\mathbf{2}}\theta\times\mathbf{\Omega}+\frac{\mathbf{1}-\mathbf{g}}{\theta^{\mathbf{2}}}\theta\times(\theta\times\mathbf{\Omega}) \tag{13}$$

The system of equations $(12)-(13)$ contains eight unknown functions: the vectors $\theta,\mathbf{\Omega}$ and scalars α,β. Therefore we need two scalar equations to add to the system $(12)-(13)$. In order to derive these additional equations we have to consider the rotors and motors inside the gyrostat. For the moments M_b and M_c of motors, rotating rotors B and C, we accept the simplest expressions

$$M_b=-\eta_b(\dot{\alpha}-\omega_b)\,,\quad M_c=-\eta_c\left(\dot{\beta}-\omega_c\right),\quad \eta_b\geq 0,\quad \eta_c\geq 0 \tag{14}$$

where ω_b and ω_c are the nominal velocities of motors, the coefficients η_b and η_c are determined by power of motors. If $\eta_b=\eta_c=0$ then motors are absent. If $\eta_b=\eta_c=\infty$ then the motors have infinitely big power. The equations of motion of rotors have the form

$$\lambda_b\ddot{\alpha}+\eta_b(\dot{\alpha}-\omega_b)+\lambda_b\dot{\mathbf{\Omega}}\cdot\mathbf{p}=\mathbf{0},\quad \lambda_{\mathbf{c}}\ddot{\beta}+\eta_{\mathbf{c}}\left(\dot{\beta}-\omega_{\mathbf{c}}\right)+\lambda_{\mathbf{c}}\dot{\mathbf{\Omega}}\cdot\mathbf{q}=\mathbf{0} \tag{15}$$

The system of equations (12), (13) and (14) gives to us the equations of motion of two rotors gyrostat on an elastic foundation. The important advantage of this equations is that they contain the first derivatives of the vectors θ and $\mathbf{\Omega}$.

3. Rotor with disbalance

A gyrostat is such a construction in which the distribution of mass does not change under its operation. For this rotors must be perfectly balanced. In reality any rotor has a small disbalance. Strictly speaking such construction can't be named a gyrostat. Nevertheless we are going to discuss this case since it is important for practical aims for example for centrifuges of different kind. Let us accept that the vector $\mathbf{m}$ coincides with the vector of gravity field. Besides we shall consider that the axes of rotor B coincides with the vector $\mathbf{m}$, i.e. $\mathbf{p}=\mathbf{m}$. Let us accept that the disbalance arises due to addition of the mass-point m to the rotor B. Let the position of the mass-point with respect to fixed point O in the reference position is determined by the vector $\mathbf{a}:\mathbf{a}=\mathbf{l_1m}+\mathbf{l_2n},\ \mathbf{m}\cdot\mathbf{n}=\mathbf{0}$. Then the inertia tensor $\mathbf{B}_*$ of the mass-point has the form

$$\mathbf{B}_*=\mathbf{m}\left(\mathbf{a}^{\mathbf{2}}\mathbf{E}-\mathbf{a}\otimes\mathbf{a}\right)=\mathbf{m}\left(\mathbf{l}_{\mathbf{1}}^{\mathbf{2}}+\mathbf{l}_{\mathbf{2}}^{\mathbf{2}}\right)\mathbf{E}-\mathbf{ml}_{\mathbf{1}}^{\mathbf{2}}\mathbf{m}\otimes\mathbf{m}-\mathbf{ml_1l_2}(\mathbf{m}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{m})-\mathbf{ml}_{\mathbf{2}}^{\mathbf{2}}\mathbf{n}\otimes\mathbf{n}$$

In what follows we accept $m\left(l_1^2+l_2^2\right)<<\mu_b$, $ml_1^2<<\lambda_b$ and

$$\mathbf{B}+\mathbf{B}_*=(\lambda_{\mathbf{b}}-\mu_{\mathbf{b}})\,\mathbf{m}\otimes\mathbf{m}+\mu_{\mathbf{b}}\mathbf{E}-\mathbf{ml_1l_2}(\mathbf{m}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{m})-\mathbf{ml}_{\mathbf{2}}^{\mathbf{2}}\mathbf{n}\otimes\mathbf{n} \tag{16}$$

In this case the equation of motion (12) must be replaced by the equation

$$\dot{\mathbf{J}}_*+\mathbf{\Omega}\times\mathbf{J}_*+\mathbf{2}\frac{\partial\mathbf{F}}{\partial\left(\theta^{\mathbf{2}}\right)}\theta+\frac{\partial\mathbf{F}}{\partial\left(\mathbf{m}\cdot\theta\right)}\mathbf{Z}\cdot\mathbf{m}+\mathbf{Z}\cdot\frac{\mathbf{dV}}{\mathbf{d}\theta}=\mathbf{P}^{\mathbf{T}}\cdot\mathbf{M_{ex}} \tag{17}$$

$$\dot{\mathbf{J}}_*=\mathbf{J}-\mathbf{m}\left[\mathbf{l_1l_2}(\mathbf{m}\otimes\mathbf{n}_*+\mathbf{n}_*\otimes\mathbf{m})+\mathbf{l}_{\mathbf{2}}^{\mathbf{2}}\mathbf{n}_*\otimes\mathbf{n}_*\right]\cdot\mathbf{\Omega}-\mathbf{ml_1l_2}\dot{\alpha}\mathbf{n}_*,\quad \mathbf{n}_*=\mathbf{Q}(\alpha\mathbf{m})\cdot\mathbf{n} \tag{18}$$

The equations (15) must be replaced by the next equations

$$\lambda_b\ddot{\alpha}+\eta_b(\dot{\alpha}-\omega_b)+\lambda_b\dot{\mathbf{\Omega}}\cdot\mathbf{m}-\mathbf{ml_1l_2}(\mathbf{\Omega}\cdot\mathbf{n}_*)^{\cdot}=\mathbf{0},\quad \lambda_{\mathbf{c}}\ddot{\beta}+\eta_{\mathbf{c}}\left(\dot{\beta}-\omega_{\mathbf{c}}\right)+\lambda_{\mathbf{c}}\dot{\mathbf{\Omega}}\cdot\mathbf{q}=\mathbf{0} \tag{19}$$

4. Linearization of basic equations. An example

The system of equations $(13), (17), (18)$ and (19) is rather complicated for analytical solution. However there are many cases when it is possible. For practical aims the basic system may be simplified. Because of elastic foundation the turns of carrying body A are small. Thus the system can be linearized with respect to the turn-vector $\theta : |\theta| << \mathbf{1}$.

$$\omega = \mathbf{\Omega} = \dot{\theta}, \quad \mathbf{P} = \mathbf{E} + \theta \times \mathbf{E}, \quad \mathbf{Z} = \mathbf{E} \tag{20}$$

In such a case the elastic energy may be defined by the expression (11). For the sake of simplicity let us accept that the rotors are coaxial. In addition the tensor of inertia $\mathbf{A}$ is supposed to be transversally isotropic. The equation (17) takes the form

$$\dot{\mathbf{J}}_* + \dot{\theta} \times \left[\left(\lambda_{\mathbf{b}} \dot{\alpha} + \underline{\lambda_{\mathbf{c}} \dot{\beta}} \right) \mathbf{m} - \underline{\mathbf{m l_1 l_2} \dot{\alpha} \mathbf{n}_*} \right] + \mathbf{C}\theta + (\mathbf{D} - \mathbf{C}) (\mathbf{m} \cdot \theta) \mathbf{m} + \underline{\frac{\mathbf{dV}}{\mathbf{d}\theta}} = \underline{\mathbf{M_{ex}} + \theta \times \mathbf{M_{ex}}} \tag{21}$$

$$\mathbf{J}_* = \mu \dot{\theta} + \left[\lambda_{\mathbf{b}} \dot{\alpha} + \underline{\lambda_{\mathbf{c}} \dot{\beta} + (\lambda - \mu) \mathbf{m} \cdot \dot{\theta} - \mathbf{m l_1 l_2 n}_* \cdot \dot{\theta}} \right] \mathbf{m} - \mathbf{m l_1 l_2} \left(\dot{\alpha} + \underline{\mathbf{m} \cdot \dot{\theta}} \right) \mathbf{n}_* \tag{22}$$

where $\lambda = \lambda_a + \lambda_b + \lambda_c$, $\mu = \mu_a + \mu_b + \mu_c$. The underlined terms here and below are absent in known applied investigations. The equations (19) take such a form

$$\lambda_b \ddot{\alpha} + \eta_b (\dot{\alpha} - \omega_b) + \underline{\lambda_b \ddot{\theta} \cdot \mathbf{m} - \mathbf{m l_1 l_2} \left(\dot{\theta} \cdot \mathbf{n}_* \right)^{\cdot}} = 0, \quad \underline{\lambda_c \ddot{\beta} + \eta_c \left(\dot{\beta} - \omega_c \right) + \lambda \ddot{\theta} \cdot \mathbf{m} = \mathbf{0}} \tag{23}$$

From the equations (23) it follows that in linear approximation the (21) must be replaced by the next equation

$$\dot{\mathbf{J}}_* + \dot{\theta} \times \left[\left(\lambda_{\mathbf{b}} \omega_{\mathbf{b}} + \underline{\lambda_{\mathbf{c}} \omega_{\mathbf{c}}} \right) \mathbf{m} - \underline{\mathbf{m l_1 l_2} \omega_{\mathbf{b}} \mathbf{n}_*} \right] + \mathbf{C}\theta + (\mathbf{D} - \mathbf{C}) (\mathbf{m} \cdot \theta) \mathbf{m} + \underline{\frac{\mathbf{dV}}{\mathbf{d}\theta}} = \underline{\mathbf{M_{ex}} + \theta \times \mathbf{M_{ex}}} \tag{24}$$

We are forced to drop the analysis of this equations. However let us remark that the difference from the known results may be very essential.

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